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**A RANDOM FIXED POINT THEOREM FOR A MULTIVALUED
CONTRACTION MAPPING**

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Some results on measurability of multivalued mappings are given. Then using them, the following random fixed point theorem is proved: **Theorem.** Let X be a Polish space, (T, \mathcal{A}) a measurable space. Let $F: T \times X \rightarrow CB(X)$ be a mapping such that for each $x \in X$, $F(\cdot, x)$ is measurable and for each $t \in T$, $F(t, \cdot)$ is $k(t)$ -contraction, where $k: T \rightarrow [0, 1)$ is measurable. Then there exists a measurable mapping $u: T \rightarrow X$ such that for every $t \in T$, $u(t) \in F(t, u(t))$.

1. Introduction. Random fixed point theorems for contraction mappings in Polish spaces were proved by Špaček [8], Hanš [2, 3], etc. For a brief survey of them and related results, we refer the reader to Bharucha–Reid [1, Chapter 3]. On the other hand, fixed point theorems for multivalued contraction mappings in complete metric spaces were obtained by Nadler [7], etc.

In this paper, in §3 we give some results on measurability and measurable selectors of multivalued mappings. Then in §4, using them we prove a random fixed point theorem for a multivalued contraction mapping in a Polish space.

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2. Preliminaries. Throughout this paper, let (X, d) be a Polish space, i.e., a separable complete metric space, and (T, \mathcal{A}) a measurable space. For any $x \in X$, $B \subset X$, we denote $d(x, B) = \inf\{d(x, y) : y \in B\}$. Let 2^X be the family of all subsets of X , $CB(X)$ the family of all nonempty bounded closed subsets of X , \mathcal{B} the σ -algebra of Borel subsets of X , respectively. Let D be the Hausdorff metric on $CB(X)$ induced by d . A mapping $S: X \rightarrow CB(X)$ is called k -Lipschitz, where $k \geq 0$, if for every $x, y \in X$, $D(S(x), S(y)) \leq kd(x, y)$. When $k < 1$, then S is called k -contraction. A mapping $F: T \rightarrow 2^X$ is called (\mathcal{A}) -measurable if for any open subset B of X , $F^{-1}(B) \in \mathcal{A}$, where $F^{-1}(B) = \{t \in T : F(t) \cap B \neq \emptyset\}$. Notice that in Himmelberg [5] this is called weakly measurable, but in this paper we use only this type of measurability for multivalued mappings, hence we omit the term 'weakly' for the sake of simplicity. A mapping $u: T \rightarrow X$ is said to be a measurable selector of a measurable mapping $F: T \rightarrow 2^X$ if u is measurable and for any $t \in T$, $u(t) \in F(t)$.

3. Some results on measurability. In this section we give important results related to the concept of measurability and measurable selector. They play a crucial role in proving a random fixed point theorem in §4.

PROPOSITION 1. *Let $\{F_n\}$ be a sequence of measurable mappings $F_n: T \rightarrow CB(X)$, and $F: T \rightarrow CB(X)$ a mapping such that for each $t \in T$, $D(F_n(t), F(t)) \rightarrow 0$ as $n \rightarrow \infty$. Then F is measurable.*

Proof. By Himmelberg [5, Theorem 3.5], it suffices to show that for each $x \in X$, the real-valued function on T by $t \rightarrow d(x, F(t))$ is measurable. For any $B, C \in CB(X)$, we have

$$|d(x, B) - d(x, C)| \leq D(B, C),$$

thus

$$|d(x, F_n(t)) - d(x, F(t))| \leq D(F_n(t), F(t)).$$

Since for any $t \in T$, $D(F_n(t), F(t)) \rightarrow 0$ as $n \rightarrow \infty$, $d(x, F_n(t)) \rightarrow d(x, F(t))$ as $n \rightarrow \infty$. Therefore, $d(x, F(\cdot))$ is the pointwise limit of measurable functions $\{d(x, F_n(\cdot))\}$, hence measurable.

PROPOSITION 2. *Let $F: T \times X \rightarrow CB(X)$ be a mapping such that for each $t \in T$, $F(t, \cdot)$ is $k(t)$ -Lipschitz and for each $x \in X$, $F(\cdot, x)$ is measurable. Let $u: T \rightarrow X$ be a measurable mapping, then the mapping $G: T \rightarrow CB(X)$ defined by $G(t) = F(t, u(t))$ ($t \in T$) is measurable.*

