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**ABSOLUTELY DIVERGENT SERIES AND ISOMORPHISM OF  
SUBSPACES. II**

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The following relations between a Banach space  $E$  and a Banach space  $X$  are, roughly speaking, generalizations of the relation " $E$  is a closed subspace of  $X$ ."

(LIX) The finite dimensional subspaces of  $E$  are uniformly isomorphic to subspaces of  $X$  under isomorphisms which extend to all of  $E$  without increase of norm.

(SpX) Finite rank mappings from any Banach space into  $E$  can be uniformly factored through subspaces of  $X$ .

(ASX) The continuous linear mappings from  $E$  into  $X$  distinguish the absolutely summing mappings from any Banach space into  $E$ .

(SIX) For each absolutely divergent series  $\sum_n x_n$  in  $E$  there is a continuous linear mapping  $T$  from  $E$  into  $X$  such that  $\sum_n Tx_n$  diverges absolutely.

Our main result is that these four conditions are equivalent if  $X$  contains a subspace isomorphic to  $\lambda[X]$  where  $\lambda$  is a normal  $BK$ -space. A related result of some interest is that the class of continuous linear mappings which factor through spaces which contain a complemented copy of  $\lambda[X]$  forms a Banach operator ideal.

The consideration of the above relations continues the theme begun in [2] and [7]. A similar result to our main result is proven in [7] under a different assumption on the space  $X$  — an isometric assumption. We do not know whether the hypothesis on  $X$  in the present paper is strictly weaker than that in the previous paper. But in this case it is an isomorphic assumption and easier to verify. For example, it is satisfied by any space with a symmetric basis.

**1. Some prerequisites.** A. *Sequence spaces.* The space of all sequences of scalars  $(s_i)$  (real or complex) with the product topology is denoted by  $\omega$ . The subspace of  $\omega$  which contains all sequences which are eventually 0 is denoted by  $\varphi$ . A Banach space  $\lambda$  of sequences is called a  $BK$ -space if the inclusion from  $\lambda$  into  $\omega$  is continuous. A space of sequences  $\lambda$  is called normal if whenever  $(s_i)$  is in  $\lambda$  and  $(t_i)$  is in  $m$ , the  $BK$ -space of bounded sequences it follows that  $(t_i s_i)$  is also in  $\lambda$ . It is known that if  $\lambda$  is a  $BK$  space there is an equivalent norm  $\| \cdot \|$  on  $\lambda$  for which

$$\|(t, s_i)\| \leq \sup_i |t_i| \|(s_i)\|$$

for  $(t_i)$  in  $m$ .

If  $\lambda$  is any set of sequences  $\lambda^\alpha$  consists of all sequences  $(t_i)$  such that  $(t, s_i)$  is in  $1$  for each  $(s_i)$  in  $\lambda$ . Here  $1$  denotes the *BK*-space of all sequences  $(u_i)$  such that  $\sum_i |u_i| < \infty$ . If  $\lambda$  is a normal *BK*-space in which  $\varphi$  is dense then  $\lambda^\alpha$  is isomorphic to the topological dual space of  $\lambda$  by means of the correspondence of  $f$  in  $\lambda'$  to  $(f(e_n))$  in  $\lambda^\alpha$ . Here  $e_n$  denotes the sequence with  $1$  in the  $n$ th place and  $0$ 's elsewhere. The closed unit ball  $U_{\lambda^\alpha}$  of  $\lambda^\alpha$  for  $\lambda$  a normal *BK* space is equal to the set of all  $(t_i)$  in  $\omega$  such that  $|\sum_i t_i s_i| \leq 1$  for each  $(s_i)$  in  $\varphi$  with  $\|(s_i)\|_\lambda \leq 1$ . Therefore,  $U_{\lambda^\alpha}$  is compact in  $\omega$ . For  $\lambda$  a normal *BK* space and  $X$  any Banach space  $\lambda[X]$  denotes the space of all sequences  $(x_n)$  in  $X$  such that  $(\|x_n\|_X)$  is in  $\lambda$ . With the norm

$$\|(x_n)\|_\lambda = \|(\|x_n\|_X)\|_\lambda$$

$\lambda[X]$  is known to be a Banach space. The closed linear span of  $\varphi$  in  $m$  consists of all sequences which converge to  $0$  and is denoted by  $c_0$ . References: [5], [6], [7].

**B. Operator ideals.** Let  $L$  denote the class of all continuous linear mappings between Banach spaces. For two Banach spaces  $E$  and  $F$  let  $L(E, F)$  denote the space of all continuous linear mappings between  $E$  and  $F$ . A subclass  $\mathcal{A}$  of  $L$  is called an *operator ideal* if it is closed under sums and by multiplication on the left and right by members of  $L$  where multiplication and addition is restricted to pairs of operators for which these operations are meaningful. An operator ideal  $\mathcal{A}$  is called a *Banach operator ideal* if there is a nonnegative correspondence  $\alpha$  defined on  $\mathcal{A}$  such that

(1) For every pair  $E, F$ ,  $\mathcal{A}(E, F) = (\mathcal{A} \cap L(E, F))$  is a Banach space with norm  $\alpha$ .

