NORMS OF COMPACT PERTURBATIONS OF OPERATORS

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Let $\mathcal{B}(\mathcal{H})$ denote the algebra of all bounded linear operators on a complex separable Hilbert space. This paper is concerned with reducing the norm of a product of operators by compact perturbations of one or more of the factors. For any $T$ in $\mathcal{B}(\mathcal{H})$, it is well known that the infimum,

$$\| T \|_e = \inf \{ \| T + K \| : K \text{ is a compact operator} \}$$

is attained by some compact perturbation $T + K_0$. For $T$ a noncompact product of $n$ operators, $T = T_1 \cdots T_n$, it is proved that this infimum can be obtained by a compact perturbation of any one of the factors. If $T$ is a compact product, so that the infimum is zero, it is shown that there are compact perturbations $T_i + K_{i_1} \cdots T_n + K_{n_1}$ of the factors of $T$ such that the product $(T_i + K_{i_1}) \cdots (T_n + K_{n_1})$ is zero; furthermore, it may be necessary to perturb every factor of $T$ in order to obtain this zero infimum.

These results are applied to an arbitrary operator $T$ to find a compact perturbation $T + K$ with $\| (T + K)^2 \| = \| T^2 \|_e$ and $\| (T + K)^3 \| = \| T^3 \|_e$; here the identical factors are perturbed in identical fashion to achieve both infima. Stronger theorems of this latter sort are proved for special classes of operators.

For any $T$ in $\mathcal{B}(\mathcal{H})$, let $\| T \|_e$ as defined above, be called the essential norm of $T$ [7]. I. C. Gohberg and M. G. Krein first showed in [4] that for any $T$ in $\mathcal{B}(\mathcal{H})$ there is a compact perturbation $T + K_0$ which realizes the essential norm (so $\| T + K_0 \| = \| T \|_e$). The case $n = 2$ of the theorem stated above for compact products was proved in a different way in [6]: for any compact product $T = T_1 T_2$ of two factors, a projection $E$ was constructed so that $T_1 E$ and $(I - E) T_2$ are both compact (and so that the product of perturbations $T_1 I((I - E) T_2$ is zero).

This study was motivated partly by questions considered by J. K. Plastiras and the author in [7]: if $T$ is a bounded operator on $\mathcal{H}$, is there a compact $K$ with $\| p(T + K) \| = \| p(T) \|_e$ for all complex polynomials $p$? Less ambitiously, if $T$ and $p$ are both given, is there a compact $K_p$ such that $\| p(T + K_p) \| = \| p(T) \|_e$? We know of no examples where either of these questions has a negative answer.

It follows from the results proved here on perturbations of products that for each $T$ in $\mathcal{B}(\mathcal{H})$, there is a compact $K$ with $\| T + K \| = \| T \|_e$ and $\| (T + K)^2 \| = \| T^2 \|_e$; and a compact $L$ with $\| (T + L)^2 \| = \| T^2 \|_e$ and $\| (T + L)^3 \| = \| T^3 \|_e$. If $T^3$ is not compact we can take $K = L$, to get one
perturbation achieving all three essential norms. There appear to be serious difficulties in passing from $T^3$ to $T^4$. The existence of an operator $K$ as above was proved in [7] for any partial isometry $T$, and for certain other operators.

Stronger results are obtainable for special classes of operators. In [7] it was shown that for operators $T$ which are subnormal or essentially normal, there is one compact $K$ such that $\|p(T + K)\| = \|p(T)\|$, for every complex polynomial $p$. Here we prove this for $n$-normal operators. Turning to operators with no normality properties, we show that for any weighted shift $T$, there is one compact $K$ with $\|(T + K)^n\| = \|T^n\|$, for all $n$. If in addition $T$ is nilpotent, then $\|p(T + K)\| = \|p(T)\|$, for every polynomial $p$. In [6] it was shown that for any $T$ in $B(\mathcal{H})$ with $p(T)$ compact, there is a compact $K_p$ with $\|p(T + K_p)\| = \|p(T)\|_e = 0$.

If it were true that every $T$ in $B(\mathcal{H})$ could be perturbed by $K$, to simultaneously obtain $\|p(T + K)\| = \|p(T)\|_e$ for every polynomial $p$, this would have significant consequences. It would immediately imply the theorem of T. T. West [11] that every Riesz operator is a compact perturbation of a quasinilpotent, and would also answer a question of W. Arveson: if $\pi(T)$ is quasialgebraic in the Calkin algebra, so that $\|p_n(\pi(T))\|^{1/\deg p_n} \to 0$, then is there a compact $K$ so that $\|p_n(T + K)\|^{1/\deg p_n} \to 0$, for the same sequence $\{p_n\}_n$ of monic polynomials? A partial answer to this latter question, and further discussion is given in [7]. See also the question raised by S. R. Caradus [3].

In a recent communication we have learned that D. Legg, P. Smith, and J. Ward have proved using Banach space techniques, that for any $T$ in $B(\mathcal{H})$, there is one compact $K$ with $\|T + K + \lambda I\| = \|T + \lambda I\|_e$, for all complex $\lambda$. Thus it is possible to simultaneously attain the essential norm for all linear polynomials in $T$.

A related result in a more general setting has been obtained by G. K. Pedersen [8]. In [7] it was shown for any $T$ in $B(\mathcal{H})$ and for any polynomial $p$, that

$$\|p(T)\|_e = \text{inf}\|p(T + K)\|, \quad K \text{ compact.}$$

Pedersen has proved that if $\mathcal{A}$ is a $C^*$-algebra and $\mathcal{I}$ is a closed ideal in $\mathcal{A}$ then for any $A \in \mathcal{A}$ and for any $n$,

$$\|A^n + \mathcal{I}\| = \text{inf}\|(A + B)^n\|, \quad B \in \mathcal{I}.$$
orthonormal basis for $\mathcal{H}$ consisting of eigenvectors for $D$. A finite rank operator is one with finite-dimensional range. The range projection of $T \in \mathcal{B}(\mathcal{H})$ is the smallest projection $Q$ such that $QT = T$, and the support projection $P$ is the smallest projection such that $TP = T$. Throughout the paper we use $|T|$ to denote $(T^*T)^{1/2}$ and $\sigma(T)$ to denote the spectrum of $T \in \mathcal{B}(\mathcal{H})$. The reader is referred to [S] for general facts about Hilbert space operators.

1. Reducing the norm of a product by perturbing its factors. This first theorem is the heart of the paper.

**Theorem 1.** Let $A, B$ in $\mathcal{B}(\mathcal{H})$ be such that the product $AB$ is not compact. Then there is a compact operator $K$ such that

$$\|A(I-K)B\| = \|AB\|.$$  

Furthermore, if $\{e_n\}_n$ is any orthonormal basis for $\mathcal{H}$, then $K$ can be constructed to be diagonal relative to that basis, with $0 \leq K \leq I$.

