E-UNITARY COVERS FOR INVERSE SEMIGROUPS

D. B. McALISTER AND NORMAN R. REILLY
E-UNITARY COVERS FOR INVERSE SEMIGROUPS

D. B. McALISTER AND N. R. REILLY

An inverse semigroup is called $E$-unitary if the equations $ea = e = e^2$ together imply $a^2 = a$. In a previous paper, the first author showed that every inverse semigroup has an $E$-unitary cover. That is, if $S$ is an inverse semigroup, there is an $E$-unitary inverse semigroup $P$ and an idempotent separating homomorphism of $P$ onto $S$. The purpose of this paper is to consider the problem of constructing $E$-unitary covers for $S$.

Let $S$ be an inverse semigroup and let $F$ be an inverse semigroup, with group of units $G$, containing $S$ as an inverse subsemigroup and suppose that, for each $s \in S$, there exists $g \in G$ such that $s \leq g$. Then $\{(s, g) \in S \times G : s \leq g\}$ is an $E$-unitary cover of $S$. The main result of §1 shows that every $E$-unitary cover of $S$ can be obtained in this way. It follows from this that the problem of finding $E$-unitary covers for $S$ can be reduced to an embedding problem. A further corollary to this result is the fact that, if $P$ is an $E$-unitary cover of $S$ and $P$ has maximal group homomorphic image $G$, then $P$ is a subdirect product of $S$ and $G$ and so can be described in terms of $S$ and $G$ alone. The remainder of this paper is concerned with giving such a description.

1. $E$-unitary covers. An inverse semigroup is called $E$-unitary if the equations $ea = e = e^2$ together imply $a^2 = a$. It was shown in [4] that every inverse semigroup $S$ has an $E$-unitary cover in the sense that there is an $E$-unitary inverse semigroup $P$ together with an idempotent separating homomorphism $\theta$ of $P$ onto $S$. It was further shown in [5] that every $E$-unitary inverse semigroup is isomorphic to a $P(G, \mathcal{X}, \mathcal{Y})$ where $\mathcal{X}$ is a down directed partially ordered set with $\mathcal{X}$ an ideal and subsemilattice of $\mathcal{X}$ and where $G$ acts on $\mathcal{X}$ by order automorphisms in such a way that $\mathcal{X} = G\mathcal{Y}$; see [5] for details. The group $G$ in $P = P(G, \mathcal{X}, \mathcal{Y})$ is isomorphic to the maximum group homomorphic image $P/\sigma$ of $P$ where

$$\sigma = \{(a, b) \in P \times P : ea = eb \text{ for some } e^2 = e \in P\}.$$

Definition 1.1. Let $S$ be an inverse semigroup and let $G$ be a group. Then an $E$-unitary inverse semigroup $P$ is an $E$-unitary cover of $S$ through $G$ if

(i) $P/\sigma \cong G$

(ii) there is an idempotent separating homomorphism $\theta$ of $P$ onto $S$. 

Thus, if $P = P(G, \mathcal{I}, \mathcal{U})$ is an $E$-unitary cover of $S$ then $P$ is an $E$-unitary cover of $S$ through $G$. As stated, the problem of finding $E$-unitary covers of an inverse semigroup $S$ consists of finding homomorphisms onto $S$. The main result of this section shows that this problem can be replaced by an embedding problem.

**Definition 1.2.** [2] Let $S = S'\upharpoonright$ be an inverse semigroup, with group of units $G$. Then $S$ is a **factorizable** inverse semigroup if and only if, for each $a \in S$ there exists $g \in G$ such that $a \leq g$.

Chen and Hsieh showed in [1] that every inverse semigroup $S$ can be embedded in a factorizable inverse semigroup. Indeed, let $\theta$ be a homomorphism of $S$ into the symmetric inverse semigroup $\mathcal{S}_X$ on a set $X$. Let $Y = X$ if $X$ is finite and $Y = X \cup X'$, with $X \cap X' = \emptyset$, $|X| = |X'|$, otherwise. Then $F = \{\alpha \in \mathcal{S}_Y : \alpha \leq \gamma \text{ for some permutation } \gamma \text{ of } Y\}$ is a factorizable inverse semigroup which contains $S\theta$.

**Proposition 1.3.** Let $F$ be a factorizable inverse semigroup with group of units $G$ and let $\theta$ be a one-to-one homomorphism of an inverse semigroup $S$ into $F$. Suppose that for each $g \in G$ there exists $s \in S$ such that $s\theta \leq g$. Then

$$P = \{(s, g) \in S \times G : s\theta \leq g\}$$

is an $E$-unitary cover of $S$ through $G$.

**Proof.** It is clear that $P$ is an inverse subsemigroup of $S \times G$ and, because $F$ is factorizable, that the first projection $\pi_1: P \to S$ is an idempotent separating homomorphism of $P$ onto $S$. Likewise, because of the condition on the homomorphism $\theta: S \to F$, the second projection $\pi_2: P \to G$ is a homomorphism of $P$ onto $G$.

Now $(s, g), (t, h) \in P$ with $(s, g)\pi_2 = (t, h)\pi_2$ implies $g = h$ thus $(st^{-1})\theta = s\theta(t\theta)^{-1} \leq gh^{-1} = 1$. It follows from this that $st^{-1}$ is idempotent so that $es = et$, for some idempotent $e$ in $S$. But then $(e, 1) \in P$ and $(e, 1)(s, g) = (e, 1)(t, h)$ so that $(s, g)\sigma(t, h)$. This shows that $\pi_2 \circ \pi_2^{-1} \subseteq \sigma$. On the other hand, since $\pi_2$ is a homomorphism onto a group, $\sigma \subseteq \pi_2 \circ \pi_2^{-1}$. Hence $P$ has maximum group homomorphic image $G$.

