A RANDOM FIXED POINT THEOREM FOR A MULTIVALUED
CONTRACTION MAPPING

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Some results on measurability of multivalued mappings are given. Then using them, the following random fixed point theorem is proved: Theorem. Let $X$ be a Polish space, $(T, \mathcal{A})$ a measurable space. Let $F: T \times X \to CB(X)$ be a mapping such that for each $x \in X$, $F(\cdot, x)$ is measurable and for each $t \in T$, $F(t, \cdot)$ is $k(t)$-contraction, where $k: T \to [0, 1)$ is measurable. Then there exists a measurable mapping $u: T \to X$ such that for every $t \in T$, $u(t) \in F(t, u(t))$.

1. Introduction. Random fixed point theorems for contraction mappings in Polish spaces were proved by Špaček [8], Hanš [2,3], etc. For a brief survey of them and related results, we refer the reader to Bharucha-Reid [1, Chapter 3]. On the other hand, fixed point theorems for multivalued contraction mappings in complete metric spaces were obtained by Nadler [7], etc.

In this paper, in §3 we give some results on measurability and measurable selectors of multivalued mappings. Then in §4, using them we prove a random fixed point theorem for a multivalued contraction mapping in a Polish space.

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2. Preliminaries. Throughout this paper, let $(X, d)$ be a Polish space, i.e., a separable complete metric space, and $(T, \mathcal{A})$ a measurable space. For any $x \in X$, $B \subset X$, we denote $d(x, B) = \inf \{d(x, y) : y \in B\}$. Let $2^X$ be the family of all subsets of $X$, $CB(X)$ the family of all nonempty bounded closed subsets of $X$, $\mathcal{B}$ the $\sigma$-algebra of Borel subsets of $X$, respectively. Let $D$ be the Hausdorff metric on $CB(X)$ induced by $d$. A mapping $S: X \to CB(X)$ is called $k$-Lipschitz, where $k \geq 0$, if for every $x, y \in X$, $D(S(x), S(y)) \leq kd(x, y)$. When $k < 1$, then $S$ is called $k$-contraction. A mapping $F: T \to 2^X$ is called $(\mathcal{A}$-)measurable if for any open subset $B$ of $X$, $F^{-1}(B) \in \mathcal{A}$, where $F^{-1}(B) = \{t \in T : F(t) \cap B \neq \emptyset\}$. Notice that in Himmelberg [5] this is called weakly measurable, but in this paper we use only this type of measurability for multivalued mappings, hence we omit the term 'weakly' for the sake of simplicity. A mapping $u: T \to X$ is said to be a measurable selector of a measurable mapping $F: T \to 2^X$ if $u$ is measurable and for any $t \in T$, $u(t) \in F(t)$.
3. Some results on measurability. In this section we give important results related to the concept of measurability and measurable selector. They play a crucial role in proving a random fixed point theorem in §4.

Proposition 1. Let \( \{F_n\} \) be a sequence of measurable mappings \( F_n: T \to CB(X) \), and \( F: T \to CB(X) \) a mapping such that for each \( t \in T \),
\[ D(F_n(t), F(t)) \to 0 \quad \text{as} \quad n \to \infty. \]
Then \( F \) is measurable.

Proof. By Himmelberg [5, Theorem 3.5], it suffices to show that for each \( x \in X \), the real-valued function on \( T \) by \( t \to d(x, F(t)) \) is measurable. For any \( B, C \subseteq CB(X) \), we have
\[ |d(x, B) - d(x, C)| \leq D(B, C), \]
thus
\[ |d(x, F_n(t)) - d(x, F(t))| \leq D(F_n(t), F(t)). \]
Since for any \( t \in T \), \( D(F_n(t), F(t)) \to 0 \) as \( n \to \infty \), \( d(x, F_n(t)) \to d(x, F(t)) \) as \( n \to \infty \). Therefore, \( d(x, F(\cdot)) \) is the pointwise limit of measurable functions \( \{d(x, F_n(\cdot))\} \), hence measurable.

Proposition 2. Let \( F: T \times X \to CB(X) \) be a mapping such that for each \( t \in T \), \( F(t, \cdot) \) is \( k(t) \)-Lipschitz and for each \( x \in X \), \( F(\cdot, x) \) is measurable. Let \( u: T \to X \) be a measurable mapping, then the mapping \( G: T \to CB(X) \) defined by \( G(t) = F(t, u(t)) \) \( (t \in T) \) is measurable.