(2)  $\alpha(ST) \leq \alpha(S) \|T\|$  if  $S \in \mathcal{A}(F, G)$ ,  $T \in L(E, F)$ .

(3)  $\alpha(ST) \leq \|S\| \alpha(T)$  if  $S \in L(F, G)$ ,  $T \in \mathcal{A}(E, F)$ .

Here  $\| \cdot \|$  denotes the uniform operator. With the uniform operator topology,  $L$  is a Banach operator ideal. Let  $U(E, F)$  denote the unit ball of  $L(E, F)$  with the uniform norm.

The class  $\mathcal{F}$  of finite rank mappings between Banach spaces is an operator ideal. Every finite rank mapping  $T$  from  $E$  to  $F$  has a nonunique representation

$$Tx = \sum_{i=1}^n x'_i(x) y_i$$

with each  $x'$  in  $E'$  the topological dual space of  $E$  and each  $y_i$  in  $F$ . If  $E = F$  the number

$$\text{tr}(T) = \sum_{i=1}^n x'_i(y_i)$$

does not depend on the representation of  $T$ ; it is called the trace of  $T$ .

In this paper we shall refer to the following three operator ideals:

(1) The class  $\text{int}$  of all integral mappings between Banach spaces. A mapping  $T$  in  $L(E, F)$  is integral if there is  $K > 0$  such that

$$|\text{tr}(ST)| \leq K \|S\| \quad S \in \mathcal{F}(F, E).$$

The norm on  $L(E, F)$  is given by

$$\|T\| = \sup\{\text{tr}(ST): S \in \mathcal{F}(F, E), \|S\| \leq 1\}.$$

(2) The class  $N$  of nuclear mappings. A mapping  $T$  in  $L(E, F)$  is nuclear if  $T = \sum_{i=1}^{\infty} T_i$  has rank one and  $\sum_{i=1}^{\infty} \|T_i\| < \infty$ . The norm on  $N(E, F)$  is given by

$$\|T\|_N = \inf\left\{ \sum_{i=1}^{\infty} \|T_i\|: \sum_{i=1}^{\infty} T_i = T, \text{ each } T_i \text{ has rank one} \right\}.$$

(3) The class  $AS$  of absolutely summing mappings. A mapping  $T$  in  $L(E, F)$  is absolutely summing if

$$\sum_n \|Tx_n\|$$

whenever

$$\sum_i |x'(x_n)| < \infty \text{ for all } x' \text{ in } E'.$$

The norm on  $AS(E, F)$  is given by

$$\|T\|_{AS} = \sup\left\{ \sum_n \|Tx_n\|: \sum_n |x'(x_n)| \leq 1 \quad \forall x' \in U_{E'} \right\}.$$

Here  $U_{E'}$  denotes the unit ball in  $E'$ .

References: [1], [4], [8].

**2. Mappings which factor through  $X$ .** For  $X$  a Banach space let  $\langle X \rangle$  denote the class of all continuous linear mappings which factor through  $X$ . That is, a mapping  $T$  in  $L(E, F)$  is in  $\langle X \rangle$  if  $T = T_1 T_2$

where  $T_1$  is in  $L(X, F)$  and  $T_2$  is in  $L(E, X)$ . One can show that  $\langle X \rangle$  is an operator ideal if and only if  $X \times X$  is isomorphic to a complemented subspace of  $X$ .

For  $X$  a Banach space and  $(t_n)$  a sequence of scalars the sequence  $(t_n x_n)$  is in  $l[X]$  for all  $(x_n)$  in  $m[X]$  if and only if  $(t_n)$  is in  $l$ . We denote by  $\text{diag}(X)$  the collection of all such scalar diagonal mappings from  $m[X]$  to  $l[X]$ .

2.1. PROPOSITION. *The smallest Banach operator ideal which contains  $\langle X \rangle$  is equal to the class of all  $T$  in  $L$  which have as a factor a mapping from  $\text{diag}(X)$ . In other words,  $T$  in  $L(E, F)$  is in this ideal if and only if*

$$T = T_1 \Delta T_2$$

where  $T_2$  is in  $L(E, m[X])$ ,  $\Delta$  is in  $\text{diag}(X)$  and  $T_1$  is in  $L(l[X], F)$ .

