GENERALISED QUASI-NÖRLUND SUMMABILITY

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Just as \((N, p, q)\) generalises Nörlund methods, so also, in this paper we define generalised quasi-Nörlund Method \((N^*, p, q)\) generalising the quasi-Nörlund method due to Thorpe.

To begin with, we have determined the inverse of a generalised quasi-Nörlund matrix in a limited case. Besides, limitation Theorems for both ordinary and absolute \((N^*, p, q)\) summability have been established.

Finally we have established an Abelian Theorem (the main theorem) for \((N^*, p, q) \Rightarrow (J, q)\), where \((J, q)\) is a power series method which reduces to the Abel method \((A)\) for \(q_n = 1\) (all \(n\)).

1. Vermes [10] pointed out that there is a close relation between the summability properties of a matrix \(A = (a_{nk})\) regarded as a sequence to sequence transformation and those of its transpose \(A^* = (a_{kn})\) regarded as a series to series transformation.

Suppose that \(A\) is a sequence to sequence transformation and further that

\[
\sum_{k=0}^{\infty} a_{nk} = 1 \quad \text{for all } n,
\]

then by using Theorems of regularity (see Hardy [5], Theorem 2) and absolute regularity (see Knopp and Lorentz [6]) we see that \(A^*\) is an absolutely regular series to series transformation.

Conversely, given any absolutely regular series to series method \(C = (c_{nk})\), its transpose \(C^*\) is regular as a sequence to sequence method provided that

\[
c_{nk} \to 0 \quad \text{as } k \to \infty \quad \text{for fixed } n.
\]

We can also see that if \(A\) is absolutely regular and the above condition is satisfied then \(A^*\) is regular and the converse also holds.

We shall call \(A^*\) the quasi-method associated with \(A\) and remember that, it is a series to series transformation.


Just as \((N, p, q)\) generalises Nörlund methods, so also we can define generalised quasi-Nörlund method \((N^*, p, q)\) generalising the quasi-Nörlund methods. We give the definition in the following manner:
Given \( p_n \) and \( q_n \) we define \( r_n = \sum_{v=0}^{n} p_{n-v} q_v \) and suppose that \( r_n \neq 0 \) for \( n \geq 0 \). We say that the \((N^*, p, q)\) method is applicable to the given infinite series \( \sum a_n \) if

\[
(1.1) \quad b_n = q_n \sum_{k=n}^{\infty} \frac{p_{k-n} a_k}{r_k}
\]

exists for each \( n \geq 0 \). If further, \( \sum b_n = s \), then we say that \( \sum a_n \) is summable by \((N^*, p, q)\) method to sum \( s \) and if \( \sum |b_n| < \infty \) then \( \sum a_n \) is said to be absolutely summable by \(|N^*, p, q|\) method.

The method \((N^*, p, q)\) reduces to the quasi-Nörlund method \((N^*, p)\) if \( q_n = 1 \), to the quasi-Riesz method \((\tilde{N}^*, q)\) if \( p_n = 1 \), to (say) quasi-Euler-Knopp method \((E^*, \sigma)\) when

\[
p_n = \frac{\alpha^n \sigma^n}{n!}, \quad q_n = \frac{\alpha^n}{n!} \quad (\alpha > 0, \ \sigma > 0),
\]

to the (say) \((C^*, \alpha, \beta)\) method (let us call it generalised quasi-Cesàro method) when

\[
p_n = \left( \frac{n + \alpha - 1}{\alpha} \right), \quad q_n = \left( \frac{n + \beta}{\beta} \right).
\]

It may be recalled that \((N, p, q)\) matrix is given by

\[
a_{nk} = \begin{cases} 
  \frac{p_{n-k} q_k}{r_n} & (k \leq n), \\
  0 & (k > n).
\end{cases}
\]

and the \((N^*, p, q)\) is given by its transpose matrix:

\[
a^*_{nk} = \begin{cases} 
  \frac{q_n p_{k-n}}{r_k} & (k \geq n), \\
  0 & (k < n).
\end{cases}
\]

Since for the \((a_{nk})\) defined above we have

\[
\sum_{k=0}^{n} a_{nk} = 1,
\]

it follows from the above discussion that if

\[
p_{k-n} = o(r_k) \quad \text{as} \quad k \to \infty,
\]
for each fixed $n$, then $(N^*, p, q)$ is regular if and only if $(N, p, q)$ is absolutely regular, and $(N^*, p, q)$ is absolutely regular if and only if $(N, p, q)$ is regular.

The main object of this paper is to obtain certain conditions for which $\sum a_n \in (N^*, p, q) \Rightarrow \sum a_n \in (J, q)$.

The method $(J, q)$ is defined as follows. Suppose that $q_n \geq 0$ and $q_n \neq 0$ for an infinity values of $n$. Let $\rho_q (\rho_q < \infty)$ be the radius of convergence of the power series

$$q(z) = \sum_{n=0}^\infty q_n z^n.$$

If the sequence to function transformation,

$$J(x) = \frac{\sum_{n=0}^\infty q_n s_n x^n}{\sum_{n=0}^\infty q_n x^n}$$

exists for $0 \leq x \leq \rho_q$, we say that $(J, q)$ method is applicable to $\sum a_n$ (or $\{s_n\}$), and if further $J(x) \to s$ as $x \to \rho_q - 0$, we say that $\sum a_n$ (or $\{s_n\}$) is summable $(J, q)$ to $s$. See Hardy [5], Das [4].