Before beginning the proof we make some relevant observations. If $D$ is any diagonal operator with $\|D\| \leq 1$, then it is trivial that $\|DABx\| \leq \|ABx\|$, for any $x \in \mathcal{H}$ and any $A, B$ in $\mathcal{B}(\mathcal{H})$. It is also obvious that $\|ABD\| \leq \|AB\|$, although $\|ABDx\| \leq \|ABx\|$ may not hold for every $x$. On the other hand, there is no general relationship between $\|ADB\|$ and $\|AB\|$.

We remark also that this theorem is false if the product $AB$ is compact. To see this let $A$ be any injective compact operator and let $B = I$. Then $\|AB\|_e = 0$, but $A(I-K)B$ cannot be zero if $K$ is any compact operator.

**Proof.** We may assume that $\|A\| \leq 1$ and $\|B\| \leq 1$. Let $\{P_k\}_k$ be the increasing sequence of finite rank projections with range $(P_k) = \text{span}\{e_1, \ldots, e_k\}$.

Let $\mu$ be any number with $\|AB\|_e < \mu < \|AB\| = \mu_0$. We will first construct a finite rank perturbation $D$ of $I$ so that $0 \leq D \leq I$; $D$ will be $\{e_n\}_n$-diagonal; and with $\|ADB\| \leq \mu$. Then we will show how this construction is repeated, to define by induction the desired operator $I - K$. In order to be able later to set up the induction, we will write in the factor $I$, which is being perturbed.

Let $E(\lambda)$ be the spectral resolution for $|AIB|$. Set $E = E((\mu - 2\delta, \mu_0))$, where $\delta > 0$ is a small number with $\mu - 2\delta > \|AB\|_e$. Then $E$ must be a finite rank projection: otherwise, we could find an infinite orthonormal set $\{x_n\}_n$ such that $x_n \in \text{ran}(E)$, and hence for which
\[ \| AIBx_n \| = \| AIB \| x_n \| > \mu - 2\delta. \]

But this would imply
\[ \| AIB \| _e \geq \mu - 2\delta > \| AIB \| _e, \]

a contradiction.

Let \( G \) be the projection onto \( \text{ran}(IBE) \), so \( G \) is finite rank. Choose \( P_k \), from the sequence \( \{P_i\} \) large enough so that
\[ \| (I - P_k)G \| < \nu, \]

where \( \nu > 0 \) is a very small number to be determined. Let \( Q_1 \) be the finite rank projection onto \( \text{ran}(AP_k) \). Let \( H_1 \) be the support projection of the finite rank operator \( Q_1A(I - P_k) \), so \( H_1 \leq I - P_k \).

Choose \( k_2 > k_1 \) sufficiently large so that
\[ \| Q_1A(I - P_k) \| = \| Q_1A(I - P_k)H_1(I - P_k) \| \leq \| H_1(I - P_k) \| < \nu. \]

Let \( Q_2 \) be the finite rank projection onto \( \text{ran}(AP_k) \). Let \( H_2 \) be the finite rank support projection of \( Q_2A(I - P_k) \), so \( H_2 \leq I - P_k \).

Choose \( k_3 > k_2 \) sufficiently large so that
\[ \| Q_2A(I - P_k) \| = \| Q_2A(I - P_k)H_2(I - P_k) \| \leq \| H_2(I - P_k) \| < \nu. \]

Repeat this process \( m \) times, where \( m \) is to be determined, to get two increasing sets of finite rank projections \( \{Q_n\}_{n=1}^m, \{P_k\}_{n=1}^m \). Set \( E_1 = P_k, E_2 = P_k - P_k, \ldots, E_m = P_k - P_k, \ldots, E_{m+1} = I - P_k \). Set \( F_1 = Q_1, F_2 = Q_2 - Q_1, \ldots, F_m = Q_m - Q_m, \ldots, F_{m+1} = I - Q_m \).

Observe now that \( F_jAE_n = 0 \), if \( n < j \): for,
\[ F_jAE_n = F_jAP_kE_n = F_jQ_nAP_kE_n = (Q_j - Q_{j-1})Q_nAP_kE_n = 0, \]

whenever \( n < j \).

Observe also that \( \| F_jAE_n \| < \nu \) if \( n > j + 1 \) for then \( E_n = (I - P_{k_{j+1}})E_n \), so that
\[ \| F_jAE_n \| = \| F_jQ_jA(I - P_{k_{j+1}})E_n \| \leq \| Q_jA(I - P_{k_{j+1}}) \| < \nu. \]

Now, set \( \gamma = (\mu - 2\delta)/\| AIB \| \), so \( 0 < \gamma < 1 \).

Define \( D = I + \sum_{j=1}^{m+1} \eta_jE_j \) where \( \gamma = \eta_1 < \eta_2 < \cdots < \eta_{m+1} = 1 \), is an even partition of the interval \([\gamma, 1]\). We choose a small \( \epsilon > 0 \) to be determined, and we now determine \( m \): so that \( m\epsilon > 1 - \gamma \). In other words, \( \eta_1 - \eta_{j-1} < \epsilon \). Thus \( D \) is a finite rank perturbation of \( I \), and is \( \{e_n\}_n \)-diagonal.
We will now show that \( \| ADB \| \leq \mu \). (Note that so far we have that \( \gamma = \gamma(\delta, \mu) \), and \( m = m(\gamma, \epsilon) \); but we are free to choose \( \epsilon \) and \( \nu \) as small as we wish.)

Let \( z \) be a unit vector of \( \mathcal{H} \), and write \( z = \alpha x \oplus \beta y \), where \( |\alpha|^2 + |\beta|^2 = 1 \), \( \| x \| = \| y \| \), and \( x \in \text{ran} E((\mu - 2\delta, \mu_0]) \), \( y \in \text{ran} E([0, \mu - 2\delta]) \). Since these are orthogonal spectral projections for \( |AIB| \), this means \( AIBx \) is orthogonal to \( AIBy \). Now,

\[
\| ADBz \|^2 \leq |\alpha|^2 \| ADBx \|^2 + |\beta|^2 \| ADBy \|^2 + 2|\alpha\beta| \langle ADBx, ADBy \rangle,
\]

and we consider the three summands separately.

First,

\[
\| ADBx \| = \left\| A \sum_{j=1}^{m+1} \eta_j E_j IBx \right\| \\
\leq \left\| A \sum_{j=1}^{m+1} \eta_j E_j GIBx \right\| \\
\leq \| A \eta_1 E_1 GIBx \| + \nu \quad (\| (I - E_1)G \| < \nu) \\
\leq \eta_1 \| AGIBx \| + 2\nu \\
= \gamma \| AIBx \| + 2\nu \quad (\eta_1 = \gamma) \\
\leq \frac{\mu - 2\delta}{\| AIB \|} \| AIBx \| + 2\nu \\
\leq \mu - 2\delta + 2\nu.
\]