*Proof.* Let  $\{X\}$  denote the smallest Banach operator ideal which contains  $\langle X \rangle$ . We first show that  $T$  in  $L(E, F)$  is in  $\{X\}$  if and only if

$$(2.1) \quad T = \sum_n S_n V_n \quad \sum_n \|S_n\| \|V_n\| < \infty$$

where each  $S_n$  is in  $L(X, F)$  and each  $V_n$  is in  $L(E, X)$ . It is a routine task to verify that the class  $[X]$  of all such mappings does form a Banach operator ideal with the norm

$$\|T\| = \inf \left\{ \sum_n \|S_n\| \|V_n\| : \sum_n S_n V_n = T \right\},$$

and it is clear that  $[X]$  contains  $\langle X \rangle$  and hence  $\{X\}$ .

On the other hand, for each  $S$  in  $L(X, F)$  the correspondence  $V$  to  $SV$  is a continuous linear mapping from  $L(E, X)$  into  $\{X\}(E, F)$  so that

$$\sup \{ \|SV\|_{\{X\}} : V \in U(E, X) \} < \infty$$

by the Uniform Boundedness Principle. A second application of this principle shows that the set

$$\{SV : V \in U(E, X), S \in U(X, F)\}$$

is bounded in  $\{X\}$ . Thus if

$$\sum_n \|S_n\| \|V_n\| < \infty, \quad S_n \in L(X, F), \quad V_n \in L(E, X)$$

it follows that

$$\sum_n \|S_n V_n\|_{\{X\}} < \infty.$$

Therefore, every  $T$  of the form (2.1) is in  $\{X\}$ .

If  $T$  has the form (3.1) let  $t_n = \|S_n\| \|V_n\|$  for each  $n$ . Define  $T_2$  in  $L(E, m[X])$  by

$$T_2 x = (V_n(x) / \|V_n\|),$$

$\Delta$  from  $m[X]$  into  $l(X)$  by

$$\Delta(u_n) = (t_n u_n)$$

and  $T_1$  from  $l[X]$  into  $F$  by

$$T_1(u_n) = \sum_n S_n u_n / \|S_n\|.$$

Then  $T = T_1 \Delta T_2$  where  $\Delta$  is in  $\text{diag}(X)$ . On the other hand we can verify that every mapping  $T$  of the form  $T = T_1 \Delta T_2$  has the form (2.1) by a routine inversion of the above argument.

The following theorem is proven in [3].

2.2. THEOREM. *Let  $\lambda$  be a normal BK-space containing  $\varphi$ . For each sequence  $(r_n)$  in  $l^1$  we can find sequences  $(s_n)$  in  $\lambda^\circ$  (the closure of  $\varphi$  in  $S$ ) and  $(t_n)$  in  $\lambda^{\alpha^\circ}$  such that  $s_n t_n = r_n$  for all  $n$ .*

2.3. THEOREM. *If  $X$  contains a complemented subspace isomorphic to  $\lambda[X]$  for  $\lambda$  a normal BK-space containing  $\varphi$  then  $\langle X \rangle$  is a Banach operator ideal. ( $\langle X \rangle = \{X\}$ )*

*Proof.* Given  $T$  in  $\{X\}(E, F)$  we show that  $T$  factors through  $X$ . Since  $\lambda[X]$  is complemented in  $X$  it suffices to show that  $T$  factors through  $\lambda[X]$ . By Proposition 2.1 there are  $T_1$  in  $L(l^1[X], F)$ ,  $T_2$  in  $L(E, m[X])$  and  $(r_n)$  in  $l$  such that

$$Tx = \sum_n r_n T_1 T_2 x \quad x \in E.$$

We may assume that  $r_n \geq 0$  for all  $n$ . By Theorem 2.2 there is  $(s_n)$  in  $\lambda$  and  $(t_n)$  in  $\lambda^{\alpha^\circ}$  such that  $s_n t_n = r_n$  for all  $n$ . Define  $R_2$  from  $m[X]$  into  $\lambda[X]$  by

$$R_2(u_n) = (s_n u_n)$$

and  $R_1$  from  $\lambda[X]$  into  $l[X]$  by

$$R_1(v_n) = (t_n v_n).$$

Then  $T = T_1 R_1 R_2 T_3$  so  $T$  factors through  $\lambda[X]$ .

2.4. COROLLARY. *If  $\lambda$  is a symmetric BK space then  $\langle \lambda[X] \rangle$  is an ideal for every Banach space  $X$ .*

*Proof.* If  $\lambda$  is a symmetric BK space it is not hard to show that  $\lambda[\lambda[X]]$  is isomorphic to  $\lambda[X]$ .

The following fact is needed later.

