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**SKEW LINEAR VECTOR FIELDS ON SPHERES IN THE  
STABLE RANGE**

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# SKEW LINEAR VECTOR FIELDS ON SPHERES IN THE STABLE RANGE

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**THEOREM.** *Assume  $n > 2k$ . Then every  $(k - 1)$ -field on  $S^{n-1}$  is skew linear.*

**1. Introduction.** Skew linear vector fields on spheres have been studied by Strutt [6], Zvengrowski [8] and Milgram and Zvengrowski [4, 5]. Extensive calculation of projective homotopy classes in [5] led Milgram and Zvengrowski to conjecture that every  $r$ -field on  $S^{n-1}$  is skew linear. Here we will prove this conjecture in the stable range, as stated above.

After a reformulation using a construction of L. Woodward [7] and the results of [1], the theorem will follow from the Kahn–Priddy theorem [3].

Since proving this theorem I have learned that Milgram and Zvengrowski had already obtained the result using different methods [9]. They have also shown that 7 and 8-fields on  $S^{15}$  are skew linear, the two remaining cases excluded by the condition  $n > 2k$  and not already dealt with in [8]. L. Woodward has also proved the theorem by methods similar to those used here.

**2. Proof of the theorem.** If  $p: E \rightarrow B$  is a fibration let  $C(B; E)$  denote the set of vertical homotopy classes of cross sections to  $p$ . If  $Z_2$  acts freely on  $B$  and  $E$  in such a way that  $p$  is equivariant let  $C_{Z_2}(B; E)$  denote the set of equivariant vertical homotopy classes of equivariant cross sections to  $p$ .

Let  $V_{n,k}$  denote the Stiefel manifold of  $k$ -frames in  $R^n$  with the involution  $[v_1, \dots, v_k] \rightarrow [-v_1, \dots, -v_k]$ . Recall that a *skew linear*  $(k - 1)$ -field on  $S^{n-1}$  is a cross section to the bundle  $V_{n,k} \rightarrow S^{n-1}$  which is vertically homotopic to an equivariant cross section. Let  $L_{n,k}$  denote the space of equivariant maps  $S^{k-1} \rightarrow S^{n-1}$ . Fixing  $x_0 = (1, 0, \dots, 0) \in S^{k-1}$  as base point we have a fibration  $L_{n,k} \rightarrow S^{n-1}$  by evaluating at  $x_0$  and a commutative square

$$\begin{array}{ccc}
 V_{n,k} & \xrightarrow{\sigma} & L_{n,k} \\
 \downarrow & & \downarrow \\
 S^{n-1} & \xleftarrow{\quad} & S^{n-1}
 \end{array}$$

where  $\sigma$  is the natural inclusion. The antipodal map on  $S^{n-1}$  induces an involution on  $L_{n,k}$  such that the maps in the above diagram are equivariant. As is well known [2],  $\sigma$  is a  $(2(n-k)-1)$ -equivalence. Hence

$$C(S^{n-1}; V_{n,k}) \simeq C(S^{n-1}; L_{n,k})$$

and

$$C_{Z_2}(S^{n-1}; V_{n,k}) \simeq C_{Z_2}(S^{n-1}; L_{n,k}).$$

Let  $P_k$  denote  $(k-1)$ -dimensional real projective space and  $\eta_k$  the Hopf bundle over  $P_k$ . Let  $\text{Tr}(n\eta_k)$  (respectively,  $\text{Tr}_{Z_2}(n\eta_k)$ ) denote the set of fiber homotopy classes of fiber preserving maps (respectively, equivariant fiber homotopy classes of equivariant fiber preserving maps)  $P_k \times S^{n-1} \rightarrow S(n\eta_k)$ , whose restriction to the fiber over  $[x_0]$  is the identity map. Here  $S(n\eta_k)$  is the unit sphere bundle of  $n\eta_k$ . Define a map