Finally, since $(s, 1) \in P$ implies $s\theta \leq 1$, and so $s^2 = s$, if $(s, g)\pi_2 = 1$ then $(s, g)$ is idempotent. Hence $P$ is $E$-unitary.

It follows from the remarks before Proposition 1.3 that every inverse semigroup has an $E$-unitary cover. The main result of this section shows that every $E$-unitary cover of $S$ through $G$ is constructed as in Proposition 1.3 for some factorizable inverse semigroup $F$, with group of
units $G$, containing $S$ as an inverse subsemigroup. In order to prove this, we need some lemmas.

**Lemma 1.4.** Let $\theta: P(G, \mathcal{X}, \mathcal{Y}) \to S$ be an idempotent separating homomorphism of an $E$-unitary inverse semigroup $P(G, \mathcal{X}, \mathcal{Y})$ onto $S$. For each $A \in \mathcal{Y}$ set $N_A = \{g \in G: g^{-1}A \in \mathcal{Y} \text{ and } (A, g) \theta = (A, 1) \theta\}$; if $X = gA$ with $g \in G$, $A \in \mathcal{Y}$, set $N_x = gN_Ag^{-1}$. Then $N_X$ is well defined. Further, the relation $\pi$ on $\mathcal{X} \times G$ defined by

$$(A, g) \pi (B, h) \text{ if and only if } A = B \text{ and } gh^{-1} \in N_A$$

is an equivalence on $\mathcal{X} \times G$ inducing $\theta \circ \theta^{-1}$ on $P(G, \mathcal{X}, \mathcal{Y})$.

**Proof.** It was shown in [5] that the subgroups $N_A, A \in \mathcal{Y}$ satisfy the following three conditions

(i) $N_A \triangleleft C_A = \{g \in G: gB = B \text{ for all } B \subseteq A\}$ for $A \in \mathcal{Y}$;
(ii) $A, gA \in \mathcal{Y}$ implies $N_{gA} = gN_Ag^{-1}$;
(iii) $A \subseteq B \in \mathcal{Y}$ implies $N_B \subseteq N_A$.

We use (ii), to show that $N_X$ is well defined. Suppose $gA = hB$, $A, B \in \mathcal{Y}$. Then $B = h^{-1}gA$ so that, $N_B = (h^{-1}g)N_A(h^{-1}g)^{-1}$ by (ii). Thus $hN_Bh^{-1} = gN_Ag^{-1}$. When the $N_X, X \in \mathcal{X}$ are defined in this way, it is easy to see that they obey the analogs of (i), (ii), (iii) and the remainder of the lemma follows easily.

For each $C \in \mathcal{X}, g \in G$, we shall denote by $[C, g]$ the $\pi$-class containing $(C, g)$. Further, we shall denote by $\mathcal{I}$ the set $(\mathcal{X} \times G)/\pi \cup G$.

**Lemma 1.5.** For each $s \in S$ such that $s = (A, g) \theta$, set

$$\Delta_{\rho_s} = \{[C, h]: h^{-1}C \subseteq A\}$$

and $[C, h]_{\rho_s} = [C, hg]$ for each $[C, h] \in \Delta_{\rho_s}$.

Then $\rho: s \to \rho_s$ is a faithful representation of $S$ by one-to-one, partial transformations of $\mathcal{X}$.

**Proof.** This follows straightforwardly, using the fact that $\{N_X: X \in \mathcal{X}\}$ obeys (i), (ii), (iii) of Lemma 1.4.

For each $g \in G$, define $\alpha_g: \mathcal{X} \to \mathcal{X}$ by

$$h\alpha_g = hg \text{ and } [C, h]_{\alpha_g} = [C, hg]$$

for each $h \in G, [C, h] \in (\mathcal{X} \times G)/\pi$. Then, as in Lemma 1.5, it follows that $\alpha: h \to \alpha_h$ is a faithful representation of $G$ by permutations of $\mathcal{X}$. Let $F = \{g \in \mathcal{F}_G: \gamma \leq \alpha_g \text{ for some } g \in G\}$. 


PROPOSITION 1.6. \( F \) is a factorizable inverse semigroup which contains \( S_p \). Further, \( F \) has group of units \( \{ \alpha_g : g \in G \} \) and for each \( \alpha_g \) there exists \( \rho, \leq \alpha_g \); also

\[
P(G, \mathcal{X}, \mathcal{Y}) \cong \{(s, g) \in S \times G : \rho \leq \alpha_g \}.
\]

\( \text{Proof.} \) The only part requiring verification is that \( P(G, \mathcal{X}, \mathcal{Y}) \cong P \) where \( P = \{(s, g) \in S \times G : \rho \leq \alpha_g \} \).

Define \( \phi : P(G, \mathcal{X}, \mathcal{Y}) \to P \) by \( (A, g) \phi = ((A, g) \theta, \alpha_g) \). Then, since \( \theta \) is idempotent separating and \( \alpha \) is faithful, \( \phi \) is one-to-one. Further

\[
(A, g) \phi (B, h) \phi = ((A, g) \theta, \alpha_g)((B, h) \theta, \alpha_h)
\]

\[
= ((A \wedge gB, gh) \theta, \alpha_g, \alpha_h)
\]

\[
= ((A, g)(B, h)) \phi
\]

so \( \phi \) is a homomorphism.