Now consider

\[
\| ADBy \| = \left\| A \sum_{n=1}^{m+1} \eta_n E_n IBy \right\| = \left\| \sum_{j=1}^{m+1} F_j A \sum_{n=1}^{m+1} \eta_n E_n IBy \right\| \\
= \left\| F_1 A \eta_1 E_1 IBy + F_1 A \eta_2 E_2 IBy + F_1 A \sum_{n=3}^{m+1} \eta_n E_n IBy \\
+ F_2 A \eta_3 E_2 IBy + F_2 A \eta_3 E_3 IBy + F_2 A \sum_{n=4}^{m+1} \eta_n E_n IBy \\
+ \cdots + F_{m-1} A \eta_{m-1} E_{m-1} IBy + F_{m-1} A \eta_m E_m IBy \\
+ F_{m-1} A \eta_{m+1} E_{m+1} IBy \\
+ F_m A \eta_m E_m IBy + F_m A \eta_{m+1} E_{m+1} IBy \\
+ F_m A \eta_{m+1} E_{m+1} IBy \right\|,
\]

since \( F_j A E_n = 0 \) if \( n < j \);
\[
\leq \left\| \sum_{j=1}^{m} F_j A \eta_j (E_j + E_{j+1}) IB_y + \sum_{j=1}^{m} (\eta_{j+1} - \eta_j) F_j A E_{j+1} IB_y \\
+ F_{m+1} A \eta_{m+1} E_{m+1} IB_y \right\| + \frac{m(m-1)}{2} \nu,
\]

since \( \left\| F_j A E_n \right\| < \nu \) if \( n > j + 1 \);

\[
\leq \left\| \sum_{j=1}^{m+1} \eta_j F_j A IB_y - \sum_{j=1}^{n-1} \eta_j F_j A \sum_{n=j+2}^{m+1} E_n IB_y \right\| + \epsilon \left\| \sum_{j=1}^{m} F_j A E_{j+1} IB_y \right\| \\
+ \frac{m(m-1)}{2} \nu,
\]

since \( \eta_{j+1} - \eta_j < \epsilon \);

\[
\leq \left\| \sum_{j=1}^{m+1} \eta_j F_j A IB_y \right\| + m(m-1) \nu + \epsilon
\]

\[
\leq \left\| A IB_y \right\| + m(m-1) \nu + \epsilon
\]

\[
\leq \mu - 2 \delta + m(m-1) \nu + \epsilon.
\]

Finally, consider

\[
|\langle ADBx, ADBy \rangle| = \left| \left\langle A \sum_{j=1}^{m+1} \eta_j E_j IBx, ADBy \right\rangle \right|
\]

\[
\leq \left| \langle A \eta_1 E_1 GIBx, ADBy \rangle \right|
\]

\[
+ \left| \left\langle A \sum_{j=2}^{m+1} \eta_j E_j (I - P_k) GIBx, ADBy \right\rangle \right|
\]

\[
\leq \left| \left\langle F_1 A \eta_1 E_1 GIBx, F_1 A \sum_{n=1}^{m+1} \eta_n E_n IBy \right\rangle \right| + \nu,
\]

since \( F_1 A E_1 = AE_1 \), and \( \left\| (1 - P_k) G \right\| < \nu \);

\[
= \left| \langle F_1 A \eta_1 E_1 GIBx, F_1 A (\eta_1 E_1 + \eta_2 E_2) IBy \rangle \right| + (m - 1) \nu,
\]

since \( \left\| F_j A E_n \right\| < \nu \) if \( n \geq j + 2 \);

\[
\leq \left| \langle F_1 A \eta_1 E_1 GIBx, F_1 A \eta_1 (E_1 + E_2) IBy \rangle \right| + \epsilon + (m - 1) \nu
\]

\[
\leq \left| \langle F_1 A E_1 GIBx, F_1 A (E_1 + E_2) IBy \rangle \right| + \epsilon + (m - 1) \nu
\]

\[
\leq \left| \left\langle F_1 A E_1 GIBx, F_1 A \sum_{n=1}^{m+1} E_n IBy \right\rangle \right| + \epsilon + (2m - 2) \nu
\]

\[
= \left| \langle AE_1 GIBx, AIBy \rangle \right| + \epsilon + (2m - 2) \nu
\]

\[
\leq \left| \langle AGIBx, AIBy \rangle \right| + \epsilon + 2m \nu
\]

\[
= \epsilon + 2m \nu.
\]
Now determine $\epsilon = \epsilon(\delta, \mu)$ sufficiently small ($2\epsilon < \delta$ and $(\mu - \delta)^2 + 2\epsilon < \mu^2$), and $\nu = \nu(m, \epsilon)$ sufficiently small ($2m\nu < \epsilon$, $\nu < \epsilon$, $m^2\nu < \epsilon$), so that

\[
\|ADBz\| \leq |\alpha| \|ADbx\|^2 + |\beta| \|ADby\|^2 + 2|\alpha\beta|\langle ADBx, ADBy \rangle
\]
\[
< |\alpha|^2(\mu - \delta)^2 + |\beta|^2(\mu - \delta)^2 + 2|\alpha\beta|2\epsilon
\]
\[
< (\mu - \delta)^2 + 2\epsilon
\]
\[
< \mu^2.
\]

Thus we have $D$ with the desired properties.

This construction is the first step in an induction. To view it as such, rename $D = D_1$, $\mu = \mu_1$, $\delta = \delta_1$, $\gamma = \gamma_1$, $m = m_1$, $\epsilon = \epsilon_1$, $\nu = \nu_1$, $\{E_{ij}\}_{i=1}^{n+1}$ as $\{E_{ij}\}_{i=1}^{n_1+1}$, and $\{\eta_{ij}\}_{i=1}^{n+1}$ as $\{\eta_{ij}\}_{i=1}^{n_1+1}$. A decreasing sequence $\{D_n\}_n$ of $\{e_n\}_n$-diagonal operators will be constructed by induction; each a finite rank perturbation of $I$. Then the operator $D_0 = \inf D_n$ will be the desired compact perturbation, $D_0 = I - K$.

We specify the sequences of constants to be used (the first terms as above):

1. Choose a strictly decreasing sequence of positive numbers $\{u_n\}_n$ with $\mu_1$ (as above) $< \mu_0 = \|AB\|$ and $\lim \mu_n = \|AB\| \neq 0$. The sequence $\{D_n\}_n$ will satisfy $\|AD_nB\| \leq \mu_n$.

2. Choose $\{\delta_n\}_n$ positive numbers decreasing to zero, so that $2\delta_n < \mu_n - \mu_{n+1}$.

3. Let $\{\gamma_n\}_n$ be the positive sequence converging to 1 given by $\gamma_n = (\mu_n - 2\delta_n)/\mu_{n+1}$.

Then from (2) we have

\[
\frac{\mu_{n+1}}{\mu_{n-1}} < \gamma_n < \frac{\mu_n}{\mu_{n-1}}
\]

so that the infinite product $\prod \gamma_n$ converges to a nonzero limit precisely when the operator $AB$ is not compact; i.e., when the $\lim \mu_n = \|AB\| \neq 0$.

4. Choose $\{\epsilon_n\}_n$ decreasing to zero, such that $2\epsilon_n < \delta_n$ and $(\mu_n - \delta_n)^2 + 2\epsilon_n < \mu_n^2$.