As well-known particular cases of the $(J, q)$ method, we have the Abel method when $q_n = 1$, the logarithmic method or $(L)$ method when $q_n = 1/n + 1$ (Borwein [1], Hardy [5] p. 81), the $A_\alpha$ method when $q_n = \binom{n + \alpha}{\alpha} (\text{Borwein [2]}$ ($A_\alpha$ is the same as Abel method $A$), the Borel method where $q_n = 1/n !$ (see Hardy [5]). We write $p_n \in \mathbb{N}$, when $p_n > 0$ and $p_n/p_{n-1} \leq p_{n+1}/p_n \leq 1 (n > 0)$.

Let $P_n = \sum_{v=0}^n p_v$, $Q_n = \sum_{v=0}^n q_v$.

Let $c_n$ be defined formally by the identity,

$$\left(\sum_{n=0}^\infty p_n x^n\right)\left(\sum_{n=0}^\infty c_n x^n\right) = 1.$$

2. Statements of the theorems. As in the case of quasi-Nörlund, it is not always possible to obtain an inverse to the transformation (1.1) but we have succeeded in getting an inverse for a class of sequences $p_n \in \mathbb{N}$ and $q_n \neq 0 (n \geq 0)$.

This is embodied in.

Theorem 1. Suppose that $p_n \in \mathbb{N}$ and $q_n \neq 0 (n \geq 0)$. Then $(N^*, p, q)$ (where applicable) has an inverse transformation, whose matrix
is given by the transpose of the inverse of \((N, p, q)\), that is, if \(b_n\) is given by transformation \((1.1)\), then

\[
a_n = r_n \sum_{k=n}^\infty \frac{b_k c_{k-n}}{q_k}.
\]

This is our basic theorem in the sense that it is widely used here and elsewhere and it may be noted that this theorem yields a result due to Thorpe [8] in the case \(q_n = 1\).

The next couple of theorems are limitation theorems which assert that the method can not sum too rapidly divergent series.

**Theorem 2.** Suppose \(p_n \in \mathcal{M}\), \(q_n \neq 0\) \((n \geq 0)\) and that \(|q_n|\) is non-decreasing. If \(\Sigma a_n\) be summable \((N^*, p, q)\) to \(s\) then

\[
a_n = o\left(\frac{|r_n|}{|q_n|}\right).
\]

If further \(r_n \geq 0\), then

\[
s_n = s + o\left(Q_n/|q_n|\right).
\]

**Theorem 3.** Suppose \(p_n \in \mathcal{M}\), \(q_n\) is positive, \(\{q_n\}\) is nondecreasing and \(\{q_n/r_n\}\) is nonincreasing. Then if \(\Sigma a_n\) is summable \(|N^*, p, q|\), then

\[
\left\{\frac{q_n s_n}{r_n}\right\} \in BV.
\]

The main theorem in this paper is the Abelian theorem which is stated as:

**Theorem 4.** Suppose \(p_n \in \mathcal{M}\), \(q_n > 0\) and that \(\{q_n\}\) and \(\{q_n/q_{n+1}\}\) are nondecreasing. Also let

\[
(2.2) \quad r_n(q_{n+1} - q_n) = O(q_{n+1}(r_{n+1} - r_n)).
\]

Then

\[
\Sigma a_n = s(N^*, p, q) \Rightarrow \Sigma a_n = s(J, q).
\]

It may be remarked that the relationship between \((N, p, q)\) and \((J, q)\) was studied by Das (4). Putting \(q_n = 1\) in Theorem 4, we obtain the result of Thorpe regarding \((N^*, p) \Rightarrow (A)\). We need the following lemma for the proof of the theorem.
Lemma 1. Let $p_n \in \mathcal{M}$. Then

(i) $\sum_{n=0}^{\infty} |c_n| < \infty$,
(ii) $c_0 > 0$, $c_n \leq 0 \ (n \geq 1)$,
(iii) $\sum c_n \geq 0$,
(iv) $\sum c_n = 0$, if and only if $P_n \to \infty$ as $n \to \infty$.

The above theorem is due to Kaluza. The proof of the theorem appears in Hardy (5), Theorem 22.

3. Proof of Theorem 1. We know from the identity:

$$(\sum c_n x^n)(\sum p_n x^n) = 1$$

that

$$(3.1) \sum_{n=0}^{k} p_n c_{k-n} = \begin{cases} 1 & (k = 0), \\ 0 & (k > 0). \end{cases}$$

Hence

$$(3.2) \sum_{k=n}^{N} c_{k-n} p_{v-k} = - \sum_{k=N+1}^{v} c_{k-n} p_{v-k} \quad (v > n).$$