Finally, if \( (s, \alpha_g) \in P \), where \( s = (B, h) \theta \), then \( \rho, \leq \alpha_g \) implies \( [C, 1] \rho = [C, 1] \alpha_g \) for each \( C \subseteq B \); that is, \( [C, h] = [C, g] \) for each \( C \subseteq B \). In particular, \( [B, h] = [B, g] \) so that \( s = (B, h) \theta = (B, g) \theta \). Hence \( (s, \alpha_g) = (B, g) \phi \) so that \( \phi \) is onto.

Summing up, we have the following theorem.

THEOREM 1.7. Let \( G \) be a group and let \( S \) be an inverse semigroup. Let \( F \) be a factorizable inverse semigroup with group of units \( G \) which contains \( S \) as an inverse subsemigroup. Suppose that, for each \( g \in G \), there exists \( s \in S \) such that \( s \leq g \). Then

\[
\{(s, g) \in S \times G : s \leq g \}
\]

is an \( E \)-unitary cover of \( S \) through \( G \). Conversely, each \( E \)-unitary cover is isomorphic to a cover obtained in this way.

COROLLARY 1.8. Let \( P \) be an \( E \)-unitary cover of \( S \) through \( G \). Then \( P \) is a subdirect product of \( S \) and \( G \).

Theorem 1.7 shows that the problem of finding \( E \)-unitary covers of \( S \) through \( G \) is equivalent to finding an embedding of \( S \) into a factorizable inverse semigroup. Such embeddings are hard to classify as they may be much larger than \( S \) and \( G \). On the other hand, Corollary 1.8 shows that every \( E \)-unitary cover of \( S \) through \( G \) is a subdirect product of \( S \) and \( G \) and therefore depends only on \( S \) and \( G \). In the next section, we turn to the problem of constructing those subdirect products of \( S \) and \( G \) which are \( E \)-unitary covers of \( S \) through \( G \).
2. Subdirect products of inverse semigroups.

DEFINITION 2.1. Let $S$ and $T$ be inverse semigroups. Then a mapping $\phi: S \rightarrow 2^T$ is a subhomomorphism of $S$ into $T$ if

(i) $\phi(s) \neq \emptyset$ for each $s \in S$;
(ii) $\phi(s)\phi(t) \subseteq \phi(st)$ for all $s, t \in S$;
(iii) $\phi(s^{-1}) = \phi(s)^{-1}$ for each $s \in S$,

where, for any $A \subseteq T$, $A^{-1} = \{a^{-1} : a \in A\}$.

The set $\phi(S) = \{t \in T : t \in \phi(s) \text{ for some } s \in S\}$ is, from (ii) and (iii), an inverse subsemigroup of $T$. We say that $\phi$ is surjective if $T = \phi(S)$.

PROPOSITION 2.2. Let $S$ and $T$ be inverse semigroups and let $\phi$ be a surjective subhomomorphism of $S$ into $T$. Then

$$\Pi(S, T, \phi) = \{(s, t) \in S \times T : t \in \phi(s)\}$$

is an inverse semigroup which is a subdirect product of $S$ and $T$.

Conversely, suppose that $V$ is an inverse semigroup which is a subdirect product of $S$ and $T$ and let $\psi$ be the induced homomorphism of $V$ into $S \times T$. Then $\phi$ defined by

$$\phi(s) = \{t \in T : (s, t) \in V\psi\}$$

is a surjective subhomomorphism of $S$ into $T$. Further

$$V\psi = \Pi(S, T, \phi).$$

Proof. This is straightforward.

Proposition 2.2 shows that every $E$-unitary cover of $S$ through $G$ is determined by a subhomomorphism of $S$ into $G$; and, dually, by a subhomomorphism of $G$ into $S$. We shall consider these two approaches and the relationships between them in the later sections of the paper.

3. Subhomomorphisms into a group. In this section we shall describe the subhomomorphisms of an inverse semigroup $S$ into a group $G$. Note, however, that not every subhomomorphism $\phi$ of $S$ into $G$ gives an $E$-unitary cover of $S$ through $G$.

DEFINITION 3.1. Let $S$ be an inverse semigroup and let $G$ be a group. Then a subhomomorphism $\phi: S \rightarrow G$ is unitary if

$$1 \in \phi(s) \quad \text{implies} \quad s^2 = s.$$
Note that, if \( s \) is an idempotent then
\[
\phi(s)\phi(s)^{-1} = \phi(s)\phi(s^{-1}) \subseteq \phi(ss^{-1}) = \phi(s)
\]
so that \( 1 \in \phi(s) \).

**Proposition 3.2.** Let \( S \) be an inverse semigroup and let \( G \) be a group. Suppose that \( \phi \) is a surjective unitary subhomomorphism of \( S \) into \( G \). Then \( \Pi(S,G,\phi) \) is an \( E \)-unitary cover of \( S \) through \( G \). Conversely, let \( P \) be an \( E \)-unitary cover of \( S \) through \( G \) with \( \psi \) the induced homomorphism \( P \rightarrow S \times G \). Then \( \phi \) defined by
\[
\phi(s) = \{ g \in G : (s, g) \in P\psi \}
\]
is a surjective unitary subhomomorphism of \( S \) into \( G \).