$$\mu: C(S^{n-1}; L_{n,k}) \rightarrow \text{Tr}(n\eta_k)$$

by  $\Delta \rightarrow \tilde{\Delta}$  where  $\tilde{\Delta}([x], y) = [x, \Delta(y)(x)]$ ,  $x \in S^{k-1}$ ,  $y \in S^{n-1}$ . This map is a bijection; in fact the underlying function spaces are homeomorphic (see Woodward [7, Lemma 1,2]). Similarly we have a bijection

$$\mu_{Z_2}: C_{Z_2}(S^{n-1}; L_{n,k}) \rightarrow \text{Tr}_{Z_2}(n\eta_k).$$

Let  $G(S^{n-1})$  denote the identity component of the space of maps  $S^{n-1} \rightarrow S^{n-1}$  and let  $G = \text{inj. lim. } G(S^{n-1})$ . Let  $G_{Z_2} = \text{inj. lim. } G_{Z_2}(S^{n-1})$  where  $G_{Z_2}(S^{n-1})$  is the identity component of the space of equivariant maps  $S^{n-1} \rightarrow S^{n-1}$ . Fixing an equivariant fiber map  $f: S(n\eta_k) \rightarrow P_k \times S^{n-1}$  whose restriction to the fiber over  $[x_0]$  is the identity, we have equivalences

$$\nu: \text{Tr}(n\eta_k) \rightarrow [P_k; G]$$

and

$$\nu_{Z_2}: \text{Tr}_{Z_2}(n\eta_k) \rightarrow [P_k; G_{Z_2}].$$

Each of these is defined by sending  $h: P_k \times S^{n-1} \rightarrow S(n\eta_k)$  to the adjoint of

$$P_k \times S^{n-1} \xrightarrow{h} S(n\eta_k) \xrightarrow{f} P_k \times S^{n-1} \rightarrow S^{n-1}.$$

Here  $[\ ; ]$  denotes homotopy classes of base point preserving maps. Summarizing, let

$$\psi: C(S^{n-1}; V_{n,k}) \rightarrow [P_k; G]$$

denote the composite

$$C(S^{n-1}; V_{n,k}) \xrightarrow{\sigma^*} C(S^{n-1}; L_{n,k}) \xrightarrow{\mu} \text{Tr}(n\eta_k) \xrightarrow{\nu} [P_k; G]$$

and let  $\psi_{Z_2}$  denote its equivariant analogue.

LEMMA. *Assume  $n > 2k$ . There is a commutative square*

$$\begin{array}{ccc} C_{Z_2}(S^{n-1}; V_{n,k}) & \xrightarrow{\psi_{Z_2}} & [P_k; G_{Z_2}] \\ \downarrow \phi & & \downarrow \phi_* \\ C(S^{n-1}; V_{n,k}) & \xrightarrow{\psi} & [P_k; G] \end{array}$$

in which  $\psi$  and  $\psi_{Z_2}$  are equivalences and  $\phi$  is the forgetful map.

If  $X$  is a connected space let  $Q^0(X^+)$  denote the  $o$ -component of  $Q(X^+) = \Omega^\infty S^\infty(X^+)$ . By the main result of [1] there is a commutative square

$$\begin{array}{ccc} G_{Z_2} & \xrightarrow{\lambda} & Q^0(RP^{\infty+}) \\ \downarrow \phi & & \downarrow \tau \\ G & \xrightarrow{\lambda} & Q^0(S^0) \end{array}$$

in which the horizontal maps are homotopy equivalences and  $\tau$  is the transfer map associated with the double cover  $S^\infty \rightarrow RP^\infty$ . In view of this and the above lemma, our theorem will follow by showing that

$$\tau_*: [P_k; Q^0(RP^{\infty+})] \rightarrow [P_k; Q^0(S^0)]$$

is epimorphic. This is a consequence of the Kahn–Priddy theorem [3]. First note that both of these groups are finite and  $\tau_*$  is clearly onto

the odd primary part. The Kahn–Priddy result states that  $\tau_*$  also maps onto the 2–primary part. (Although they only consider the morphisms  $\tau_*: [S^m; Q^0(RP^{\infty+})] \rightarrow [S^m; Q^0(S^0)]$ , for all  $m$ , their proof is valid with  $S^m$  replaced by any finite complex.)

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