Now for $N > n$ and by (1.1) we have,

$$r_n \sum_{k=n}^{N} b_k c_{k-n} q_k = r_n \sum_{k=n}^{N} \frac{c_{k-n} q_k}{q_k} \sum_{v=k}^{\infty} \frac{a_v p_{v-k}}{r_v}$$

$$= r_n \sum_{k=n}^{N} \frac{c_{k-n}}{q_k} \left( \sum_{v=k}^{N} + \sum_{v=N+1}^{\infty} \right) \frac{a_v p_{v-k}}{r_v}$$

$$= r_n \sum_{v=n}^{N} \frac{a_v}{r_v} \sum_{k=n}^{V} \frac{c_{k-n} p_{v-k}}{r_v} + r_n \sum_{v=N+1}^{\infty} \frac{a_v}{r_v} \sum_{k=n}^{N} \frac{c_{k-n} p_{v-k}}{r_v}$$

$$= a_n + r_n \sum_{v=N+1}^{\infty} \frac{a_v}{r_v} \sum_{k=n}^{N} c_{k-n} p_{v-k}$$

by (3.1). Thus the necessary and sufficient condition for the validity of (2.1) is that, for each fixed $n$,

$$\sum_{v=N+1}^{\infty} \frac{a_v}{r_v} \sum_{k=n}^{N} c_{k-n} p_{v-k} \to 0, \quad \text{as} \quad N \to \infty,$$
which is the same thing as, for each fixed $n$,

$$\phi_N = \sum_{v=N+1}^{\infty} \frac{a_v}{r_v} \sum_{k=N+1}^{v} c_{k-n}p_{v-k} \to 0, \quad \text{as} \quad N \to \infty$$

in view of (3.2).

Let us write

$$b_0 = q_0 \sum_{k=0}^{\infty} \frac{p_k a_k}{r_k}, \quad (3.4)$$

$$\omega_v = q_0 \sum_{k=v}^{\infty} \frac{p_k a_k}{r_k}.$$ 

Since $(N^*, p, q)$ method is applicable to $\sum a_n$, $b_0$ is finite and hence, $\omega_v$ is well defined and tends to zero as $v \to \infty$. Now from (3.4)

$$\frac{a_v}{r_v} = \frac{\omega_v - \omega_{v+1}}{q_0 p_v}.$$ 

Hence

$$\phi_N = \frac{1}{q_0} \sum_{v=N+1}^{\infty} \frac{\omega_v - \omega_{v+1}}{q_0 p_v} \sum_{k=N+1}^{v} c_{k-n}p_{v-k}.$$ 

Now for $M > N$,

$$\frac{1}{q_0} \sum_{v=N+1}^{M} \frac{\omega_v - \omega_{v+1}}{p_v} \sum_{k=N+1}^{v} c_{k-n}p_{v-k} = \frac{1}{q_0} \sum_{v=N+1}^{M} \omega_v \left[ \sum_{k=N+1}^{v} \frac{p_{v-k}c_{v-k}}{p_v} - \sum_{k=N+1}^{v-1} \frac{p_{v-k-1}c_{v-k-1}}{p_{v-1}} \right]$$ 

$$- \frac{1}{q_0} \omega_{M+1} \sum_{k=N+1}^{M} p_{M-k}c_{k-n}.$$ 

Since $p_n \in \mathcal{M}$ (by Lemma 1)

$$\left| \sum_{k=N+1}^{M} p_{M-k}c_{k-n} \right| = O(1), \quad \text{as} \quad M \to \infty,$$

and by definition,

$$\omega_M = o(1), \quad \text{as} \quad M \to \infty,$$

we see that,

$$\phi_N = \frac{1}{q_0} \sum_{v=N+1}^{\infty} \omega_v \sum_{k=N+1}^{v} c_{k-n} \left( \frac{p_{v-k}}{p_v} - \frac{p_{v-k-1}}{p_{v-1}} \right).$$
Since \( \{\omega_v\} \) is an arbitrary sequence tending to 0, hence (3.3) is valid, that is, \( \phi_N \to 0 \) if and only if, (see Hardy (5), Theorem 8) for fixed \( n \),

\[
J_N = \sum_{v=n+1}^{\infty} \left| \sum_{k=n+1}^{v} \left( \frac{p_{v-k}}{p_v} - \frac{p_{v-k-1}}{p_{v-1}} \right) c_{k-n} \right| = O(1)
\]
as \( N \to \infty \). But by virtue of (3.1)

\[
\sum_{k=n+1}^{v} \left( \frac{p_{v-k}}{p_v} - \frac{p_{v-k-1}}{p_{v-1}} \right) c_{k-n} = - \sum_{k=n}^{N} \left( \frac{p_{v-k}}{p_v} - \frac{p_{v-k-1}}{p_{v-1}} \right) c_{k-n}
\]

for \( v > n \) and also,

\[
\frac{p_{v-k}}{p_v} - \frac{p_{v-k-1}}{p_{v-1}} \leq 1, \quad \text{for} \quad k \leq v - 1.
\]

Hence

\[
J_N = \sum_{v=n+1}^{\infty} \left| \sum_{k=n}^{N} \left( \frac{p_{v-k}}{p_v} - \frac{p_{v-k-1}}{p_{v-1}} \right) c_{k-n} \right|
\]

\[
\leq \sum_{v=n+1}^{\infty} c_0 \left| \frac{p_{v-n}}{p_v} - \frac{p_{v-n-1}}{p_{v-1}} \right|
\]

\[
+ \sum_{v=n+1}^{\infty} \sum_{k=n+1}^{N} \left| c_{k-n} \left( \frac{p_{v-k}}{p_v} - \frac{p_{v-k-1}}{p_{v-1}} \right) \right|
\]

\[
= J^{(1)}_N + J^{(2)}_N, \quad \text{(say)}.
\]

Since \( p_n \in \mathbb{W} \), \( \{p_n/p_{n+1}\} \) is nonincreasing and so,

\[
J^{(1)}_N = O(1), \quad \text{as} \quad N \to \infty.
\]