5. Choose integers $\{m_n\}_n$ such that $1 - \gamma_n < m_n\epsilon_n$.

6. Finally, choose positive $\{\nu_n\}_n$ converging to zero, so that $\nu_n < \epsilon_n$, $m_n^2\nu_n < \epsilon_n$ and $2m_n\nu_n < \epsilon_n$.

Now repeat the above construction, line for line, with $D_1$ in place of $I$, using $\mu_2$, $\delta_2$, $\epsilon_2$, $m_2$, $\nu_2$, and specifying $\{E_{2i}\}_{i=1}^{n+1}$ and $\{\eta_{2i}\}_{i=1}^{n+1}$; the only additional stipulation being that we choose $E_{21} > \sum_{i=1}^{n_1} E_{1i}$. Thus we obtain a $D_2 \leq D_1$, $D_2$ a finite rank perturbation of $D_1$, and hence of $I$, with $\|AD_2B\| < \mu_2$:
\[ D_2 = D_1 \sum_{j=1}^{m_2-1} \eta_{2j} E_{2j} = \left[ \sum_{i=1}^{m_1} \eta_{1i} E_{1i} \right] \left[ \sum_{j=1}^{m_2-1} \eta_{2j} E_{2j} \right] \]

\[ = \left[ \sum_{i=1}^{m_1} \eta_{1i} E_{1i} \right] \eta_{21} E_{21} + E_{1,m_1+1} \left[ \sum_{j=1}^{m_2-1} \eta_{2j} E_{2j} \right] \]

\[ = \gamma_2 \sum_{i=1}^{m_1} \eta_{1i} E_{1i} + \eta_{21} \left( E_{21} - \sum_{i=1}^{m_1} E_{1i} \right) + \sum_{j=2}^{m_2-1} \eta_{2j} E_{2j} \]

recalling that \( \gamma_2 = \eta_{21} \). The point of this equation is to exhibit the diagonal operator \( D_2 \) as a linear combination of orthogonal projections.

Assume for induction we have recursively constructed \( D_1 \equiv D_2 \equiv \cdots \equiv D_{k-1} \) as above using in turn the specified constants and such that

\[ E_{j1} > \sum_{i=1}^{m_2-1} E_{j-1,i}, \quad j = 1, \ldots, k-1. \]

Then repeat the above construction with the \( k \)th constants, choosing

\[ E_{k1} > \sum_{i=1}^{m_k-1} E_{k-1,i}, \]

to obtain \( \| AD_k B \| \leq \mu_k \), and \( D_k \leq D_{k-1} \), where, as an orthogonal sum, we have

\[ D_k = \prod_{j=2}^{k} \gamma_j \left[ \eta_{11} \left( E_{11} - 0 \right) + \sum_{i=2}^{m_1} \eta_{1i} E_{1i} \right] \]

\[ + \prod_{j=3}^{k} \gamma_j \left[ \eta_{21} \left( E_{21} - \sum_{i=1}^{m_1} E_{1i} \right) + \sum_{i=2}^{m_2} \eta_{2i} E_{2i} \right] \]

\[ \vdots \]

\[ + 1 \left[ \eta_{k1} \left( E_{k1} - \sum_{i=1}^{m_{k-1}} E_{k-1,i} \right) + \sum_{i=2}^{m_k} \eta_{ki} E_{ki} \right] \]

\[ + \left[ I - \sum_{i=1}^{m_k} E_{ki} \right], \]

noting that the last summand equals \( \eta_{k,m_k+1} E_{k,m_k+1} \).

By induction we now have the desired sequence \( \{ D_n \}_n \) defined. We show that \( \{ D_n \}_n \) converges uniformly to \( \inf D_n = D_0 \), with

\[ D_0 = \sum_{n=1}^{\infty} \left( \prod_{j=n+1}^{\infty} \gamma_j \right) \left[ \eta_{n1} \left( E_{n1} - \sum_{i=1}^{m_{n-1}} E_{n-1,i} \right) + \sum_{i=2}^{m_n} \eta_{ni} E_{ni} \right] \]

(where \( E_{0i} = 0 \), all \( i \)). This will complete the proof: for then, \( AD_n B \) converges to \( AD_0 B \), so that \( \| AD_n B \| \leq \mu_n \) each \( n \), implying that
\[ \|AD_0B\| \leq \lim_{n \to \infty} \mu_n = \mu. \]  And since \( I - D_n \) is finite rank for each \( n \), therefore \( I - D_0 \) must be compact. Then \( K = I - D_0 \) will satisfy the conclusion of the theorem.

The convergence of \( \{D_n\} \) follows simply because the product \( \Pi \gamma_i \) converges. That is,

\[
D_k - D_0 = \left\{ \left( 1 - \prod_{j=k+1}^{\infty} \gamma_j \right) \prod_{j=1}^{k} \gamma_j \left[ \eta_{11} E_{11} + \sum_{i=2}^{m_1} \eta_{1i} E_{1i} \right] + \left( 1 - \prod_{j=k+1}^{\infty} \gamma_j \right) \prod_{j=1}^{k} \gamma_j \left[ \eta_{21} \left( E_{21} - \sum_{i=1}^{m_2} E_{1i} \right) + \sum_{i=2}^{m_2} \eta_{2i} E_{2i} \right] + \cdots + \left( 1 - \prod_{j=k+1}^{\infty} \gamma_j \right) \left( E_{k1} - \sum_{i=1}^{m_k} E_{k-1,i} \right) + \sum_{i=2}^{m_k} \eta_{k1} E_{k1} \right) + \sum_{n=k+1}^{\infty} \left\{ \left( 1 - \prod_{j=n+1}^{\infty} \gamma_j \right) \left( E_{n1} - \sum_{i=1}^{m_n} E_{n-1,i} \right) + \sum_{i=2}^{m_n} \left[ 1 - \left( \prod_{j=n+1}^{\infty} \gamma_j \right) \eta_{ni} \right] E_{ni} \right\},
\]

(recall \( \gamma_n = \eta_{ni} \)). Note that for each \( n \),

\[
1 - \left( \prod_{j=n+1}^{\infty} \gamma_j \right) \eta_{ni} < 1 - \prod_{j=n}^{\infty} \gamma_j.
\]

Thus,

\[
\|D_k - D_0\| \leq \sup_{n \geq k+1} \left( 1 - \prod_{j=n}^{\infty} \gamma_j \right) \|R\|,
\]

where \( R \) is a sum of orthogonal projections multiplied by constants that are between zero and one. Thus \( \lim_k \|D_k - D_0\| = 0 \), and the theorem is proved.

As immediate corollaries, we get the following:

**Theorem 2.** For any \( A, B \) in \( \mathcal{B}(\mathcal{H}) \), and any \( \epsilon > 0 \), there is a finite rank operator \( F \) with \( 0 \leq F \leq I \) such that

\[
\|A(I - F)B\| < \|AB\| + \epsilon.
\]

Furthermore, given any orthonormal basis, \( F \) can be constructed to be diagonal relative to that basis.