Proof. Since X is separable, there exists a countable subset $\{x_i\}$ of X such that $\text{cl}(\{x_i\}) = X$, where $\text{cl}(Y)$ is the closure of Y . For each n , denote

$$B_{1n} = \{x \in X: d(x, x_1) \leq 1/n\}$$

and

$$B_{in} = \{x \in X: d(x, x_i) \leq 1/n\} - \bigcup_{j=1}^{i-1} B_{jn} \quad (i = 2, 3, \dots),$$

then $\{B_{in}\}$ is a countable measurable partition of X , that is, $B_{in} \in \mathcal{B}$, $\bigcup_{i=1}^{\infty} B_{in} = X$ and if $i \neq j$, then $B_{in} \cap B_{jn} = \emptyset$. Define $F_n: T \times X \rightarrow CB(X)$ as follows:

$$F_n(t, x) = F(t, x) \quad \text{if } t \in T, \quad x \in B_{in}.$$

Then for any open subset B of X ,

$$\begin{aligned} & \{(t, x): F_n(t, x) \cap B \neq \emptyset\} \\ &= \bigcup_{i=1}^{\infty} \{t \in T: F(t, x_i) \cap B \neq \emptyset\} \times B_{i_n} \in \mathcal{A} \times \mathcal{B}, \end{aligned}$$

where $\mathcal{A} \times \mathcal{B}$ is the product σ -algebra on $T \times X$. Thus, for any n , F_n is $\mathcal{A} \times \mathcal{B}$ -measurable. For each $t \in T$, $x \in X$, there exists a unique i such that $x \in B_{i_n}$ and

$$\begin{aligned} D(F_n(t, x), F(t, x)) &= D(F(t, x_i), F(t, x)) \\ &\leq k(t)d(x_i, x) \leq k(t)/n. \end{aligned}$$

Hence $D(F_n(t, x), F(t, x)) \rightarrow 0$ as $n \rightarrow \infty$. By Proposition 1, F is $\mathcal{A} \times \mathcal{B}$ -measurable. The mapping $g: T \rightarrow T \times X$ defined by $g(t) = (t, u(t))$ ($t \in T$) is measurable in the sense that $g^{-1}(\mathcal{A} \times \mathcal{B}) \subset \mathcal{A}$. It follows that for any open subset B of X ,

$$G^{-1}(B) = g^{-1}(\{(t, x): F(t, x) \cap B \neq \emptyset\}) \in \mathcal{A},$$

and G is measurable.

PROPOSITION 3. *Let Y be a metric space, $f: T \times X \rightarrow Y$ a mapping such that for any $t \in T$, $f(t, \cdot)$ is continuous and for any $x \in X$, $f(\cdot, x)$ is measurable. Let $F: T \rightarrow 2^X$ be a measurable mapping such that for each $t \in T$, $F(t)$ is nonempty closed, and U an open subset of Y . Then the mapping $G: T \rightarrow 2^X$ by $G(t) = \{x \in F(t): f(t, x) \in U\}$ ($t \in T$) is measurable.*

Proof. By Himmelberg [5, Theorem 5.6], there exists a countable family $\{u_n\}$ of measurable selectors of F such that for each $t \in T$, $\text{cl}(\{u_n(t)\}) = F(t)$. Let B be an open subset of X , then

$$\begin{aligned} G^{-1}(B) &= \{t \in T: f(t, x) \in U \text{ for some } x \in F(t) \cap B\} \\ &= \{t \in T: f(t, u_n(t)) \in U, u_n(t) \in B \text{ for some } n\} \\ &= \bigcup_{n=1}^{\infty} \{t \in T: f(t, u_n(t)) \in U\} \cap u_n^{-1}(B). \end{aligned}$$

As in the proof of [5, Theorem 6.5],

$$\{t \in T: f(t, u_n(t)) \in U\} \in \mathcal{A},$$

hence $G^{-1}(B) \in \mathcal{A}$ and G is measurable.

PROPOSITION 4. *Let $F, G: T \rightarrow CB(X)$ be measurable mappings, $u: T \rightarrow X$ a measurable selector of F , $r: T \rightarrow (0, \infty)$ a measurable function. Then there exists a measurable selector $v: T \rightarrow X$ of G such that for any $t \in T$,*

$$d(u(t), v(t)) \leq D(F(t), G(t)) + r(t).$$

Proof. By Himmelberg [5, Theorem 5.6], there exist a countable family $\{u_n\}$ of measurable selectors of F and a countable family $\{v_n\}$ of measurable selectors of G such that for each $t \in T$, $\text{cl}(\{u_n(t)\}) = F(t)$ and $\text{cl}(\{v_n(t)\}) = G(t)$, respectively. It follows that

$$D(F(t), G(t)) = \max \left\{ \sup_i \inf_j d(u_i(t), v_j(t)), \sup_j \inf_i d(u_i(t), v_j(t)) \right\},$$

hence the real-valued function $D(F(\cdot), G(\cdot))$ on T is measurable. Define mappings $f: T \times X \rightarrow \mathbb{R}$ and $G_1: T \rightarrow 2^X$ by

$$f(t, x) = d(u(t), x) - D(F(t), G(t)) - r(t)$$

and

$$G_1(t) = \{x \in G(t): f(t, x) < 0\},$$

then by Proposition 3, G_1 is measurable, and by definition of the Hausdorff metric, $G_1(t)$ is nonempty for all $t \in T$. Thus, the mapping $G_2: T \rightarrow CB(X)$ by $G_2(t) = \text{cl}(G_1(t))$ ($t \in T$) is measurable and has a measurable selector $v: T \rightarrow X$ by Kuratowski and Ryll-Nardzewski [6, Theorem, p. 398]. For this v , we have the desired conclusion.