**Proof.** Suppose that \( \phi \) is unitary. Then, since the idempotents of \( P = \Pi(S,G,\phi) \) are the elements \((e, 1)\) with \( e^2 = e \) in \( S \), the projection \( \pi_s \) of \( P \) onto \( S \) is idempotent separating.

The projection \( \pi_G : P \rightarrow G \) is onto, so, to prove that \( G \cong P/\sigma \), we need only show that \( \pi_G \circ \pi_s^1 = \sigma \). In fact, since \( \pi_G \) is a homomorphism onto a group, so that \( \sigma \subseteq \pi_G \circ \pi_s^1 \), we need only show that \( \pi_G \circ \pi_s^1 \subseteq \sigma \).

Suppose that \((s, g)\pi_G = (t, h)\pi_G\) so that \( g = h \). Then \((s, g), (t, g) \in P\) implies \((s^r, 1) \in P\). That is, \( 1 \in \phi(s^r) \). Since \( \phi \) is unitary, this implies \( s^r \) is idempotent so that \( es = et \) for some idempotent \( e \) in \( S \). But then \((e, 1)(s, g) = (e, 1)(t, g) = (e, 1)(t, h)\) so that \( (s, g)\sigma(t, h) \). Thus \( \pi_G \circ \pi_s^1 \subseteq \sigma \).

Finally, if \((s, g) \in P\) and \((e, 1)(s, g) = (e, 1)\) then \( g = 1 \) so that \( 1 \in \phi(s) \). Since \( \phi \) is unitary, this requires \( s^2 = s \); whence \((s, g)\) is idempotent. Hence \( P \) is an \( E \)-unitary cover of \( S \) through \( G \).

Conversely, suppose that \( P \) is an \( E \)-unitary cover of \( S \) through \( G \) and let \( 1 \in \phi(s) \). Then \((s, 1) = p\psi \) for some \( p \in P \). But \( 1 = p\psi\pi_G = p\sigma^h \) implies \( p^2 = p \), since \( P \) is \( E \)-unitary, so that \( s = p\psi\pi_s = p\theta \), where \( \theta \) is the idempotent separating homomorphism \( P \rightarrow S \), is also idempotent. Hence \( \phi \) is unitary.

The proof of Proposition 3.2 is strikingly reminiscent of that of Proposition 1.3. This is because Proposition 1.3 is a special case of Proposition 3.2. For, let \( \theta \) be a homomorphism of \( S \) into a factorizable inverse semigroup \( F \), with group of units \( G \), and set
\[
\phi(s) = \{ g \in G : s\theta \leq g \}.
\]
Then \( g \in \phi(s), h \in \phi(t) \) implies \( s\theta \leq g, t\theta \leq h \) so that \( s\theta t\theta \leq gh \). Thus
(st)θ ≤ gh; that is, gh ∈ φ(st). Hence φ(s)φ(t) ⊆ φ(st). Further sθ ≤ g if and only if sθ^{-1} ≤ g^{-1}; that is s^{-1}θ ≤ g^{-1}. Hence φ(s^{-1}) = φ(s)^{-1} so that φ is a subhomomorphism. If θ is one-to-one, then clearly φ is unitary so that Proposition 1.3 follows.

In order to obtain a subhomomorphism φ, as above, from a mapping θ: S → F one need not assume that θ is a homomorphism. Only that θ is a v-prehomomorphism in the sense of the following definition.

**Definition 3.4.** Let S, T be inverse semigroups then a mapping θ: S → T is a v-prehomomorphism if it obeys the following two conditions.

(i) (st)θ ≤ sθtθ for each s, t ∈ S;
(ii) (s^{-1})θ = (sθ)^{-1} for each s ∈ S.

If S and T are semilattices then a v-prehomomorphism is just an isotone mapping of S into T.

The results in the next lemma follow straightforwardly from the definitions.

**Lemma 3.5.** Let S be an inverse semigroup and let F be a factorizable inverse semigroup with group of units G. Suppose that θ is a v-prehomomorphism of S into F. Then φ defined by

φ(s) = \{g ∈ G: sθ ≤ g\}

is a subhomomorphism of S into G. It is surjective if and only if, for each g ∈ G, there exists s ∈ S such that sθ ≤ g; it is unitary if and only if θ is idempotent determined in the sense that aθ idempotent implies a idempotent.

Lemma 3.5 shows that v-prehomomorphisms of S into F give rise to subhomomorphisms from S into G. On the other hand, Proposition 1.6 shows that surjective unitary subhomomorphisms of S into G can be obtained from embeddings of S into factorizable inverse semigroups with groups of units G. To end this section, we show that every subhomomorphism of S into G is determined by a v-prehomomorphism θ of S into a factorizable inverse semigroup \(H(G)\) which depends only on G.

It follows from this that every subdirect product of S and G, in particular every E-unitary cover of S through G, is determined by a v-prehomomorphism of S into \(H(G)\). The problem of constructing v-prehomomorphisms between inverse semigroups is considered, in detail, in [6].

Let G be a group. Then we shall denote by \(H(G)\) the set of all cosets \(X = Ha\) of G modulo subgroups of G. The following simple
lemma characterizes the members of $\mathcal{H}(G)$ among the nonempty subsets of $G$.

**Lemma 3.6** [3]. Let $G$ be a group and let $X$ be a nonempty subset of $G$. Then $X \in \mathcal{H}(G)$ if and only if $X = XX^{-1}X$.