Since \( p_n/p_{n+1} \geq 1 \) and \( \{p_n/p_{n+1}\} \) is nonincreasing it follows that, \( \lim p_n/p_{n+1} \) exists and

\[
A = \lim p_n/p_{n+1} \geq 1.
\]

Hence,

\[
\sum_{v=n+1}^{\infty} \left( \frac{p_{v-k}}{p_v} - \frac{p_{v-k-1}}{p_{v-1}} \right)
\]

\[
= \lim_{v \to \infty} \frac{p_{v-k}}{p_v} - \frac{p_{N-k}}{p_N}
\]

\[
= \lim_{v \to \infty} \left( \frac{p_{v-k}}{p_v} \frac{p_{v+1-k}}{p_{v+2-k}} \ldots \frac{p_{v-1}}{p_v} \right) - \frac{p_{N-k}}{p_N}
\]

\[
= A^k - \frac{p_{N-k}}{p_N}.
\]
Therefore, by (3.1)

\[ J^{(2)}_N = \sum_{k=n+1}^{N} c_{k-n}A^k - \sum_{k=n+1}^{N} \frac{c_{k-n}p_{N-k}}{p_N} \]
\[ = \sum_{k=n+1}^{N} c_{k-n}A^k - \frac{1}{p_N} \left[ \sum_{k=n}^{N} c_{k-n}p_{N-k} - c_0p_{N-n} \right] \]
\[ = \sum_{k=n+1}^{N} c_{k-n}A^k + c_0 \frac{p_{N-n}}{p_N}. \]

Since,

\[ \sum_{k=n+1}^{N} c_{k-n}A^k \leq 0, \]

we get,

\[ J^{(2)}_N = \frac{c_0p_{N-n}}{p_N} \]
\[ = O(1), \quad \text{as} \quad N \to \infty. \]

This completes the proof of the theorem.

4. Proof of Theorem 2. Since \( \Sigma a_n \) is \((N^*, p, q)\) summable, \( \Sigma b_n \) is convergent and hence \( b_n = o(1) \). By using the inversion formula as given in Theorem 1 we obtain, by using hypotheses,

\[ |a_n| = \left| r_n \sum_{k=n}^{\infty} \frac{b_kc_{k-n}}{q_k} \right| \]
\[ \leq \left| \frac{r_n}{q_n} \right| \sum_{k=n}^{\infty} |b_kc_{k-n}| \]
\[ = \left| \frac{r_n}{q_n} \right| \sum_{k=n}^{\infty} o(1) |c_{k-n}| \]
\[ = o \left( \left| \frac{r_n}{q_n} \right| \right), \]

since \( \Sigma |c_n| < \infty \) and \( b_n = o(1) \).

Next, suppose that \( \Sigma b_n = s \). Since

\( (\Sigma c_nx^n)(\Sigma r_nx^n) = \Sigma q_nx^n, \)
\( (\Sigma c^{(1)}_nx^n)(\Sigma r_nx^n) = \Sigma Q_nx^n, \)
it follows that

\[(4.1) \quad \sum_{v=0}^{n} r_v c_{n-v} = q_n,\]

\[(4.2) \quad \sum_{v=0}^{n} r_v c_{n-v}^{(1)} = Q_n.\]

Thus, when \(p_n \in \mathcal{M}\) we have \(c_{n}^{(1)} \geq 0\) and if \(r_n \geq 0\), it follows from (4.2) that \(Q_n \geq 0\) whether or not \(q_n\) is positive.

Now by (4.1)

\[
\begin{align*}
    s_m &= \sum_{n=0}^{m} r_n \sum_{k=n}^{\infty} \frac{b_k c_{k-n}}{q_k} \\
    &= \sum_{n=0}^{m} r_n \left( \sum_{k=n}^{n} + \sum_{k=m+1}^{\infty} \right) \frac{b_k c_{k-n}}{q_k} \\
    &= \sum_{k=0}^{m} b_k \sum_{n=0}^{k} r_n c_{k-n} + \sum_{n=0}^{m} r_n \sum_{k=m+1}^{\infty} \frac{b_k c_{k-n}}{q_k} \\
    &= \sum_{k=0}^{m} b_k + \sum_{n=0}^{m} r_n \sum_{k=m+1}^{\infty} \frac{b_k c_{k-n}}{q_k}.
\end{align*}
\]

Hence, as \(b_k = o(1)\),

\[
\begin{align*}
    \left| s_m - \sum_{k=0}^{m} b_k \right| &\leq \sum_{n=0}^{m} r_n \sum_{k=m+1}^{\infty} o(1) \frac{|c_{k-n}|}{q_k} \\
    &= o(1) \left( \frac{1}{q_m} \sum_{n=0}^{m} r_n \sum_{k=m+1}^{\infty} |c_{k-n}| \right).
\end{align*}
\]

But when \(p_n \in \mathcal{M}\), by Lemma 1, we have

\[(4.3) \quad \sum_{k=m+1}^{\infty} |c_{k-n}| \leq c_{m-n}^{(1)};\]

and hence, by identity (4.2)

\[
\begin{align*}
    \left| s_m - \sum_{k=0}^{m} b_k \right| &= o(1) \left( \frac{1}{q_m} \sum_{n=0}^{m} r_n c_{m-n}^{(1)} \right) \\
    &= o(1) \frac{Q_m}{q_m}.
\end{align*}
\]

This completes the proof.