**Proof.** This is simply the first construction in the preceding proof, and it does not require noncompactness of the product \( AB \).
**Theorem 3.** Let $T_1, \ldots, T_n$ be in $\mathcal{B}(\mathcal{H})$ such that $\prod T_j$ is not compact. Then for any $j$ there is a compact perturbation $S_j$ of $T_j$ such that

$$\| T_1 \cdots T_{j-1} S_j T_{j+1} \cdots T_n \| = \| \prod T_j \|.$$

If $T_j$ is diagonal, $S_j$ may be obtained by reducing the moduli of some eigenvalues of $T_j$.

**Proof.** For $j = 1$, set $A = I, B = \prod T_j$ and apply Theorem 1 to get a compact $K$ with $\|(I - K)\prod T_j\| = \|\prod T_j\|$. Then set $S_1 = (I - K)T_1$. If $T_1$ is diagonal, construct $K$ to be diagonal relative to the same basis as $T_1$. If $j = 2$, set $A = T_1$ and $B = \prod_{j>1} T_j$, and proceed similarly; the other cases are the same.

In order to obtain a corresponding theorem for compact products of operators we require some preliminary results.

**Proposition 4.** Any $T$ in $\mathcal{B}(\mathcal{H})$ has a compact perturbation $S$ where $|S|$ is diagonal.

**Proof.** Let $T = U|T|$ be the polar decomposition for $T$. Let $E = U^*U$ and regard $|T|$ as a positive operator in $\mathcal{B}(\mathcal{E}\mathcal{H})$. By a theorem of H. Weyl [10], there is a compact operator $K$ in $\mathcal{B}(\mathcal{E}\mathcal{H})$ with $|T| + K$ diagonal relative to some orthonormal basis for $\mathcal{E}\mathcal{H}$. Consider this as a diagonal operator on $\mathcal{H}$: $\sigma(|T| + K)$ is the closure of the set $\{d_n\}_n$ of diagonal entries. The Weyl spectrum of $|T| + K$ is

$$\sigma_w(|T| + K) = \bigcap_{C \text{ compact}} \sigma(|T| + K + C).$$

Since $|T| + K$ is normal, by Weyl's Theorem, $\sigma_w(|T| + K)$ consists of the cluster points of $\sigma(|T| + K)$ union the eigenvalues that are repeated infinitely often [1]. Now,

$$\sigma_w(|T| + K) = \sigma_w(|T|) \subset \sigma(|T|),$$

so $\sigma_w(|T| + K)$ consists of nonnegative real numbers. Thus the subset of $\{d_n\}_n$ consisting of nonzero, nonpositive numbers has no accumulation points and no infinitely repeated numbers. If we replace such $d_n$ by the element of $\sigma_w(|T| + K)$ nearest $d_n$, the result is a positive diagonal operator $D$ which is a compact perturbation of $|T| + K$, and such that $ED = D$. Then $S = UD$ is the desired compact perturbation of $T$. 
PROPOSITION 5. Let $A, B$ be in $\mathcal{B}(H)$ and let $K$ be any compact operator. There are compact perturbations $A'$ and $B'$ of $A$ and $B$ and a projection $E$ such that

$$A'B' = (AB + K)E.$$ 

Proof. Let $U\sqrt{B}$ be the polar decomposition for $B$. Using the previous result, assume that $\sqrt{B}$ is diagonal relative to an orthonormal basis $\{e_n\}_n$ with diagonal sequence $\{b_n\}_n$.

To motivate the proof, we remark that, since $B$ may not be invertible, we cannot simply set $A' = A + KB^{-1}$, to get $A'B = AB + K$. However, if we first erase a subsequence of $\{b_n\}_n$ which converges to zero "too fast", then this approach will work.

Let $P_n$ be the finite rank projection onto span $\{e_1, \cdots, e_n\}$. Then $\{P_nK\} \to K$ uniformly, so choose a subsequence $\{P_{nk}\}_k$ with

$$\|K - P_{nk}K\| < \frac{1}{2^k}.$$ 

Define a sequence of nonnegative real numbers $\{c_m\}_m$ by

$$c_m = \begin{cases} 
0 & \text{if } b_m < \frac{1}{2^k} \\
 b_m & \text{if } b_m \geq \frac{1}{2^k}
\end{cases}$$ 

whenever $n_{k-1} < m \leq n_k$, for $k = 1, 2, \cdots$, and $n_0 = 0$. Define another sequence $\{d_m\}_m$ by

$$d_m = \begin{cases} 
1 & \text{if } c_m = 0 \\
\frac{1}{c_m} & \text{if } c_m \neq 0.
\end{cases}$$ 

Note that for $m \leq n_k$, $d_m \leq 2^k$. Let $C \in \mathcal{B}(H)$ be the diagonal operator with diagonal $\{c_m\}_m$ relative to $\{e_m\}_m$, and let $D$ be the unbounded densely defined diagonal operator with diagonal $\{d_m\}_m$ relative to $\{e_m\}_m$. Clearly $|B| - C$ is a compact operator.

Furthermore $KD$ is a compact operator: in particular, the sequence $\{P_nKDP_n\}$ is uniformly Cauchy. For, assuming $k > i,$
\[ \| P_{nK} KDP_n - P_{nK} KDP_n \| \leq \sum_{j=i+1}^{k} \| P_{nK} KDP_n - P_{nK} KDP_{n-1} \| \]
\[ \leq \sum_{j=i+1}^{k} \| P_n (K - P_{n-1} KDP_{n-1}) DP_n \| \]
\[ \leq \sum_{j=i+1}^{k} \| K - P_{n-1} KDP_{n-1} \| \| DP_n \| \]
\[ \leq \sum_{j=i+1}^{k} \frac{1}{2^{(j-1)} 2^j} = \sum_{j=i+1}^{k} \frac{1}{2^{j-2}} < \frac{1}{2^{i-2}}. \]

Since \( \{P_n\} \) converges strongly to \( I \), then \( \{P_n KDP_n\} \) converges uniformly to \( KD \).

Let \( E \) be the projection whose range is \( \text{span}\{e_n: c_n \neq 0\} \). Thus \( C = |B| E \) and \( DC = E \).

To finish the proof, set \( A' = A + KDU^* \), \( B' = UC \). Then
\[ A'B' = (A + KDU^*)(UC) = AUC + KDC = AU|B|E + KE = (AB + K)E \]
and we are done.

Using Theorem 2, it is possible to reduce the norm of a compact product by perturbing any one factor. However it may be necessary to perturb every factor to get a zero product. For example, let \( C \) be any one-to-one compact operator, and let \( A = \begin{pmatrix} I & 0 \\ 0 & C \end{pmatrix} \), \( B = \begin{pmatrix} C & 0 \\ 0 & I \end{pmatrix} \), \( AB = \begin{pmatrix} C & 0 \\ 0 & C \end{pmatrix} \), and let \( \begin{pmatrix} K & L \\ M & N \end{pmatrix} \) be any compact operator. Then
\[ \begin{pmatrix} A + \begin{pmatrix} K & L \\ M & N \end{pmatrix} \end{pmatrix} = \begin{pmatrix} C + KC & L \\ MC & N + C \end{pmatrix} \]
which equals zero only if \( C(I + K) = 0 \), an impossibility. Thus the next theorem is the best possible general result.