It follows from Lemma 3.6 that any nonempty intersection of cosets is again a coset. We may thus define a binary operation $*$ on $\mathcal{H}(G)$ as follows: for $X, Y \in \mathcal{H}(G)$,

$$X * Y = \text{smallest coset that contains } XY.$$ 

**Proposition 3.7** [8]. Let $G$ be a group. Then $\mathcal{H}(G)$ is a factorizable inverse semigroup with group of units isomorphic to $G$. The idempotents are the subgroups of $G$. Further, for $X, Y \in \mathcal{H}(G)$, $X \subseteq Y$ if and only if $X \subseteq Y$.

**Proposition 3.8.** Let $S$ be an inverse semigroup and let $G$ be a group. Suppose that $\phi$ is a subhomomorphism of $S$ into $G$. Then $\theta$ defined by

$$a \theta = \phi(a) \text{ considered as a member of } \mathcal{H}(G)$$

is a $\nu$-prehomomorphism of $S$ into $\mathcal{H}(G)$.

**Proof.** Let $X = \phi(a)$; then $X \subseteq XX^{-1}X$. On the other hand, if $g_1, g_2, g_3 \in X$ then

$$g_1g_2^{-1}g_3 \in \phi(a)\phi(a)^{-1}\phi(a) = \phi(a)\phi(a^{-1})\phi(a) \subseteq \phi(aa^{-1}a) = \phi(a),$$

since $\phi$ is a subhomomorphism. Hence $XX^{-1}X \subseteq X$ and so $X = XX^{-1}X$. This shows $X \in \mathcal{H}(G)$, so that $\theta$ is a mapping into $\mathcal{H}(G)$.

Next, since $\phi$ is a subhomomorphism, $a \theta b \theta \subseteq (ab)\theta$ for each $a, b \in S$. But $a \theta * b \theta$ is the smallest coset containing $a \theta b \theta$ so this implies $a \theta * b \theta \subseteq (ab)\theta$. That is, by Lemma 3.7, $a \theta * b \theta \geq (ab)\theta$. Finally, since $\phi$ is a subhomomorphism, $(a^{-1})\theta = \phi(a^{-1}) = \phi(a)^{-1} = (a\theta)^{-1}$ for each $a \in S$. Hence $\theta$ is a $\nu$-prehomomorphism of $S$ into $\mathcal{H}(G)$.

It follows from Proposition 3.8 that the subdirect products of $S$ and $G$ are determined by $\nu$-prehomomorphisms of $S$ into $\mathcal{H}(G)$. More precisely, we have the following theorem, which sums up the results of this section. It should be pointed out however that it may be easier to find subhomomorphisms of $S$ into $G$ directly than to find $\nu$-prehomomorphisms of $S$ into $\mathcal{H}(G)$.

**Theorem 3.9.** Let $S$ be an inverse semigroup and let $G$ be a group.
(A). Let θ be a v-prehomomorphism of S into \( X(G) \) and suppose that, for each \( g \in G \), there exists \( s \in S \) such that \( s\theta = \{g\} \); i.e. \( g \in s\theta \). Then \( \{(s, g) : g \in s\theta\} \) is an inverse semigroup which is a subdirect product of \( S \) and \( G \). Every subdirect product of \( S \) and \( G \) is of this form for some v-prehomomorphism of \( S \) into \( X(G) \).

(B). With \( \theta \) as in (A), \( \{(s, g) : g \in s\theta\} \) is an E-unitary cover of \( S \) through \( G \) if and only if \( \theta \) is idempotent determined. Every E-unitary cover of \( S \) through \( G \) is of this form for some idempotent determined v-prehomomorphism of \( S \) into \( X(G) \).

4. Subhomomorphisms from a group. In §3, we characterized the E-unitary covers of an inverse semigroup \( S \), through a group \( G \), as subdirect products \( \Pi(S, G, \phi) \) with \( \phi \) a subhomomorphism of \( S \) into \( G \). They can also be described in the form \( \Pi(G, S, \phi) \) with \( \phi \) a subhomomorphism of \( G \) into \( S \). In this section, we give such a description. As might be expected the results obtained are, in a sense, dual to those in §3.

**Definition 4.1.** Let \( S \) and \( T \) be inverse semigroups. Then a mapping \( \theta : S \to T \) is a \( \lambda \)-prehomomorphism of \( S \) into \( T \) if it obeys the following two conditions.

(i) \( \alpha \theta b \theta \leq (ab) \theta \) for each \( \alpha, b \in S \);

(ii) \( (a^{-1}) \theta = (a \theta)^{-1} \) for each \( a \in S \).

**Proposition 4.2.** Let \( S \) be an inverse semigroup and let \( G \) be a group. Suppose that \( T \) is an inverse semigroup containing \( S \) and let \( \theta \) be a \( \lambda \)-prehomomorphism of \( G \) into \( T \). Then \( \phi \) defined by

\[ \phi(g) = \{s \in S : s \leq g\theta\} \]

is a subhomomorphism of \( G \) into \( S \); \( \phi \) is surjective if and only if, for each \( s \in S \) there exists \( g \in G \) such that \( s \leq g\theta \).

The semigroup \( \Pi(G, S, \phi) \) is E-unitary. It is an E-unitary cover of \( S \) through \( G \) if \( \phi \) is surjective.

