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Some results on measurability of multivalued mappings are given. Then using them, the following random fixed point theorem is proved: Theorem. Let X be a Polish space, $(T, \mathscr{A})$ a measurable space. Let $F\colon T \times X \to CB(X)$ be a mapping such that for each $x \in X$, $F(\cdot, x)$ is measurable and for each $t \in T$, $F(t, \cdot)$ is $k(t)$-contraction, where $k\colon T \to [0,1)$ is measurable. Then there exists a measurable mapping $u\colon T \to X$ such that for every $t \in T$, $u(t) \in F(t, u(t))$.

1. Introduction. Random fixed point theorems for contraction mappings in Polish spaces were proved by Špaček [8], Hanš [2, 3], etc. For a brief survey of them and related results, we refer the reader to Bharucha-Reid [1, Chapter 3]. On the other hand, fixed point theorems for multivalued contraction mappings in complete metric spaces were obtained by Nadler [7], etc.

In this paper, in §3 we give some results on measurability and measurable selectors of multivalued mappings. Then in §4, using them we prove a random fixed point theorem for a multivalued contraction mapping in a Polish space.

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2. Preliminaries. Throughout this paper, let (X, d) be a Polish space, i.e., a separable complete metric space, and $(T, \mathscr{A})$ a measurable space. For any $x \in X$, $B \subset X$, we denote $d(x, B) = \inf\{d(x, y)\colon y \in B\}$. Let 2^X be the family of all subsets of X, $CB(X)$ the family of all nonempty bounded closed subsets of X, $\mathscr{B}$ the σ-algebra of Borel subsets of X, respectively. Let D be the Hausdorff metric on $CB(X)$ induced by d. A mapping $S\colon X \to CB(X)$ is called *k-Lipschitz*, where $k \geqq 0$, if for every $x, y \in X$, $D(S(x), S(y)) \leqq kd(x, y)$. When $k < 1$, then S is called *k-contraction.* A mapping $F\colon T \to 2^X$ is called *($\mathscr{A}$-)measurable* if for any open subset B of X, $F^{-1}(B) \in \mathscr{A}$, where $F^{-1}(B) = \{t \in T\colon F(t) \cap B \neq \varnothing\}$. Notice that in Himmelberg [5] this is called weakly measurable, but in this paper we use only this type of measurability for multivalued mappings, hence we omit the term 'weakly' for the sake of simplicity. A mapping $u\colon T \to X$ is said to be a *measurable selector* of a measurable mapping $F\colon T \to 2^X$ if u is measurable and for any $t \in T$, $u(t) \in F(t)$.

3. Some results on measurability. In this section we give important results related to the concept of measurability and measurable selector. They play a crucial role in proving a random fixed point theorem in §4.

PROPOSITION 1. *Let $\{F_n\}$ be a sequence of measurable mappings F_n: $T \to CB(X)$, and F: $T \to CB(X)$ a mapping such that for each $t \in T$, $D(F_n(t), F(t)) \to 0$ as $n \to \infty$. Then F is measurable.*

Proof. By Himmelberg [**5**, Theorem 3.5], it suffices to show that for each $x \in X$, the real-valued function on T by $t \to d(x, F(t))$ is measurable. For any $B, C \in CB(X)$, we have

$$|d(x, B) - d(x, C)| \leqq D(B, C),$$

thus

$$|d(x, F_n(t)) - d(x, F(t))| \leqq D(F_n(t), F(t)).$$

Since for any $t \in T$, $D(F_n(t), F(t)) \to 0$ as $n \to \infty$, $d(x, F_n(t)) \to d(x, F(t))$ as $n \to \infty$. Therefore, $d(x, F(\cdot))$ is the pointwise limit of measurable functions $\{d(x, F_n(\cdot))\}$, hence measurable.

PROPOSITION 2. *Let F: $T \times X \to CB(X)$ be a mapping such that for each $t \in T$, $F(t, \cdot)$ is $k(t)$-Lipschitz and for each $x \in X$, $F(\cdot, x)$ is measurable. Let u: $T \to X$ be a measurable mapping, then the mapping G: $T \to CB(X)$ defined by $G(t) = F(t, u(t))$ $(t \in T)$ is measurable.*