**Proof of Theorem 3.** We have
\[
\sum_{n=0}^{\infty} \left| \frac{s_n q_n}{r_n} - \frac{s_{n+1} q_{n+1}}{r_{n+1}} \right| = \sum_{n=0}^{\infty} \left| \Delta \left( \frac{s_n q_n}{r_n} \right) \right| \\
\leq \sum_{n=0}^{\infty} \left| a_{n+1} \right| \frac{q_{n+1}}{r_{n+1}} + \sum_{n=0}^{\infty} s_n \left| \Delta \left( \frac{q_n}{r_n} \right) \right| \\
= L_n + M_n, \text{ (say)}.
\]

By using (2.1), we get (as \( q_n \) is nondecreasing)

\[
L_n \leq \sum_{n=0}^{\infty} \frac{g_{n+1}}{r_{n+1}} \sum_{k=n+1}^{\infty} \frac{b_k}{q_k} \left| c_{k-n-1} \right| \\
= \sum_{k=0}^{\infty} \left| b_k \right| \sum_{n=0}^{k} \left| c_{k-n-1} \right| \\
= O(1),
\]

since \( \Sigma |b_k| < \infty \) and \( \Sigma |c_n| < \infty \) as \( p_n \in \mathcal{W} \). Since \( \{q_n/r_n\} \) is decreasing we have,

\[
\sum_{n=v}^{\infty} \left| \Delta \frac{q_n}{r_n} \right| = \sum_{n=v}^{\infty} \left( \frac{q_n}{r_n} - \frac{q_{n+1}}{r_{n+1}} \right) \leq \frac{q_v}{r_v}.
\]

Hence,

\[
M_n = \sum_{n=0}^{\infty} \left| \Delta \frac{q_n}{r_n} \right| \left| \sum_{v=0}^{n} r_v \sum_{k=v}^{\infty} \frac{b_k c_{k-v}}{q_k} \right| \\
\leq \sum_{n=0}^{\infty} \left| \Delta \frac{q_n}{r_n} \right| \sum_{v=0}^{\infty} r_v \sum_{k=v}^{\infty} \left| b_k \right| \left| c_{k-v} \right| q_k \\
= \sum_{v=0}^{\infty} r_v \sum_{n=v}^{\infty} \left| \Delta \frac{q_n}{r_n} \right| \sum_{k=n}^{\infty} \left| b_k \right| \left| c_{k-n} \right| q_k \\
= \sum_{v=0}^{\infty} r_v \sum_{k=v}^{\infty} \left| b_k \right| \left| c_{k-v} \right| q_v \\
\leq \sum_{v=0}^{\infty} \sum_{k=v}^{\infty} \left| b_k \right| \left| c_{k-v} \right| r_v \\
= \sum_{k=0}^{\infty} \sum_{v=0}^{k} \left| b_k \right| \left| c_{k-v} \right| \\
< \infty,
\]
by hypothesis. Hence

$$\sum \left| \Delta \left( \frac{s_n q_n}{r_n} \right) \right| \leq L_n + M_n = O(1) \quad \text{as} \quad n \to \infty$$

and therefore

$$\{s_n q_n / r_n \} \in BV.$$

This completes the proof of Theorem 3.

5. Now we will prove our main theorem and for this, we require the following lemma.

**Lemma 2.** Let $p_n \in \mathcal{M}$, $q_n > 0$ and nondecreasing. Then (2.2) implies that

$$0 \leq q^2 \leq \sum_{v=0}^{k} q_v r_v c_{k-v} = O(q^2).$$

**Proof.** Since $q_n > 0$ and nondecreasing and $p_n > 0$, it follows that $r_n > 0$ and nondecreasing. Since, as $p_n \in \mathcal{M}$, by Lemma 1, $c_0 > 0$, $c_n \leq 0$ ($n \geq 1$), when we get

$$\sum_{v=0}^{k} q_v r_v c_{k-v} \geq q_k \sum_{v=0}^{k} r_v c_{k-v} = q_k^2 \geq 0,$$

by identity (4.1). Now

$$\sum_{v=0}^{k} q_v r_v c_{k-v} = \sum_{v=0}^{k} \Delta_v(q_{k-v} r_{k-v}) c_v (1)$$

$$= \sum_{v=0}^{k} q_{k-v} (r_{k-v} - r_{k-v-1}) c_v (1)$$

$$+ \sum_{v=0}^{k} r_{k-v-1} (q_{k-v} - q_{k-v-1}) c_v (1).$$

Hence, as $c^{(1)}_n \geq 0$, we get by (4.2)

$$\sum_{v=0}^{k} q_{k-v} (r_{k-v} - r_{k-v-1}) c^{(1)}_v \leq q_k (Q_k - Q_{k-1}) = q_k^2.$$

Again by (2.2)
\[ 0 \leq \sum_{v=0}^{k} r_{k-v-1}(q_{k-v} - q_{k-v-1})c_{v}^{(1)} \]
\[ = O(1) \sum_{v=0}^{k} q_{k-v}(r_{k-v} - r_{k-v-1})c_{v}^{(1)} \]
\[ = O(1)q^{2k}, \]
as in the previous case.

Hence
\[ 0 \leq \sum_{v=0}^{k} q_{v}r_{v}c_{k-v} = O(q^{2}). \]

This completes the proof of the lemma.

**Proof of Theorem 4.** We shall first prove that whenever \( \Sigma a_{n} \) is summable \((N^{*}, p, q)\), then \((J, q)\) method is applicable to \( \Sigma a_{n} \).