**Proof.** The fact that \( \phi \) is a subhomomorphism and the statement about the surjectivity of \( \phi \) are readily verified.

Let \( a = 1\theta \); then, since \( \theta \) is a \( \lambda \)-prehomomorphism

\[ a = aa^{-1}a = aaa \leq (1.1)\theta a = a^2 \leq a \]

so that \( a \) is idempotent. Let \( (g, s) \in \Pi(G, S, \phi) \) and suppose that \( (g, s)(1, e) = (1, e) \) where \( e \) is idempotent. Then \( g = 1 \) so that \( s \leq 1\theta = a \); thus \( s \) is idempotent. Hence \( \Pi(G, S, \phi) \) is E-unitary.
Suppose that $\phi$ is surjective. Then, if we identify $S \times G$ with $G \times S$, $\Pi(G, S, \phi) = \Pi(S, G, \phi^*)$ where $g \in \phi^*(s)$ if and only if $s \in \phi(g)$. Since $1 \in \phi^*(s)$ implies $s \leq 1 = a$, which is idempotent, $\phi^*$ is unitary. Hence, by Proposition 3.2, $\Pi(G, S, \phi)$ is an $E$-unitary cover of $S$ through $G$.

Proposition 4.2 is analogous to Proposition 3.2. The next proposition is similar to Proposition 3.8; it shows that every $E$-unitary cover of $S$ through $G$ is determined by a $\land$-prehomomorphism of $G$ into a semigroup $C(S)$ depending only on $S$.

**Definition 4.3** [9]. Let $S$ be an inverse semigroup. Then a nonempty subset $H$ of $S$ is called permissible if
(i) $a \in H$, $b \leq a$ implies $b \in H$;
(ii) $a, b \in H$ implies $ab^{-1}$, $a^{-1}b$ idempotent.

Schein [9] shows that the set $C(S)$ of permissible subsets of $S$ forms an inverse semigroup under subset multiplication. Further $S$ can be embedded in $C(S)$ by means of the homomorphism $\eta$ given by

$$a\eta = \{x \in S: x \leq a\}$$

for each $a \in S$.

**Proposition 4.4.** Let $S$ be an inverse semigroup and let $G$ be a group. Suppose that $\phi$ is a surjective subhomomorphism of $G$ into $S$ such that $\Pi(G, S, \phi)$ is an $E$-unitary cover of $S$ through $G$. Then $\phi(g)$ is permissible for each $g \in G$ and $\theta$ defined by

$$g\theta = \phi(g)$$

considered as a member of $C(S)$

is a $\land$-prehomomorphism of $G$ into $C(S)$. Further

$$\Pi(G, S, \phi) = \{(g, s) \in G \times S: s \leq g\theta\};$$

here we identify $S$ with $S\eta$.

**Proof.** Suppose $a \in \phi(g)$, $b \leq a$; thus $b = ea$ for some $e^2 = e \in S$. Then $(g, a) \in P = \Pi(G, S, \phi)$ and $(1, e) \in P$ so that $(1, e)(g, a) = (g, b) \in P$. Hence $b \in \phi(g)$. Next, suppose $a, c \in \phi(g)$ then $(g, a), (g, c) \in P$ so that $(1, a^{-1}c) \in P$. Since $P$ is an $E$-unitary cover of $S$ through $G$, with $\pi_G \circ \pi_\theta = \alpha$, where $\pi_G$ denotes the projection of $P$ onto $G$, this implies that $a^{-1}c$ is an idempotent. Similarly $ac^{-1}$ is an idempotent. Hence $\phi(g)$ is permissible.

It is now easy to show that $\theta$ is a $\land$-prehomomorphism and, because $X \leq Y$ in $C(S)$ if and only if $X \subseteq Y$, that $P = \{(g, s) \in G \times S: s \leq g\theta\}$.
If we combine the results of Propositions 4.2 and 4.4, then we obtain the following dual to Theorem 3.9.

**Theorem 4.5.** Let $S$ be an inverse semigroup and let $G$ be a group. Let $\theta$ be a $\Lambda$-prehomomorphism of $G$ into $C(S)$ such that, for each $s \in S$ there exists $g \in G$ with $s \equiv g\theta$. Then

$$\{(g, s) \in G \times S : s \equiv g\theta\}$$

is an $E$-unitary cover of $S$ through $G$. Conversely, each $E$-unitary cover of $S$ through $G$ has this form for some $\Lambda$-prehomomorphism $\theta$ of $G$ into $C(S)$.

**5. Examples**

5.1. **Free group covers.** Let $S$ be an inverse semigroup and let $X$ be a set of generators for $S$ as an inverse semigroup. Let $X^1$ be a set in one to one correspondence with, but disjoint from, $X$. Then there is a homomorphism $\theta$ from the free semigroup $F_{X\cup X^1}$, on $X \cup X^1$, onto $S$ such that $x\theta^{-1} = x^{-1}\theta$ for each $x \in X$. Similarly, there is a homomorphism $\psi: F_{X\cup X^1} \to FG_X$, the free group on $X$ such that $x\psi^{-1} = x^{-1}\psi$ for each $x \in X$. Define $\phi: S \to 2^{FG_X}$ by

$$w \in \phi(s) \text{ if and only if } w = u\psi \text{ for some } u \in F_{X\cup X^1}$$

with $u\theta = s$; that is $\phi(s) = s\theta^{-1}\psi$ for some $s \in S$.