**Theorem 6.** Let \( T_1, \ldots, T_n \) be in \( B(H) \) such that \( \Pi T_j \) is compact. Then there are compact perturbations \( S_1, \ldots, S_n \) of \( T_1, \ldots, T_n \) with \( \Pi S_j = 0 \).

**Proof.** The product \( T_1(\Pi_{j=2}^n T_j) = C \), a compact operator.

By the previous proposition, there are compact \( K_1, L_1 \) and a projection \( E_1 \) with
\[ (T_1 + K_1) \left( \prod_{j=2}^n T_j + L_1 \right) = \left[ T_1 \left( \prod_{j=2}^n T_j \right) - C \right] E_1 = 0. \]
Now apply the proposition to $T_2(\Pi_{j=3}^n T_j) + L_1$ to get compact $K_2$ and $L_2$ and a projection $E_2$ with
\[
(T_2 + K_2) \left( \prod_{j=3}^n T_j + L_2 \right) = \left[ T_2 \left( \prod_{j=3}^n T_j \right) + L_1 \right] E_2.
\]
Thus
\[
(T_1 + K_1) (T_2 + K_2) \left( \prod_{j=3}^n T_j + L_2 \right) = \left( \prod_{j=1}^n T_j - C \right) E_1 E_2 = 0.
\]
Repeated applications of the proposition yield:
\[
\prod_{j=1}^{n-2} (T_j + K_j) (T_n-1 T_n + L_{n-2}) = \left( \prod_{j=1}^n T_j - C \right) \prod_{j=1}^{n-2} E_i = 0.
\]
And, a final application gives compact $K_{n-1}$ and $L_{n-1}$, and a projection $E_{n-1}$ with
\[
(T_{n-1} + K_{n-1}) (T_n + L_{n-1}) = (T_{n-1} T_n + L_{n-2}) E_{n-1}
\]
so that for $K_n = L_{n-1}$, we have
\[
\prod_{j=1}^n (T_j + K_j) = \prod_{j=1}^{n-2} (T_j + K_j) (T_n-1 T_n + L_{n-2}) E_{n-1} = 0,
\]
and the theorem is proved.

2. **Attaining the essential norm for polynomials in an operator.** In this section we first show that any bounded operator can be perturbed to attain $\|T\|_e$, $\|T^2\|_e$, or $\|T^3\|_e$; in most cases all three norms are achieved by a single compact perturbation of $T$. We then consider special classes of operators, weighted shifts and $n$-normal operators, for which stronger results are obtained. The first theorem follows by repeated applications of Theorem 3.

**Theorem 7.** Any $T$ in $\mathcal{B}(\mathcal{H})$ with $T^3$ not compact has a compact perturbation $S$ with $\|S\| = \|T\|_e$, $\|S^2\| = \|T^2\|_e$ and $\|S^3\| = \|T^3\|_e$.

**Proof.** Using Proposition 4, we may assume that $|T|$ is diagonal, where $U |T|$ is the polar decomposition for $T$. Assume also $\|T\| \leq 1$.

Let $\{\lambda_n\}_{n}$ be the sequence of diagonal entries of $|T|$ such that $\lambda_n > \|T\|_e = \|T\|_e$, then $\lim \lambda_n = \|T\|_e$. Obtain a compact perturbation $T_1$ of $T$ by replacing each $\lambda_n$ with $\|T\|_e$ to get $|T_1|$ from $|T|$, and then setting $T_1 = U |T_1|$. Clearly $\|T_1\| = \|T\|_e$. 

Now apply Theorem 3 to the product $T_1' = U | T_1 | T_1$, to get a compact perturbation $| T_1 |'$ of $| T_1 |$ by reducing some of the eigenvalues of $| T_1 |$, such that

$$\| U | T_1 |' T_1 \| = \| U | T_1 | T_1 \| = \| T_1 \|.$$  

Since reducing the eigenvalues in a diagonal first or last factor does not raise the norm of a product, we have

$$\| U | T_1 |' U | T_1 |' \| \leq \| U | T_1 |' U | T_1 | \| = \| T_1 \|.$$  

So let $T_2 = U | T_1 |'$ (then $| T_2 | = | T_1 |'$).

Finally, apply Theorem 3 to the product $| T_2 | U | T_2 | T_2$ to get a compact perturbation $| T_2 |'$ of $| T_2 |$ by reducing some of the eigenvalues of $| T_2 |$, such that

$$\| T_2 | U | T_2 |' T_2 \| = \| T_2 | U | T_2 | T_2 \| = \| U | T_2 | U | T_2 | T_2 \| = \| T_2 \|,$$

since $U^* U | T_2 | = | T_2 |$. Thus

$$\| (U | T_2 |')^3 \| = \| T_2 | U | T_2 |' U | T_2 |' \| \leq \| T_2 | U | T_2 | T_2 \| = \| T_2 \|.$$  

Now let $S = U | T_2 |'$ (so $| S | = | T_2 |'$). Then $\| S^3 \| = \| T_2 \|$, but also $\| S^2 \| = \| T_2 \|$, for,

$$\| S^2 \| = \| T_2 | U | T_2 |' \| \leq \| | T_2 | U | T_2 | \| = \| T_2 \| = \| T_2 \|.$$  

Similarly $\| S \| = \| T \|$, so the proof is complete.

**Remark 8.** One can see from this proof, that this approach does not extend to higher powers of $T$. The difficulty in simultaneously getting identical perturbations of two inside factors of $T^4$, in order to reduce the norm of $T^4$, seems to be beyond these techniques.

We have been unable to get the result in Theorem 7 only for the case where $T^3$ is compact and $T^2$ is not compact. The complication lies in finding a compact perturbation $S$ with both $S^3 = 0$ and $\| S \| = \| T \|$. On the one hand, this is a fairly special case, reducing to a $3 \times 3$ upper triangular operator matrix. On the other hand, it points up a general limitation involved in trying to combine the totally unrelated methods for perturbing compact and noncompact products. Our results are summarized in the following:

**Theorem 9.** Let $T$ be any operator in $\mathcal{B} (\mathcal{H})$. Then

(i) there is a compact perturbation $S$ with $\| S \| = \| T \|$ and $\| S^2 \| = \| T^2 \|$.  

(ii) there is a compact perturbation $R$ with $\|R^2\| = \|T^2\|$, and $\|R^3\| = \|T^3\|$. 

(iii) if $T^2$ is compact or $T^3$ is not compact we can choose $S = R$.

Proof of (i). If $T^2$ is not compact we can argue as in the beginning of the previous proof. If $T^2$ is compact, then using Theorem 2.4 of [6], we get a compact perturbation $T_1$ of $T$ with $T^2_1 = 0$. Then $T_1$ is equivalent to an operator matrix $T_1 = \begin{pmatrix} 0 & A \\ 0 & 0 \end{pmatrix}$, on a Hilbert space $\mathcal{H} \oplus \mathcal{H}$, with $\|T_1\| = \|A\|$. Let $A'$ be a compact perturbation of $A$ with $\|A'\| = \|A\|$, then $S = \begin{pmatrix} 0 & A' \\ 0 & 0 \end{pmatrix}$ satisfies (i).