Proof. Since X is separable, there exists a countable subset $\{x_i\}$ of X such that $\mathrm{cl}(\{x_i\}) = X$, where $\mathrm{cl}(Y)$ is the closure of Y. For each n, denote

$$B_{1n} = \{x \in X: d(x, x_1) \leqq 1/n\}$$

and

$$B_{in} = \{x \in X: d(x, x_i) \leqq 1/n\} - \bigcup_{j=1}^{i-1} B_{jn} \qquad (i = 2, 3, \cdots),$$

then $\{B_{in}\}$ is a countable measurable partition of X, that is, $B_{in} \in \mathcal{B}$, $\bigcup_{i=1}^{\infty} B_{in} = X$ and if $i \neq j$, then $B_{in} \cap B_{jn} = \varnothing$. Define F_n: $T \times X \to CB(X)$ as follows:

$$F_n(t, x) = F(t, x_i) \quad \text{if} \quad t \in T, \quad x \in B_{in}.$$

Then for any open subset B of X,

$$\{(t,x)\colon F_n(t,x)\cap B\neq\emptyset\}$$

$$=\bigcup_{i=1}^{\infty}\{t\in T\colon F(t,x_i)\cap B\neq\emptyset\}\times B_{in}\in\mathscr{A}\times\mathscr{B},$$

where $\mathscr{A}\times\mathscr{B}$ is the product σ-algebra on $T\times X$. Thus, for any n, F_n is $\mathscr{A}\times\mathscr{B}$-measurable. For each $t\in T$, $x\in X$, there exists a unique i such that $x\in B_{in}$ and

$$D(F_n(t,x),F(t,x))=D(F(t,x_i),F(t,x))$$

$$\leqq k(t)d(x_i,x)\leqq k(t)/n.$$

Hence $D(F_n(t,x),F(t,x))\to 0$ as $n\to\infty$. By Proposition 1, F is $\mathscr{A}\times\mathscr{B}$-measurable. The mapping $g\colon T\to T\times X$ defined by $g(t)=(t,u(t))$ $(t\in T)$ is measurable in the sense that $g^{-1}(\mathscr{A}\times\mathscr{B})\subset\mathscr{A}$. It follows that for any open subset B of X,

$$G^{-1}(B)=g^{-1}(\{(t,x)\colon F(t,x)\cap B\neq\emptyset\})\in\mathscr{A},$$

and G is measurable.

PROPOSITION 3. *Let Y be a metric space, $f\colon T\times X\to Y$ a mapping such that for any $t\in T$, $f(t,\cdot)$ is continuous and for any $x\in X$, $f(\cdot,x)$ is measurable. Let $F\colon T\to 2^X$ be a measurable mapping such that for each $t\in T$, $F(t)$ is nonempty closed, and U an open subset of Y. Then the mapping $G\colon T\to 2^X$ by $G(t)=\{x\in F(t)\colon f(t,x)\in U\}$ $(t\in T)$ is measurable.*

Proof. By Himmelberg [**5**, Theorem 5.6], there exists a countable family $\{u_n\}$ of measurable selectors of F such that for each $t\in T$, $\mathrm{cl}(\{u_n(t)\})=F(t)$. Let B be an open subset of X, then

$$G^{-1}(B)=\{t\in T\colon f(t,x)\in U \text{ for some } x\in F(t)\cap B\}$$

$$=\{t\in T\colon f(t,u_n(t))\in U,\ u_n(t)\in B \text{ for some } n\}$$

$$=\bigcup_{n=1}^{\infty}\{t\in T\colon f(t,u_n(t))\in U\}\cap u_n^{-1}(B).$$

As in the proof of [**5**, Theorem 6.5],

$$\{t\in T\colon f(t,u_n(t))\in U\}\in\mathscr{A},$$

hence $G^{-1}(B)\in\mathscr{A}$ and G is measurable.

PROPOSITION 4. *Let $F, G\colon T \to CB(X)$ be measurable mappings, $u\colon T \to X$ a measurable selector of F, $r\colon T \to (0, \infty)$ a measurable function. Then there exists a measurable selector $v\colon T \to X$ of G such that for any $t \in T$,*

$$d(u(t), v(t)) \leqq D(F(t), G(t)) + r(t).$$

Proof. By Himmelberg [**5**, Theorem 5.6], there exist a countable family $\{u_n\}$ of measurable selectors of F and a countable family $\{v_n\}$ of measurable selectors of G such that for each $t \in T$, $\mathrm{cl}(\{u_n(t)\}) = F(t)$ and $\mathrm{cl}(\{v_n(t)\}) = G(t)$, respectively. It follows that

$$D(F(t), G(t)) = \max\left\{\sup_i \inf_j d(u_i(t), v_j(t)), \sup_j \inf_i d(u_i(t), v_j(t))\right\},$$

hence the real-valued function $D(F(\cdot), G(\cdot))$ on T is measurable. Define mappings $f\colon T \times X \to R$ and $G_1\colon T \to 2^X$ by

$$f(t, x) = d(u(t), x) - D(F(t), G(t)) - r(t)$$

and

$$G_1(t) = \{x \in G(t)\colon f(t, x) < 0\},$$

then by Proposition 3, G_1 is measurable, and by definition of the Hausdorff metric, $G_1(t)$ is nonempty for all $t \in T$. Thus, the mapping $G_2\colon T \to CB(X)$ by $G_2(t) = \mathrm{cl}(G_1(t))$ $(t \in T)$ is measurable and has a measurable selector $v\colon T \to X$ by Kuratowski and Ryll-Nardzewski [**6**, Theorem, p. 398]. For this v, we have the desired conclusion.