**Proposition 5.1.** $\phi$ is a surjective unitary subhomomorphism of $S$ into $FG_X$.

**Proof.** It is straightforward to show that $\phi$ is a surjective subhomomorphism of $S$ into $FG_X$. Suppose that $1 \in \phi(s)$. Then there exists $w \in F_{X\cup X^1}$ such that $w\theta = s$, $w\psi = 1$.

Let $\eta$ be the canonical homomorphism from $F_{X\cup X^1}$ into the free inverse semigroup $FI_X$ on $S$. Then both $\theta$ and $\psi$ can be factored through $\eta$. Since $w\psi = 1$ and $FI_X$ is $E$-unitary [7] it follows that $w$, regarded as an element of $FI_X$, is idempotent. Hence $s = w\theta$ is an idempotent of $S$. This shows that $\phi$ is unitary.

As a result of the freeness of $FG_X$, $\Pi(S, FG_X, \phi)$, with $\phi$ as above, has a weak universal property.

**Proposition 5.2.** Let $S$ be an inverse semigroup and let $P = \Pi(S, FG_X, \phi)$ as above with $\alpha$ the homomorphism $P \to S$. Suppose that $Q$ is an $E$-unitary cover of $S$ through $G$ with homomorphism $\beta: Q \to S$. Then there is a homomorphism $\gamma: P \to Q$ such that $\alpha = \gamma\beta$. 
Proof. With the notation above, we have the following diagram of maps:

\[
\begin{array}{ccc}
FG_X & \rightarrow & G \\
\downarrow \psi & & \downarrow \sigma^* \\
S & \downarrow \beta & \\
F_{X\cup X^{-1}} & \rightarrow & \\
\end{array}
\]

For each \(x \in X\), choose \(y \in Q\) such that \(y\beta = x\theta\). Then there is a homomorphism \(\nu: F_{X\cup X^{-1}} \rightarrow Q\) such that \(x\nu\beta = x\theta\) and \((x\nu)^{-1} = x^{-1}\nu\) for each \(x \in X\). Then \(\nu\rho^*\) is a homomorphism of \(F_{X\cup X^{-1}}\) into \(G\) and can be factored through \(\psi\). That is, there is a homomorphism \(\delta: FG_X \rightarrow G\) such that \(\nu\sigma^* = \psi\delta\).

From the definition of \(P\), \(P = \{(w\theta, w\psi): w \in F_{X\cup X^{-1}}\}\). Define \(\gamma: P \rightarrow Q\) by \((w\theta, w\psi)\gamma = w\nu\). Then \(w\theta = u\theta, w\psi = u\psi\) implies \(w\psi\delta = u\psi\delta\), that is \(w\nu\sigma^* = u\nu\sigma^*\) and \(w\nu\beta = u\nu\beta\). Since \(Q\) is an \(E\)-unitary cover of \(S\) through \(G\), Corollary 1.8 shows that \(u\nu = w\nu\). Hence \(\gamma\) is well defined; it is clearly a homomorphism. Further, from the definition,

\[(w\theta, w\psi)\gamma\beta = w\nu\beta = w\theta = (w\theta, w\psi)\alpha,\]

for each \((w\theta, w\psi) \in P\). Hence \(\alpha = \gamma\beta\).

5.2. The Preston–Vagner cover. Let \(\rho: S \rightarrow \mathcal{I}_S\) be the Preston–Vagner representation of an inverse semigroup \(S\) and let \(Y = S\) if \(S\) is finite, if not \(Y = S \cup S'\) with \(S \cap S' = \emptyset\), \(|S| = |S'|\). Then \(F = \{\alpha \in \mathcal{I}_Y: \alpha \leq \gamma\) for some permutation \(\gamma\) of \(Y\}\) is a factorizable inverse semigroup containing \(S\rho\). It gives rise to the subhomomorphism \(\phi\) where, for each \(s \in S\),

\[\phi(s) = \{\alpha: \rho, \leq \alpha, \alpha \text{ a permutation of } Y\}\]
\[= \{\alpha: x\alpha = xs \text{ for each } x \in Ss^{-1}, \alpha \text{ a permutation of } Y\}.\]

This subhomomorphism gives an \(E\)-unitary cover of \(S\) through \(K\) where

\[K = \{\alpha \in S_Y: (xe)\alpha = x(e\alpha) \text{ for all } x \in S \text{ and some } e^2 = e \in S\},\]

where \(S_Y\) denotes the symmetric group on \(Y\).
5.3. *E*-unitary covers of bisimple inverse semigroups. A construction is given in [6] for the $\nu$-prehomomorphisms $\theta$ of a bisimple inverse semigroup $S$ into an inverse semigroup $T$. When applied to the semigroup $\mathcal{H}(G)$ of cosets of a group $G$, this construction specializes to give the following construction for the *E*-unitary covers of $S$ through $G$.

Let $H$ be a subgroup of $G$ and let $S(H) = \{a \in G : aHa^{-1} \subseteq H\}$. Then $S(H)$ is a subsemigroup of $G$ and $(G/H, S(H)/H)$ is a partial semigroup under the multiplication $*$:

$$X * Y = XY \quad \text{for each} \quad X \in S(H)/H, \ Y \in G/H.$$  

Pick an idempotent $e \in S$ and set $R_e = \{a \in S : aa^{-1} = e\}$, $P_e = R_e \cap eSe$ and let $\theta : R_e \to G/H$ be a one-to-one mapping such that the following hold

(i) $a \theta \in S(H)/H$ if $a \in P_e$
(ii) $a \theta b \theta = (ab) \theta$ for $a \in P_e$, $b \in R_e$
(iii) $G = \bigcup \{a^{-1}b \theta : a, b \in R_e\}$.