Proof of (ii). If $T^2$ is compact, (i) applies. If $T^3$ is not compact, use the preceding theorem. Otherwise, let $T_1$ be a compact perturbation of $T$ with $T^3_1 = 0$ [6], so $T_1$ is equivalent to

$$T_1 = \begin{pmatrix} 0 & A & B \\ 0 & 0 & C \\ 0 & 0 & 0 \end{pmatrix}, \quad \text{with} \quad T^2_1 = \begin{pmatrix} 0 & 0 & AC \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

where $\|T^2_1\| = \|AC\|$. Apply Theorem 3 to $AC$ to get $\|A'C\| = \|AC\|$, and set

$$S = \begin{pmatrix} 0 & A' & B \\ 0 & 0 & C \\ 0 & 0 & 0 \end{pmatrix}.$$

Proof of (iii). By (i) and the previous theorem.

We remark that the full strength of Theorem 3 (and hence of Theorem 1) is not required for part (i) of Theorem 9. In particular, Theorem 1 can be proved much more easily if the factor $A = I$; and (i) follows readily from this case.

An operator $T \in \mathcal{B}(\mathcal{H})$ is a weighted shift of multiplicity $k$ if there is an orthonormal basis $\{e_n\}_n$ for $\mathcal{H}$ on which $T$ is defined by $Te_n = a_n e_{n+k}$, $n = 1, 2, \cdots$, where $\{a_n\}_n$ is a sequence of complex numbers. In order to prove our theorem for weighted shifts we need the following elementary lemma.

Lemma 10. Assume $\alpha > \beta > 0$, and let $a_1, \cdots, a_n$ be an $n$-tuple of positive numbers with $\Pi a_i = \alpha$. Then there is an $n$-tuple of positive numbers $b_1, \cdots, b_n$ with $\Pi b_i = \beta$; $b_i \leq a_i$ all i; and $\max (a_i - b_i) \leq \alpha^{1/n} - \beta^{1/n}$.
Proof. If there is some \(i\) with \(a_i < \alpha^{1/n} - \beta^{1/n}\) the result is trivial. So assume \(a_i \geq \alpha^{1/n} - \beta^{1/n}\) all \(i\); then \(a_i \leq \alpha/(\alpha^{1/n} - \beta^{1/n})^{n-1}\).

Now, define \((c_1, \cdots, c_n)\) by \(c_i = a_i - (\alpha^{1/n} - \beta^{1/n})\). To finish the proof, it suffices to show that \(\Pi c_i \leq \beta\). For, by the continuity of the product, we can then find \((b_1, \cdots, b_n)\) with \(c_i \leq b_i \leq a_i\) all \(i\), and \(\Pi b_i = \beta\).

Set \(\gamma = \alpha^{1/n} - \beta^{1/n}\) and consider the function \(f(a_1, \cdots, a_n) = \Pi c_i = \Pi(a_i - \gamma)\) defined on the compact set \(X\) of \(\mathbb{R}^n\) where \(\gamma \leq a_i \leq \alpha/\gamma^{n-1}\) and where \(\Pi a_i = a\). Then \(f\) has a maximum value \(M\) on \(X\): suppose it occurs at \((a_1, \cdots, a_n)\) with \(a_i > a_2\). Consider \((\sqrt{a_1a_2}, \sqrt{a_1a_2}, a_3, \cdots, a_n) \in X\). Note that \(\sqrt{a_1a_2} < \frac{1}{2}(a_1 + a_2)\). Thus

\[
f(\sqrt{a_1a_2}, \sqrt{a_1a_2}, a_3, \cdots, a_n) = (\sqrt{a_1a_2} - \gamma)^n \prod_{3}^{n} (a_i - \gamma) \]

\[
\geq (a_1a_2 - \gamma(a_1 + a_2) + \gamma^2) \prod_{3}^{n} (a_i - \gamma) \]

\[
= \Pi(a_i - \gamma) = M,
\]

a contradiction. Hence \(a_1 = a_2\), and by symmetry, \(f\) takes its maximum at \((\alpha^{1/n}, \cdots, \alpha^{1/n})\), so \(M = \beta\). The lemma is proved.

Theorem 11. Let \(T\) be a weighted shift. Then there is a compact perturbation \(S\) of \(T\) with \(\|S^n\| = \|T^n\|_e\) for all \(n = 1, 2, \cdots\).

Proof. Let \(T\) be a shift with weight sequence \((a_i)_i\). We give the proof for a shift of multiplicity 1: in this case,

\[
\|T^n\| = \sup_i |a_ia_{i+1} \cdots a_{i+n-1}|.
\]

The proof for \(T\) of multiplicity \(k\) is similar, where

\[
\|T^n\| = \sup_j |a_ia_{j+k} \cdots a_{j+(n-1)k}|.
\]

A straightforward computation allows us to assume \(\|T\| \leq 1\). Let \(\mu_n = \|T^n\|_n\), \(\nu_n = \|T^n\|_e\), \(n = 1, 2, \cdots\). We will define by induction a sequence \(\{S_n\}_n\) of weighted shifts, each obtained by reducing the moduli of the weights of the preceding, and which converges to the desired perturbation \(S\) of \(T\).

Let \(S_1\) be the shift with weights \(\{a_{1j}\}_j\), where
Then $\|S_1\| = \nu_1 = \|T\|_e$ and $|a_{ij} - a_i| \leq |a_i| \leq \nu_1$. Assume for induction, that $S_1, S_2, \cdots, S_{n-1}$ have been constructed so that for each $k = 1, 2, \cdots, n-1$, and for each $j \leq k$:

(i) $T - S_k$ is compact;

(ii) $\|S_k\| = \nu_k$;

(iii) $\|S_j - S_k\| \leq \max \{\mu_j^{1/j} - \nu_j^{1/j}, \cdots, \mu_k^{1/k} - \nu_k^{1/k}\}$;

(iv) if $S_j$ and $S_k$ have weights $\{a_i\}$ and $\{a_{ki}\}$ resp., then $|a_j| \geq |a_{ki}|$, each $i$.

Construct $S_n$ as follows: note that $\|T^n\| = \sup_j |a_ja_{j+1} \cdots a_{j+n-1}| = \mu_n$. Let $\Lambda$ be the set of $j$ with $|a_j \cdots a_{j+n-1}| > \nu_n$. Define $\gamma_j = |a_j \cdots a_{j+n-1}|$, for $j \in \Lambda$. Then

$$\lim_{j} \gamma_j = \|T^n\|_e = \nu_n.$$ 

Applying the preceding Lemma, we see that for each $j \in \Lambda$, there is an $n$-tuple $(b_{j,1}, b_{j+1,1}, \cdots, b_{j+n-1,1})$ satisfying:

1. $|b_j| \leq |a_j|$, $|b_{i+1,j}| \leq |a_{i+1,j}|$, $i = 1, 2, \cdots, n-1$;

2. $\max \{|a_i - b_j|, \cdots, |a_{i+n-1} - b_{i+j}|\} \leq \gamma_j^{1/n} - \nu_n^{1/n}$.