4. A random fixed point theorem. Now we prove a random fixed point theorem for a multivalued contraction mapping.

THEOREM. *Let $F: T \times X \rightarrow CB(X)$ be a mapping such that for each $x \in X$, $F(\cdot, x)$ is measurable and for each $t \in T$, $F(t, \cdot)$ is $k(t)$ -contraction, where $k: T \rightarrow [0, 1)$ is a measurable function. Then there exists a measurable mapping $u: T \rightarrow X$ such that for any $t \in T$, $u(t) \in F(t, u(t))$.*

Proof. Denote $A_1 = \{t \in T: 0 < k(t)\}$ and $A_2 = T - A_1$, then $A_1, A_2 \in \mathcal{A}$. We first consider on A_1 . Take a measurable mapping $v_0: A_1 \rightarrow X$. By Proposition 2, the mapping $F(\cdot, v_0(\cdot)): A_1 \rightarrow CB(X)$ is measurable, hence there exists a measurable selector $v_1: A_1 \rightarrow X$ of $F(\cdot, v_0(\cdot))$ by Kuratowski and Ryll-Nardzewski [6]. Then by Proposi-

tion 4, there exists a measurable selector $v_2: A_1 \rightarrow X$ of $F(\cdot, v_1(\cdot))$ such that for any $t \in T$,

$$d(v_1(t), v_2(t)) \leq D(F(t, v_0(t)), F(t, v_1(t))) + k(t).$$

By Proposition 4 again, there exists a measurable selector $v_3: A_1 \rightarrow X$ of $F(\cdot, v_2(\cdot))$ such that for any $t \in T$,

$$d(v_2(t), v_3(t)) \leq D(F(t, v_1(t)), F(t, v_2(t))) + k(t)^2.$$

By induction, we can choose a sequence of measurable mappings $v_n: A_1 \rightarrow X$ such that for each $t \in T$,

$$v_n(t) \in F(t, v_{n-1}(t))$$

and

$$d(v_n(t), v_{n+1}(t)) \leq D(F(t, v_{n-1}(t)), F(t, v_n(t))) + k(t)^n \quad (n = 1, 2, \dots).$$

Let $t \in A_1$ be arbitrarily fixed. For any n , we have

$$\begin{aligned} d(v_n(t), v_{n+1}(t)) &\leq D(F(t, v_{n-1}(t)), F(t, v_n(t))) + k(t)^n \\ &\leq k(t)d(v_{n-1}(t), v_n(t)) + k(t)^n \\ &\leq k(t)\{D(F(t, v_{n-2}(t)), F(t, v_{n-1}(t))) + k(t)^{n-1}\} + k(t)^n \\ &\leq k(t)^2d(v_{n-2}(t), v_{n-1}(t)) + 2k(t)^n \\ &\quad \text{---} \\ &\leq k(t)^nd(v_0(t), v_1(t)) + nk(t)^n. \end{aligned}$$

Thus, for every $n \leq m$,

$$\begin{aligned} d(v_n(t), v_{m+1}(t)) &\leq \sum_{i=n}^m d(v_i(t), v_{i+1}(t)) \\ &\leq \sum_{i=n}^m k(t)^i d(v_0(t), v_1(t)) + \sum_{i=n}^m ik(t)^i. \end{aligned}$$

Since $0 < k(t) < 1$, $\{v_n(t)\}$ is a Cauchy sequence in X , hence converges to some $v(t) \in X$. It follows that for any n ,

$$\begin{aligned} d(v(t), F(t, v(t))) &\leq d(v(t), v_n(t)) + d(v_n(t), F(t, v(t))) \\ &\leq d(v(t), v_n(t)) + D(F(t, v_{n-1}(t)), F(t, v(t))) \\ &\leq d(v(t), v_n(t)) + k(t)d(v_{n-1}(t), v(t)). \end{aligned}$$

This implies that $d(v(t), F(t, v(t))) = 0$. Since $F(t, v(t))$ is closed, $v(t) \in F(t, v(t))$. The mapping $v: A_1 \rightarrow X$ is the pointwise limit of measurable mappings $\{v_n\}$, hence measurable. Now we consider on A_2 . If $t \in A_2$, then for every $x, y \in X$,

$$D(F(t, x), F(t, y)) \leq k(t)d(x, y) = 0.$$

Thus we can set $F(t, x) = F_0(t)$ for all $t \in A_2$, $x \in X$, where $F_0: A_2 \rightarrow CB(X)$ is measurable. By Kuratowski and Ryll-Nardzewski [6], there exists a measurable selector $w: A_2 \rightarrow X$ of F_0 . Then for any $t \in A_2$, $w(t) \in T(t, w(t))$. Define $u: T \rightarrow X$ by

$$u(t) = \begin{cases} v(t) & \text{if } t \in A_1 \\ \text{or} \\ w(t) & \text{if } t \in A_2, \end{cases}$$

then u is measurable and for each $t \in T$, $u(t) \in F(t, u(t))$.

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