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**REPRESENTING CODIMENSION-ONE HOMOLOGY CLASSES  
BY EMBEDDED SUBMANIFOLDS**

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## REPRESENTING CODIMENSION-ONE HOMOLOGY CLASSES BY EMBEDDED SUBMANIFOLDS

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**In this paper we prove a general theorem on representing codimension-one homology classes on compact manifolds. Our theorem gives an elementary proof of the classical result that a one-dimensional homology class on a compact orientable surface can be represented by an embedded circle precisely when the class is primitive. We will call a homology class primitive if it is the zero class or if it is not a nontrivial multiple of another class.**

Our representation theorem is motivated by our earlier work [2] on the classical situation. For compact orientable surfaces we developed a simple algorithm for representing primitive homology classes by embedded circles. This algorithm is also useful for proving other results related to two and three dimensional topology which are not implied by the general result here. Recently Mark Meyerson [3] has given another proof of the two dimensional case by applying Lickorish "twist" homeomorphisms to a fixed homology class.

The following theorem deals with the decomposition of a  $n$ -dimensional manifold  $M^n$  by an embedded submanifold  $N^{n-1}$  which represents a homology class  $\delta \in H_{n-1}(M, Z)$ . We will say that such a representation  $N$  is minimal if there is no other representation  $N'$  of  $\delta$  having fewer path components.

**REPRESENTATION THEOREM.** *Suppose  $M$  is a compact orientable piecewise linear  $n$ -dimensional manifold and  $\gamma \in H_{n-1}(M, Z)$  is a primitive nonzero class.*

1. *The class  $\delta = \kappa\gamma$  has a minimal representation by a submanifold.*
2. *If  $N$  is a minimal representation for  $\delta = \kappa\gamma$  then each path component of  $M - N$  has two ends and the number of path components of  $N = |N|$  is  $\kappa$ .*

*Proof.* The Poincaré dual to  $\delta = \kappa\gamma$  can be represented by a piecewise linear mapping  $P(\delta): M \rightarrow S^1$ . The standard duality theorems imply that wherever  $r \in S^1$  is a regular value for  $P(\delta)$ , then  $P^{-1}(r) \subset M$  is an embedded submanifold representing  $\delta$ . Hence any  $\delta \in H_{n-1}(M, Z)$  has a minimal representative.

Suppose now that  $N$  is an oriented path-connected submanifold of  $M$  with  $[N] \neq 0 \in H_{n-1}(M, Z)$ . Since  $N$  is path connected and nontrivial

on homology then  $M - N$  is path connected. Hence there is an embedded circle  $\sigma \subset M$  with  $[\sigma] \cap [N] = +1$ , where  $\cap$  is the intersection pairing on homology. This implies that  $[N]$  is primitive.

DEFINITION. If  $N \subset M$  is an embedded submanifold, and  $U$  is a path component of  $M - N$ , then  $T = \text{end closure}$  of  $U$  is formed by placing a compact boundary on each end of  $U$ .

Let  $N$  be an embedded submanifold representing  $\delta \in H_{n-1}(M, Z)$ . Suppose  $T$  is the end closure of an oriented path component of  $M - N$  with at least three ends,  $E_1, E_2, E_3$ , coming from cuts along distinct path components  $N_1, N_2, N_3$  of  $N$ . Orient the end  $E_i$  as the respective component  $N_i$  is oriented. Let  $\gamma_1$  be a path joining  $p_1 \in E_1$  to  $p_2 \in E_2$ , and let  $\gamma_2$  be a path joining  $p_2 \in E_2$  to  $p_3 \in E_3$  with  $\gamma_1 \cap \gamma_2 = \{p_2\}$ . If  $\text{sgn}(\gamma_1 \cap E_1) \neq \text{sgn}(\gamma_1 \cap E_2)$ , then we can take the connected sum of  $N_1$  and  $N_2$  in  $M$  along  $\gamma_1$ , and similarly, if  $\text{sgn}(\gamma_2 \cap E_2) \neq \text{sgn}(\gamma_2 \cap E_3)$ , we can connect  $N_2$  and  $N_3$  in  $M$  along  $\gamma_2$ . If neither of these cases hold, form the composite path  $\gamma_3 = \gamma_1 \circ \gamma_2^{-1}$ . Clearly,  $\text{sgn}(\gamma_3 \cap E_1) \neq \text{sgn}(\gamma_3 \cap E_3)$ . Since the normal bundle to  $E_2$  is trivial, we may push  $\gamma_3$  off of  $E_2$  and take the connected sum of  $N_1$  and  $N_3$  in  $M$  along this variation of  $\gamma_3$ .

A slight change in the above argument shows that the condition that  $E_1, E_2, E_3$  come from cuts along distinct path components of  $N$  is not needed. Hence, if  $N$  is a minimal representative of  $\delta$ , then each path component of  $M - N$  has two ends. This implies that  $[N] = |N| \cdot [N_1]$ , where  $N_1$  is a path component of  $N$ . Since  $N_1$  is primitive, the theorem is proved.

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