By Theorem 2, we have
\[ s_{n} = s + o\left(\frac{Q_{n}}{q_{n}}\right) = O\left(\frac{Q_{n}}{q_{n}}\right). \]

Hence
\[ J(x) = \frac{\sum q_{n}s_{n}x^{n}}{\sum q_{n}x^{n}} \]
\[ = O(1)\frac{\sum Q_{n}x^{n}}{\sum q_{n}x^{n}} \]
\[ = O(1)\sum x^{n}. \]

Since \( \sum x^{n} = 1/(1 - x) \) for \(|x| < 1\), it follows that \( J(x) \) exists for \(|x| < 1\) and hence \((J, q)\) method is applicable. Now for \(|x| < 1\),
\[ J(x) = \frac{1}{q(x)} \sum_{v=0}^{\infty} r_{v} \sum_{n=v}^{\infty} q_{n}x^{n} \sum_{k=v}^{\infty} \frac{b_{k}c_{k-v}}{q_{k}} \]
\[ = \frac{1}{q(x)} \sum_{v=0}^{\infty} r_{v} \sum_{k=v}^{\infty} \frac{b_{k}c_{k-v}}{q_{k}} \sum_{n=v}^{\infty} q_{n}x^{n} \]
\[ = \frac{1}{q(x)} \sum_{k=0}^{\infty} b_{k} \sum_{v=0}^{k} r_{v}c_{k-v} \sum_{n=v}^{\infty} q_{n}x^{n} \]
\[ = \sum_{k=0}^{\infty} g_{k}(x)b_{k}, \]
where,
The change of order of summation involved in obtaining (5.1) is justified in the range \( |x| < 1 \), by the absolute convergence of the double sum.

Now (5.1) is a series to function transformation, transforming the series \( \sum b_n \) to the function \( J(x) \). To prove the theorem, we have to show that the transformation (5.1) is regular, that is, we have to show that the conditions of regularity (see Cooke [3], page 65) are satisfied. Note that

\[
g_k(x) = \frac{\sum_{v=0}^{k} r_v c_{k-v} \left( q(x) - \sum_{n=0}^{v-1} q_n x^n \right)}{q_k q(x)}
\]

(5.2)

\[= \frac{1}{q_k} \sum_{v=0}^{k} r_v c_{k-v} \left( 1 - \sum_{n=0}^{v-1} q_n x^n / q(x) \right)
\]

\[= 1 - \left( \sum_{v=0}^{k} r_v c_{k-v} \sum_{n=0}^{v-1} q_n x^n \right) / (q(x)q_k)
\]

by identity (4.1).

Since \( q_n > 0 \) is increasing, we have

\[\sum q_n x^n \geq q_0 \sum x^n \to \infty \quad \text{as} \quad x \to 1 - 0.
\]

Hence from (5.2), we obtain

\[g_k(x) \to 1, \quad \text{as} \quad x \to 1 - 0.
\]

We have only to show that

(5.3)

\[\sum_{k=1}^{\infty} |g_k(x) - g_{k+1}(x)| \leq M,
\]

for \( 0 < x < 1 \), where \( M \) is a positive number.

Now let us write

\[\phi_v(x) = \sum_{k=v}^{\infty} q_k x^k / q(x).
\]

It is obvious that, \( \phi_0(x) = 1 \). Hence

\[g_v(x) - g_{v+1}(x) = \sum_{v=0}^{k+1} \phi_v(x) r_v \left( c_{k-v} - c_{k+1-v} \right) q_k / q_{k+1}
\]

\[= \sum_{v=0}^{k} c_{k-v} \left( \phi_v(x) r_v / q_k - \phi_{v+1}(x) r_{v+1} / q_{k+1} \right) - r_0 c_{k+1} / q_{k+1}.
\]
Since by hypothesis $\sum |c_n| < \infty$ and $\{1/q_n\}$ decreases as $n$ increases, we have,

$$\sum_{k=0}^{\infty} \frac{|c_{k+1}|}{q_{k+1}} \leq \frac{1}{q_0} \sum_{k=0}^{\infty} |c_{k+1}| < \infty.$$ 

Hence in order to show that (5.3) holds it is enough to show that,

$$\theta(x) = \sum_{k=0}^{\infty} \left| \sum_{v=0}^{k} c_{k-v} \left( \phi_v(x) - \phi_{v+1}(x) \right) \frac{r_v}{q_k} - \phi_{v+1}(x) \left( \frac{r_v}{q_k} - \frac{r_{v+1}}{q_{k+1}} \right) \right| < M,$$

for $0 < x < 1$.