Choose $c_j$ to be one among $a_j$ and those of $b_j, b_{j+1}, \cdots, b_{j+n-1}$ which are defined, having a minimum modulus (note that since $\gamma_j$ is only defined for $j \in \Lambda$, some of the $b_j, b_{j+1}$ may not be defined). Let $T_n$ be the shift with weights $\{c_j\}$.

Note that $T - T_n$ is compact, since either $a_j = c_j$ or

$$|a_j - c_j| \leq \max_k \{\gamma_k^{1/n} - \nu_n^{1/n}: k \in \Lambda \text{ with } k = j - n + 1, \cdots, j\},$$

where $\lim_j \gamma_j = \nu_n$. This inequality also shows that $\|T - T_n\| \leq \mu_n^{1/n} - \nu_n^{1/n}$, since $\mu_n = \sup \gamma_k$. Also, $\|T^n\| = \nu_n$.

Define $S_n$ to be the shift with weights $\{a_n\}$, where $a_n$ is the one of $a_{n-1,i}$ and $c_j$ having minimum modulus.

Clearly $T - S_n$ is compact; $|a_n| \leq |a_{n-1,j}|$ all $i = 1, 2, \cdots$; and $\|S_n\| = \nu_n$. Also, we see that

$$\|S_j - S_n\| \leq \max \{\mu_j^{1/j} - \nu_n^{1/n}, \cdots, \mu_n^{1/n} - \nu_n^{1/n}\},$$

by comparing the $i$th weights of these operators: since $\|T - S_n\| \leq \mu_n^{1/n} - \nu_n^{1/n}$, since $|a_n| \leq |a_{n-1,i}|$ all $i$, and by induction hypothesis (iii).
So, the sequence \( \{S_n\}_n \) is constructed, and we will now see that it converges uniformly to some bounded operator \( S \). The spectrum of any shift is circularly symmetric about the origin [5, p. 43]. Thus \( \partial \sigma(T) \) consists of one or more circles. Now \( \sigma(T) \) contains the spectrum of \( \pi(T) \) in the Calkin algebra. By a theorem of C. Putnam [9], \( \partial \sigma(T) \subset \sigma(\pi(T)) \cup \{\text{isolated eigenvalues of } T \text{ of finite multiplicity} \} \). Thus we conclude that \( \sigma(T) \) and \( \sigma(\pi(T)) \) have the same radius. Thus

\[
\lim \| T^n \|^{|1/n|} = \lim \| \pi(T)^n \|^{|1/n|} = \lim \| T^n \|^{|1/n|},
\]

so \( \lim \mu_n^{|1/n|} - \nu_n^{|1/n|} = 0 \). Hence property (iii) implies that \( \{S_n\}_n \) is uniformly Cauchy; so set \( \lim S_n = S \). Then \( T - S_n \) converges to a compact operator, \( T - S \). From the construction of \( \{S_n\}_n \) in particular property (iv), it is clear that \( S \) is a shift whose \( j \)th weight has modulus \( \leq \) the modulus of the \( j \)th weight of each \( S_n \). Thus, \( \|S^n\| \leq \|S_n^n\| = \nu_n \), each \( n = 1, 2, \ldots \), and the result is proved.

The best possible result is attainable for operators which are direct sums of matrices of bounded degree.

**Theorem 12.** Let \( T = \sum_{k=1}^\infty T_k \), a direct sum of \( m \times m \) matrices. Then there is a compact perturbation \( S \) of \( T \) such that \( \|p(S)\| = \|p(T)\|_{\text{Calkin}} \) for every complex polynomial \( p \).

**Proof.** Consider each \( T_k \) as an element of \( \mathbb{C}^{m^2} \). Since \( T \) is a bounded operator, the set \( \{T_k\}_k \) is a bounded set in \( \mathbb{C}^{m^2} \), so that the set \( X \subset \mathbb{C}^{m^2} \) of accumulation points of \( \{T_k\}_k \) is a compact set. We include in \( X \) any \( T_k \) which are repeated infinitely many times. Then \( \{T_k\} \setminus X \) has no accumulation points, so that if

\[
d_k = \text{distance}(T_k, X),
\]

then \( \lim d_k = 0 \). For each \( T_k \) choose some \( S_k \in X \) with \( \|T_k - S_k\| = d_k \) (since all topologies on \( \mathbb{C}^{m^2} \) are equivalent, we simply use the operator norm).

Let \( S = \sum_{k=1}^\infty \oplus S_k \). Clearly \( S \) is a compact perturbation of \( T \). Furthermore, every element of the set \( \{S_k\}_k \subset \mathbb{C}^{m^2} \) is an accumulation point of that set or occurs infinitely often, and thus for any complex polynomial \( p \), the same is true for the set \( \{p(S_k)\}_k \). Therefore

\[
\|p(S)\| = \left\| \sum_k \oplus p(S_k) \right\| = \sup_k \|p(S_k)\| = \limsup_k \|p(S_k)\| = \limsup_k \|p(T_k)\|.
\]
Let \( E_n = \sum_{k=1}^n \oplus I_k \). Now, any compact operator \( K \) satisfies
\[
\lim \| (I - E_n) K (I - E_n) \| = 0.
\]

Thus
\[
\| p(T) + K \| \geq \lim sup \| (I - E_n) (p(T) + K) (I - E_n) \|
\]
\[
= \lim sup \| (I - E_n) p(T) (I - E_n) \|
\]
\[
= \lim sup \| p(T_k) \|
\]
\[
= \| p(S) \|
\]

so \( \| p(S) \| = \| p(T) \| \).

**Corollary 13.** If \( T \in \mathcal{B}(\mathcal{H}) \) is a nilpotent weighted shift, there is a compact perturbation \( S \) with \( \| p(S) \| = \| p(T) \| \), for every complex polynomial \( p \).

**Proof.** Any such \( T \) satisfies the hypotheses of Theorem 12.

**Corollary 14.** Let \( T \) be an \( n \)-normal operator. Then there is a compact perturbation \( S \) such that \( \| p(S) \| = \| p(T) \| \), for every complex polynomial \( p \).

**Proof.** The operator \( T \) may be regarded as an \( n \times n \) operator matrix whose entries are commuting normal operators \( \{ T_j \} \) on a Hilbert space \( \mathcal{E} \). It follows by a theorem of L. G. Brown, R. G. Douglas, and P. A. Fillmore [2, Corollary 5.4, p. 83] that there is an orthonormal basis of \( \mathcal{E} \) with each \( T_j = D_j + K_j \) where \( D_j \) is diagonal relative to this basis, for every \( j = 1, 2, \cdots, n^2 \), and where \( K_j \) is compact. Let \( K \) be the \( n \times n \) operator matrix whose entries are the \( K_j, j = 1, \cdots, n^2 \). Then \( S = T - K \) is an \( n \times n \) operator matrix with simultaneously diagonal entries \( D_j \), so that \( S \) is unitarily equivalent to an infinite direct sum of \( n \times n \) matrices, and the previous theorem applies.

**References**


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