4. A random fixed point theorem. Now we prove a random fixed point theorem for a multivalued contraction mapping.

THEOREM. *Let $F\colon T \times X \to CB(X)$ be a mapping such that for each $x \in X$, $F(\cdot, x)$ is measurable and for each $t \in T$, $F(t, \cdot)$ is $k(t)$-contraction, where $k\colon T \to [0, 1)$ is a measurable function. Then there exists a measurable mapping $u\colon T \to X$ such that for any $t \in T$, $u(t) \in F(t, u(t))$.*

Proof. Denote $A_1 = \{t \in T\colon 0 < k(t)\}$ and $A_2 = T - A_1$, then $A_1, A_2 \in \mathcal{A}$. We first consider on A_1. Take a measurable mapping $v_0\colon A_1 \to X$. By Proposition 2, the mapping $F(\cdot, v_0(\cdot))\colon A_1 \to CB(X)$ is measurable, hence there exists a measurable selector $v_1\colon A_1 \to X$ of $F(\cdot, v_0(\cdot))$ by Kuratowski and Ryll-Nardzewski [**6**]. Then by Proposition 4, there exists a measurable selector v_2: $A_1 \to X$ of $F(\cdot, v_1(\cdot))$ such that for any $t \in T$,

$$d(v_1(t), v_2(t)) \leqq D(F(t, v_0(t)), F(t, v_1(t))) + k(t).$$

By Proposition 4 again, there exists a measurable selector v_3: $A_1 \to X$ of $F(\cdot, v_2(\cdot))$ such that for any $t \in T$,

$$d(v_2(t), v_3(t)) \leqq D(F(t, v_1(t)), F(t, v_2(t))) + k(t)^2.$$

By induction, we can choose a sequence of measurable mappings v_n: $A_1 \to X$ such that for each $t \in T$,

$$v_n(t) \in F(t, v_{n-1}(t))$$

and

$$d(v_n(t), v_{n+1}(t)) \leqq D(F(t, v_{n-1}(t)), F(t, v_n(t))) + k(t)^n \quad (n = 1, 2, \cdots).$$

Let $t \in A_1$ be arbitrarily fixed. For any n, we have

$$\begin{aligned} d(v_n(t), v_{n+1}(t)) &\leqq D(F(t, v_{n-1}(t)), F(t, v_n(t))) + k(t)^n \\ &\leqq k(t) d(v_{n-1}(t), v_n(t)) + k(t)^n \\ &\leqq k(t)\{D(F(t, v_{n-2}(t)), F(t, v_{n-1}(t))) + k(t)^{n-1}\} + k(t)^n \\ &\leqq k(t)^2 d(v_{n-2}(t), v_{n-1}(t)) + 2k(t)^n \\ &\quad \text{--- --- ---} \\ &\leqq k(t)^n d(v_0(t), v_1(t)) + nk(t)^n. \end{aligned}$$

Thus, for every $n \leqq m$,

$$\begin{aligned} d(v_n(t), v_{m+1}(t)) &\leqq \sum_{i=n}^{m} d(v_i(t), v_{i+1}(t)) \\ &\leqq \sum_{i=n}^{m} k(t)^i d(v_0(t), v_1(t)) + \sum_{i=n}^{m} ik(t)^i. \end{aligned}$$

Since $0 < k(t) < 1$, $\{v_n(t)\}$ is a Cauchy sequence in X, hence converges to some $v(t) \in X$. It follows that for any n,

$$\begin{aligned} d(v(t), F(t, v(t))) &\leqq d(v(t), v_n(t)) + d(v_n(t), F(t, v(t))) \\ &\leqq d(v(t), v_n(t)) + D(F(t, v_{n-1}(t)), F(t, v(t))) \\ &\leqq d(v(t), v_n(t)) + k(t) d(v_{n-1}(t), v(t)). \end{aligned}$$

This implies that $d(v(t), F(t, v(t))) = 0$. Since $F(t, v(t))$ is closed, $v(t) \in F(t, v(t))$. The mapping $v: A_1 \to X$ is the pointwise limit of measurable mappings $\{v_n\}$, hence measurable. Now we consider on A_2. If $t \in A_2$, then for every $x, y \in X$,

$$D(F(t, x), F(t, y)) \leqq k(t) d(x, y) = 0.$$

Thus we can set $F(t, x) = F_0(t)$ for all $t \in A_2$, $x \in X$, where F_0: $A_2 \to CB(X)$ is measurable. By Kuratowski and Ryll-Nardzewski **[6]**, there exists a measurable selector $w: A_2 \to X$ of F_0. Then for any $t \in A_2$, $w(t) \in T(t, w(t))$. Define $u: T \to X$ by

$$u(t) = \begin{cases} v(t) & \text{if} \quad t \in A_1 \\ \text{or} & \\ w(t) & \text{if} \quad t \in A_2, \end{cases}$$

then u is measurable and for each $t \in T$, $u(t) \in F(t, u(t))$.

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