Proof. Since \( X \) is separable, there exists a countable subset \( \{x_i\} \) of \( X \) such that \( \text{cl}(\{x_i\}) = X \), where \( \text{cl}(Y) \) is the closure of \( Y \). For each \( n \), denote
\[ B_{1n} = \{x \in X: d(x, x_i) \leq 1/n\} \]
and
\[ B_{in} = \{x \in X: d(x, x_i) \leq 1/n\} - \bigcup_{j=1}^{i-1} B_{jn} \quad (i = 2, 3, \ldots), \]
then \( \{B_{in}\} \) is a countable measurable partition of \( X \), that is, \( B_{in} \in \mathcal{B} \), \( \bigcup_{i=1}^{\infty} B_{in} = X \) and if \( i \neq j \), then \( B_{in} \cap B_{jn} = \emptyset \). Define \( F_n: T \times X \to CB(X) \) as follows:
\[ F_n(t, x) = F(t, x_i) \quad \text{if} \quad t \in T, \quad x \in B_{in}. \]
Then for any open subset $B$ of $X$,

$$
\{(t, x): F_n(t, x) \cap B \neq \emptyset\}
$$

$$
= \bigcup_{i=1}^{\infty} \{t \in T: F(t, x_i) \cap B \neq \emptyset\} \times B_m \in \mathcal{A} \times \mathcal{B},
$$

where $\mathcal{A} \times \mathcal{B}$ is the product $\sigma$-algebra on $T \times X$. Thus, for any $n$, $F_n$ is $\mathcal{A} \times \mathcal{B}$-measurable. For each $t \in T$, $x \in X$, there exists a unique $i$ such that $x \in B_m$ and

$$
D(F_n(t, x), F(t, x)) = D(F(t, x_i), F(t, x))
$$

$$
\leq k(t) d(x, x) \leq k(t)/n.
$$

Hence $D(F_n(t, x), F(t, x)) \to 0$ as $n \to \infty$. By Proposition 1, $F$ is $\mathcal{A} \times \mathcal{B}$-measurable. The mapping $g: T \to T \times X$ defined by $g(t) = (t, u(t))$ ($t \in T$) is measurable in the sense that $g^{-1}(A \times B) \subset A$. It follows that for any open subset $B$ of $X$,

$$
G^{-1}(B) = g^{-1}(\{(t, x): F(t, x) \cap B \neq \emptyset\}) \in \mathcal{A},
$$

and $G$ is measurable.

**Proposition 3.** Let $Y$ be a metric space, $f: T \times X \to Y$ a mapping such that for any $t \in T$, $f(t, \cdot)$ is continuous and for any $x \in X$, $f(\cdot, x)$ is measurable. Let $F: T \to 2^X$ be a measurable mapping such that for each $t \in T$, $F(t)$ is nonempty closed, and $U$ an open subset of $Y$. Then the mapping $G: T \to 2^X$ by $G(t) = \{x \in F(t): f(t, x) \in U\}$ ($t \in T$) is measurable.

**Proof.** By Himmelberg [5, Theorem 5.6], there exists a countable family $\{u_n\}$ of measurable selectors of $F$ such that for each $t \in T$, $\text{cl}(\{u_n(t)\}) = F(t)$. Let $B$ be an open subset of $X$, then

$$
G^{-1}(B) = \{t \in T: f(t, x) \in U \text{ for some } x \in F(t) \cap B\}
$$

$$
= \{t \in T: f(t, u_n(t)) \in U, u_n(t) \in B \text{ for some } n\}
$$

$$
= \bigcup_{n=1}^{\infty} \{t \in T: f(t, u_n(t)) \in U\} \cap u_n^{-1}(B).
$$

As in the proof of [5, Theorem 6.5],

$$
\{t \in T: f(t, u_n(t)) \in U\} \in \mathcal{A},
$$

hence $G^{-1}(B) \in \mathcal{A}$ and $G$ is measurable.
**Proposition 4.** Let $F, G: T \to CB(X)$ be measurable mappings, $u: T \to X$ a measurable selector of $F$, $r: T \to (0, \infty)$ a measurable function. Then there exists a measurable selector $v: T \to X$ of $G$ such that for any $t \in T$,

$$d(u(t), v(t)) \leq D(F(t), G(t)) + r(t).$$

**Proof.** By Himmelberg [5, Theorem 5.6], there exist a countable family $\{u_n\}$ of measurable selectors of $F$ and a countable family $\{v_n\}$ of measurable selectors of $G$ such that for each $t \in T$, $\text{cl}(\{u_n(t)\}) = F(t)$ and $\text{cl}(\{v_n(t)\}) = G(t)$, respectively. It follows that

$$D(F(t), G(t)) = \max \left\{ \sup_i \inf_j d(u_i(t), v_j(t)), \sup_i \inf_j d(u_i(t), v_i(t)) \right\},$$

hence the real-valued function $D(F(\cdot), G(\cdot))$ on $T$ is measurable.

Define mappings $f: T \times X \to \mathbb{R}$ and $G_1: T \to 2^X$ by

$$f(t, x) = d(u(t), x) - D(F(t), G(t)) - r(t)$$

and

$$G_1(t) = \{x \in G(t): f(t, x) < 0\},$$

then by Proposition 3, $G_1$ is measurable, and by definition of the Hausdorff metric, $G_1(t)$ is nonempty for all $t \in T$. Thus, the mapping $G_2: T \to CB(X)$ by $G_2(t) = \text{cl}(G_1(t))$ ($t \in T$) is measurable and has a measurable selector $v: T \to X$ by Kuratowski and Ryll-Nardzewski [6, Theorem, p. 398]. For this $v$, we have the desired conclusion.