Now since

$$\phi_v(x) - \phi_{v+1}(x) = \frac{q_v x^v}{q(x)},$$

it follows that,

$$(5.5) \quad \theta(x) = \sum_{k=0}^{\infty} \left| \sum_{v=0}^{k} c_{k-v} \left( \phi_v(x) - \phi_{v+1}(x) \right) \frac{r_v}{q_k} + \phi_{v+1}(x) \left( \frac{r_v}{q_k} - \frac{r_{v+1}}{q_{k+1}} \right) \right| \leq M(x) + N(x),$$

where,

$$M(x) = \frac{1}{q(x)} \sum_{k=0}^{\infty} \frac{1}{q_k} \left| \sum_{v=0}^{k} c_{k-v} q_v r_v x^v \right|$$

and

$$N(x) = \sum_{k=0}^{\infty} \left| \sum_{v=0}^{k} c_{k-v} \phi_{v+1}(x) \left( \frac{r_v}{q_k} - \frac{r_{v+1}}{q_{k+1}} \right) \right|.$$ 

Since

$$\sum_{v=0}^{k} c_{k-v} q_v r_v x^v = \sum_{v=0}^{k-1} c_{k-v} q_v r_v (x^v - x^k) + x^k \sum_{v=0}^{k} c_{k-v} q_v r_v,$$

to prove $M(x) = O(1)$ we need only show that,

$$M'(x) = \frac{1}{q(x)} \sum_{k=0}^{\infty} \frac{1}{q_k} \sum_{v=0}^{k-1} c_{k-v} r_v (x^v - x^k) = O(1),$$

in view of Lemma 2.

Since $c_n \leq 0$ ($n \geq 1$) and $\{1/q_n\}$ is decreasing, we get,
\[ M'(x) = -\frac{1}{q(x)} \sum_{k=0}^{\infty} \frac{1}{q_k} \sum_{v=0}^{k-1} q_v r_v c_{k-v}(x^v - x^k) \]
\[ = -\frac{1}{q(x)} \sum_{k=0}^{\infty} q_v r_v x^v \sum_{k=0}^{\infty} c_{k-v} \left(1 - x^{k-v}\right) q_k \]
\[ \leq -\frac{1}{q(x)} \sum_{v=0}^{\infty} q_v r_v x^v \sum_{k=v+1}^{\infty} c_{k-v} \left(1 - x^{k-v}\right) q_k \]
\[ = -\frac{1}{q(x)} \sum_{v=0}^{\infty} r_v x^v (c(1) - c(x)) \]
\[ \leq -\frac{1}{q(x)} \sum_{v=0}^{\infty} r_v x^v c(x) \]
\[ = \frac{r(x)c(x)}{q(x)} \]
\[ = 1. \]

Hence,

(5.6) \quad M(x) = O(1).

The inner sum of \( N(x) \) can be written as,

\[ \phi_{k+1}(x) \sum_{v=0}^{k} c_{k-v} \left(\frac{r_v}{q_k} - \frac{r_{v+1}}{q_{k+1}}\right) + \sum_{v=0}^{k} c_{k-v} (\phi_{v+1}(x) - \phi_{k+1}(x)) \left(\frac{r_v}{q_k} - \frac{r_{v+1}}{q_{k+1}}\right) \]
\[ = \phi_{k+1}(x) \sum_{v=0}^{k} c_{k-v} \left(\frac{r_v}{q_k} - \frac{r_{v+1}}{q_{k+1}}\right) \]
\[ + \sum_{v=0}^{k} c_{k-v} \frac{r_v}{q(x)} \left(\frac{r_v}{q_k} - \frac{r_{v+1}}{q_{k+1}}\right) \sum_{\mu=v+1}^{k} q_{\mu} x^\mu. \]

Hence,

(5.7) \quad N(x) \leq N'(x) + N''(x),

where,

\[ N'(x) = \sum_{k=0}^{\infty} \left| \phi_{k+1}(x) \sum_{v=0}^{k} c_{k-v} \left(\frac{r_v}{q_k} - \frac{r_{v+1}}{q_{k+1}}\right) \right|, \]

and

\[ N''(x) = \sum_{k=0}^{\infty} \left| \sum_{v=0}^{k} c_{k-v} \left(\frac{r_v}{q_k} - \frac{r_{v+1}}{q_{k+1}}\right) \sum_{\mu=v+1}^{k} q_{\mu} x^\mu \right| q(x). \]
By (4.1)
\[
\sum_{v=0}^{k} c_{k-v} \frac{r_v}{q_k} - \sum_{v=0}^{k} c_{k-v} \frac{r_{v+1}}{q_{k+1}}
\]
\[
= 1 - \frac{1}{q_{k+1}} \sum_{v=0}^{k} c_{k-v} r_{v+1}
\]
\[
= 1 - \frac{1}{q_{k+1}} \left( \sum_{v=0}^{k+1} c_{k+1-v} r_v - c_{k+1} r_0 \right)
\]
\[
= r_0 \frac{c_{k+1}}{q_{k+1}}.
\]

Hence,
\[
N'(x) = r_0 \sum_{k=0}^{\infty} \phi_{k+1}(x) \left| \frac{c_{k+1}}{q_{k+1}} \right|.
\]

We know from the very definition of \( \phi_k(x) \) that for \( 0 < x < 1 \),
\[
0 \leq \phi_k(x) \leq 1.
\]

Hence
\[
N'(x) \leq r_0 \sum_{k=0}^{\infty} \frac{c_{k+1}}{q_{k+1}} \leq \frac{r_0}{q_0} \sum |c_{k+1}| < \infty.
\]