Then $\{(s, g) \in S \times G : g \in a \theta^{-1}b \theta \text{ where } s = a^{-1}b\}$ is an $E$-unitary cover of $S$ through $G$. Conversely, each such has this form for some $\theta : R_e \to G/H$ as above.

5.4. *E*-unitary covers for semilattices of groups. A construction is given in [6] for the $\nu$-prehomomorphisms $\theta$ of a semilattice of groups $S$ into an inverse semigroup $T$. When applied to the semigroup $\mathcal{H}(G)$ of cosets of a group $G$, this construction specializes to give the following description of the *E*-unitary covers of $S$ through $G$.

Let $E$ be a semilattice and let $\theta$ be an anti-isotone mapping of $E$ into the lattice of subgroups of $G$. For each $e \in E$ set $G_e = e \theta$ and $C_e = \{a \in G : aG_ea^{-1} = G_e \text{ for each } f \leq e\}$. Then $G_e$ is a normal subgroup of $C_e$ and the groups $K_e = C_e/G_e$ form a semilattice of groups $SL(E, \theta, \mathcal{H}(G))$ with linking homomorphisms $\phi_{e,f} : K_e \to K_f$ given by

$$X \phi_{e,f} = G_fX \quad \text{for each} \quad X \in K_e, \ e \geq f.$$  

Suppose that $S$ is a semilattice of groups with semilattice of idempotents $E$. Suppose that $\theta$ is an anti-isotone mapping of $E$ into the lattice of subgroups of $G$ and let $\phi$ be an idempotent determined homomorphism of $S$ into $SL(E, \theta, \mathcal{H}(G))$ such that $G = \bigcup \{a \phi : a \in S\}$. Then

$$\{(s, g) \in S \times G : g \in s \phi\}$$  

is an $E$-unitary cover of $S$ through $G$. Conversely, each such has this form for some $\theta : E \to \mathcal{H}(G)$ and $\phi : S \to SL(E, \theta, \mathcal{H}(G))$. 
REFERENCES

6. ---, *v-Prehomomorphisms on inverse semigroups*.

Received May 24, 1976. This research was partly supported by a grant from the National Science Foundation and a grant from the National Research Council.

*Northern Illinois University*
DeKalb, IL 60115

*Simon Fraser University*
Burnaby, British Columbia, Canada
Richard Julian Bagby, *On $L^p$, $L^q$ multipliers of Fourier transforms* ................. 1
Robert Beauwens and Jean-Jacques Van Binnebeek, *Convergence theorems in Banach algebras* .................................................. 13
James Cyril Becker, *Skew linear vector fields on spheres in the stable range* ................................. 25
Michael James Beeson, *Continuity and comprehension in intuitionistic formal systems* ......................... 29
James K. Deveney, *Generalized primitive elements for transcendental field extensions* ................................. 41
Samuel S. Feder, Samuel Carlos Gitler and K. Y. Lam, *Composition properties of projective homotopy classes* ................................. 47
Nathan Jacob Fine, *Tensor products of function rings under composition* ............... 63
Benno Fuchssteiner, *Iterations and fixpoints* .............................................................. 73
Wolfgang H. Heil, *On punctured balls in manifolds* ....................................................... 81
Shigeru Itoh, *A random fixed point theorem for a multivalued contraction mapping* ............... 85
Nicolas P. Jewell, *Continuity of module and higher derivations* .................................................. 91
Roger Dale Konyndyk, *Residually central wreath products* ............................................. 99
Linda M. Lesniak and John A. Roberts, *On Ramsey theory and graphical parameters* .................................................. 105
Vo Thanh Liem, *Some cellular subsets of the spheres* .......................................................... 115
Dieter Lutz, *A perturbation theorem for spectral operators* .............................................. 127
P. H. Maserick, *Moments of measures on convex bodies* .................................................. 135
Stephen Joseph McAdam, *Unmixed 2-dimensional local domains* ....................................... 153
D. B. McAlister and Norman R. Reilly, *E-unitary covers for inverse semigroups* ................. 161
William H. Meeks, III and Julie Patrusky, *Representing codimension-one homology classes by embedded submanifolds* .................................................. 175
Premalata Mohapatro, *Generalised quasi-Nörlund summability* ........................................ 177
Takahiko Nakazi, *Superalgebras of weak-$*$Dirichlet algebras* ........................................ 197
Catherine Louise Olsen, *Norms of compact perturbations of operators* ................................. 209
William Henry Ruckle, *Absolutely divergent series and isomorphism of subspaces. II* ............... 229
Bernard Russo, *On the Hausdorff-Young theorem for integral operators* ........................................ 241
Arthur Argyle Sagle and J. R. Schumi, *Anti-commutative algebras and homogeneous spaces with multiplications* .................................................. 255
Robert Evert Stong, *Stiefel-Whitney classes of manifolds* .................................................. 271
D. Suryanarayana, *On a theorem of Apostol concerning Möbius functions of order $k$* .................................................. 277
Yoshio Tanaka, *On closedness of $C^*$-embeddings* ......................................................... 283
Chi Song Wong, *Characterizations of certain maps of contractive type* ................................. 293