**4. A random fixed point theorem.** Now we prove a random fixed point theorem for a multivalued contraction mapping.

**Theorem.** Let $F: T \times X \to CB(X)$ be a mapping such that for each $x \in X$, $F(\cdot, x)$ is measurable and for each $t \in T$, $F(t, \cdot)$ is $k(t)$-contraction, where $k: T \to [0,1)$ is a measurable function. Then there exists a measurable mapping $u: T \to X$ such that for any $t \in T$, $u(t) \in F(t, u(t))$.

**Proof.** Denote $A_1 = \{t \in T: 0 < k(t)\}$ and $A_2 = T - A_1$, then $A_1, A_2 \in \mathcal{A}$. We first consider on $A_1$. Take a measurable mapping $v_0: A_1 \to X$. By Proposition 2, the mapping $F(\cdot, v_0(\cdot)): A_1 \to CB(X)$ is measurable, hence there exists a measurable selector $v_1: A_1 \to X$ of $F(\cdot, v_0(\cdot))$ by Kuratowski and Ryll-Nardzewski [6]. Then by Proposi-
tion 4, there exists a measurable selector \( v_2 : A \to X \) of \( F(\cdot, v_1(\cdot)) \) such that for any \( t \in T \),

\[
d(v_1(t), v_2(t)) \leq D(F(t, v_0(t)), F(t, v_1(t))) + k(t).
\]

By Proposition 4 again, there exists a measurable selector \( v_3 : A \to X \) of \( F(\cdot, v_2(\cdot)) \) such that for any \( t \in T \),

\[
d(v_2(t), v_3(t)) \leq D(F(t, v_1(t)), F(t, v_2(t))) + k(t)^2.
\]

By induction, we can choose a sequence of measurable mappings \( v_n : A \to X \) such that for each \( t \in T \),

\[
v_n(t) \in F(t, v_{n-1}(t))
\]

and

\[
d(v_n(t), v_{n+1}(t)) \leq D(F(t, v_{n-1}(t)), F(t, v_n(t))) + k(t)^n \quad (n = 1, 2, \cdots).
\]

Let \( t \in A_1 \) be arbitrarily fixed. For any \( n \), we have

\[
d(v_n(t), v_{n+1}(t)) \leq D(F(t, v_{n-1}(t)), F(t, v_n(t))) + k(t)^n
\]

\[
\leq k(t) d(v_{n-1}(t), v_n(t)) + k(t)^n
\]

\[
\leq k(t) \{ D(F(t, v_{n-2}(t)), F(t, v_{n-1}(t))) + k(t)^{n-1} \} + k(t)^n
\]

\[
\leq k(t)^2 d(v_{n-2}(t), v_{n-1}(t)) + 2k(t)^n
\]

\[
\leq \cdots
\]

\[
\leq k(t)^n d(v_0(t), v_1(t)) + nk(t)^n.
\]

Thus, for every \( n \leq m \),

\[
d(v_n(t), v_{m+1}(t)) \leq \sum_{i=n}^{m} d(v_i(t), v_{i+1}(t))
\]

\[
\leq \sum_{i=n}^{m} k(t)^i d(v_0(t), v_1(t)) + \sum_{i=n}^{m} i k(t)^i.
\]

Since \( 0 < k(t) < 1 \), \( \{v_n(t)\} \) is a Cauchy sequence in \( X \), hence converges to some \( v(t) \in X \). It follows that for any \( n \),

\[
d(v(t), F(t, v(t)) \leq d(v(t), v_n(t)) + d(v_n(t), F(t, v(t)))
\]

\[
\leq d(v(t), v_n(t)) + D(F(t, v_{n-1}(t)), F(t, v(t)))
\]

\[
\leq d(v(t), v_n(t)) + k(t) d(v_{n-1}(t), v(t)).
\]
This implies that $d(v(t), F(t, v(t))) = 0$. Since $F(t, v(t))$ is closed, $v(t) \in F(t, v(t))$. The mapping $v : A_1 \to X$ is the pointwise limit of measurable mappings $\{v_n\}$, hence measurable. Now we consider on $A_2$. If $t \in A_2$, then for every $x, y \in X$,

$$D(F(t, x), F(t, y)) \leq k(t)d(x, y) = 0.$$ 

Thus we can set $F(t, x) = F_0(t)$ for all $t \in A_2$, $x \in X$, where $F_0 : A_2 \to CB(X)$ is measurable. By Kuratowski and Ryll-Nardzewski [6], there exists a measurable selector $w : A_2 \to X$ of $F_0$. Then for any $t \in A_2$, $w(t) \in T(t, w(t))$. Define $u : T \to X$ by

$$u(t) = \begin{cases} v(t) & \text{if } t \in A_1 \\ w(t) & \text{if } t \in A_2, \end{cases}$$

then $u$ is measurable and for each $t \in T$, $u(t) \in F(t, u(t))$.

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