And
\[
N''(x) \leq \sum_{k=0}^{\infty} \sum_{\nu=0}^{\infty} c_{k-\nu} \left| \frac{r_{\nu}}{q_k} - \frac{r_{\nu+1}}{q_{k+1}} \right| \sum_{\mu=0}^{\infty} q_{\mu} x^\mu
\]
\[
= \frac{1}{q(x)} \sum_{\nu=0}^{\infty} \sum_{k=\nu}^{\infty} c_{k-\nu} \left| \frac{r_{\nu}}{q_k} - \frac{r_{\nu+1}}{q_{k+1}} \right| \sum_{\mu=0}^{\infty} q_{\mu} x^\mu
\]
\[
= \frac{1}{q(x)} \sum_{\nu=0}^{\infty} \sum_{\mu=\nu+1}^{\infty} q_{\mu} x^\mu \sum_{k=\nu}^{\infty} c_{k-\nu} \left| \frac{r_{\nu}}{q_k} - \frac{1}{q_{k+1}} \right| + \frac{r_0 - r_{\nu+1}}{q_k+1}
\]
\[
\leq \frac{1}{q(x)} \sum_{\nu=0}^{\infty} r_{\nu} \sum_{\mu=\nu+1}^{\infty} q_{\mu} x^\mu \sum_{k=\nu}^{\infty} c_{k-\nu} \left( \frac{1}{q_k} - \frac{1}{q_{k+1}} \right) + \frac{1}{q(x)} \sum_{\nu=0}^{\infty} (r_{\nu+1} - r_{\nu}) \sum_{\mu=\nu+1}^{\infty} q_{\mu} x^\mu \sum_{k=\nu}^{\infty} c_{k-\nu} \frac{1}{q_{k+1}}
\]
\[
= \alpha(x) + \beta(x), \quad (say).
\]
Now, since \( \{q_n\} \) and \( \{q_n/q_{n+1}\} \) are increasing with \( n \) we get, by using hypothesis (2.2) and (4.3)

\[
\alpha(x) = \frac{1}{q(x)} \sum_{v=0}^{\infty} r_v \sum_{\mu=v+1}^{\infty} x^\mu \sum_{k=\mu}^{\infty} |c_{k-v}| \left(1 - \frac{q_k}{q_{k+1}}\right)
\]

\[
= \frac{1}{q(x)} \sum_{v=0}^{\infty} \frac{r_v (q_{v+1} - q_v)}{q_{v+1}} \sum_{\mu=v+1}^{\infty} c^{(1)}_{\mu-v-1} x^\mu
\]

\[
= \frac{1}{(1-x)q(x)p(x)} \sum_{v=0}^{\infty} r_v (q_{v+1} - q_v) x^{v+1}
\]

\[
= \frac{1}{(1-x)r(x)} O(1) \sum_{v=0}^{\infty} (r_{v+1} - r_v) x^{v+1}
\]

\[
= O(1),
\]

by using the identity,

\[
(1-x)p(x)\sum c^{(1)}_n x^n = 1, \quad (0 < x < 1).
\]

Again since \( \{r_n\} \) increases with \( n \) as \( \{q_n\} \) increases, we get,

\[
\beta(x) = \frac{1}{q(x)} \sum_{v=0}^{\infty} \frac{r_v (q_{v+1} - q_v)}{q_{v+1}} \sum_{\mu=v+1}^{\infty} x^\mu c^{(1)}_{\mu-v-1}
\]

\[
= \frac{1}{q(x)} \sum_{v=0}^{\infty} (r_{v+1} - r_v) x^{v+1} \sum_{n=0}^{\infty} c^{(1)}_n x^n
\]

\[
= \frac{1}{(1-x)p(x)q(x)} \sum_{v=0}^{\infty} (r_{v+1} - r_v) x^{v+1}
\]

\[
\leq 1.
\]

Hence,

\[
N''(x) = \alpha(x) + \beta(x) = O(1).
\]

Hence by (5.7), (5.6) and (5.5)

\[
\theta(x) \leq M(x) + N(x) = O(1).
\]

Hence (5.3) holds and this completes the proof of the theorem.
6. In this section, we now deduce some corollaries of Theorem 4.

**Corollary 1.** (Thorpe [9]). Suppose $p_n \in \mathcal{M}$, then $\Sigma a_n \in (N^*, p) \Rightarrow \Sigma a_n \in (A)$, where $(A)$ is the Abel method.

*Proof.* Put $q_n = 1$, for all $n$ in Theorem 4.

**Corollary 2.** Let $q_n > 0$ for all $n$, $\{q_n\}$ be increasing in $n$, such that $\{q_n/q_{n+1}\}$ is also increasing in $n$ and,

\[
Q_n(q_{n+1} - q_n) = O(q_{n+1}^{\alpha}).
\]

Then,

\[
\Sigma a_n \in (\bar{N}^*, q) \Rightarrow \Sigma a_n \in (J, q).
\]

*Proof.* Put $p_n = 1$ for all $n$, in Theorem 4. In this case we have,

\[
c_0 = 1, \quad c_1 = -1, \quad c_n = 0 \quad (n > 2).
\]

**Corollary 3.** $(C^*, \alpha, \beta) \Rightarrow A_\beta$ for $0 < \alpha \leq 1 \leq \beta$.

*Proof.* Set

\[
p_n = A_n^{\alpha - 1}, \quad q_n = A_n^{\beta - 1} \quad \text{in Theorem 4.}
\]

Then $r_n = A_n^{\alpha + \beta - 1}$ and condition (2.2) reduces to proving that

\[
n^{\alpha + \beta - 1}n^{\beta - 2} = O(n^{\beta - 1}n^{\alpha + \beta - 2}),
\]

which is valid in the present case. Also when $0 < \alpha \leq 1$, then $p_n = A_n^{\alpha - 1} \in \mathcal{M}$ and when $\beta \geq 1$, then $q_n = A_n^{\beta - 1}$ is nondecreasing.

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**References**


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