2.5. PROPOSITION. *If  $\lambda$  is a normal BK-space then there is  $K > 0$  such that for each  $(t_n)$  in  $l$  we can find  $(u_n)$  in  $\lambda$ ,  $(v_n)$  in  $\lambda^\alpha$  with  $(u_n v_n) = (t_n)$  such that*

$$(2.2) \quad \|(u_n)\|_\lambda \|(v_n)\|_{\lambda^\alpha} \leq K \sum_n |t_n|.$$

*Proof.* Let  $U_1$  denote the closed unit ball in  $\lambda^{\alpha\alpha}$  and  $U_2$  the closed unit ball in  $\lambda^\alpha$ . Then both  $U_1$  and  $U_2$  are compact in  $\omega$  so  $U_1 U_2$  is compact in  $\omega$  and thus closed in  $l$ . Since  $\lambda^{\alpha\alpha} \lambda^\alpha \supset \lambda \lambda^\alpha = l$  it follows that  $\bigcup_{n=1}^\infty n U_1 U_2 = l$ . Using the Baire Category Theorem we can find  $r > 0$  such that  $rU \subset U_1 U_2$  where  $U$  denotes the unit ball of  $l$ .

Given  $(t_n)$  in  $l$  and  $\epsilon > 0$  let  $(r_n)$  in  $c_0$  and  $(s_n)$  in  $l$  be such that  $(r_n s_n) = (t_n)$  for each  $n$  and  $|r_n| \leq 1$  for all  $n$  and  $\sum_n |s_n| \leq \sum_n |t_n| + \epsilon$ . Let  $(u'_n)$  in  $\lambda^{\alpha\alpha}$  and  $(v_n)$  in  $\lambda^\alpha$  be such that

$$(u'_n v_n) = (s_n) \quad \|(u'_n)\| \|(v_n)\| \leq 1/r.$$

For each  $(w_n)$  in  $\varphi$ ,  $(w_n u'_n)$  is in  $\varphi$ . Since  $c_0$  is the closure of  $\varphi$  in  $m$ ,  $(r_n u'_n)$  is in the closure of  $\varphi$  in  $\lambda^{\alpha\alpha}$  so  $(r_n u'_n)$  is in  $\lambda$ . Since  $\lambda^{\alpha\alpha}$  is normal

$$\|(r_n u'_n)\|_{\lambda^{\alpha\alpha}} \leq C \|(u_n)\|_{\lambda^{\alpha\alpha}}$$

where  $C$  depends only on the norm on  $\lambda$ . Thus we have

$$(r_n u'_n v_n) = (r_n s_n) = (t_n)$$

and

$$\begin{aligned} \|(r_n u'_n)\|_\lambda \| (v_n)\|_{\lambda^\alpha} &\leq C \| (u_n)\|_{\lambda^\alpha} \| (v_n)\|_{\lambda^\alpha} \\ &\leq C/r \sum_n |s_n| \\ &\leq C/r \left( \sum_n |t_n| + \epsilon \right). \end{aligned}$$

Since this inequality holds for all  $\epsilon > 0$ , (2.2) holds with  $K = C/r$ .

**3. Local immersion and series immersion.**

3.1. DEFINITION. A normed space  $E$  is said to be *locally immersed* in a normed space  $X$  if the following condition holds:

(LIX) There is a number  $K > 0$  such that for each finite dimensional subspace  $G$  of  $E$  there is a continuous linear mapping  $T$  in  $U(E, X)$  such that

$$\|Tx\| \geq K \|x\| \quad x \in G.$$

3.2. PROPOSITION. *The following property (“splits through  $X$ ”) is equivalent to (LIX).*

(SpX) There is  $K \geq 1$  such that each finite rank mapping from a normed space  $D$  to  $E$  can be factored

$$V = V_1 V_2 V_3; \|V_1\| \|V_2\| \|V_3\| \leq K \|V\|$$

with  $V_3$  in  $L(D, E)$ ,  $V_2$  in  $L(E, Y)$  where  $Y$  is a closed subspace of  $X$  and  $V_1$ , in  $L(Y, E)$ .

*Proof.* (SpX)  $\Rightarrow$  (LIX). Let  $V$  denote the inclusion map from  $G$  into  $E$ , and let  $V_1, V_2, V_3$  satisfy (SpX). If  $T = V_2 V_3 / \|V_2 V_3\|$  then  $\|T\| = 1$ , and for each  $x$  in  $G$  we have

$$\begin{aligned} \|x\| &= \|V_1 V_2 V_3 x\| \leq \|V_1\| \|V_2 V_3 x\| \\ &\leq \|V_1\| \|V_2\| \|V_3\| \|Tx\| \leq K \|Tx\|. \end{aligned}$$

(LIX)  $\Rightarrow$  (SpX). Let  $G = V(D)$ ,  $V_3 = V$  and  $V_2 = T$  where  $T$  is given by (LIX). Let  $Y = T(G)$  and define  $V_3$  on  $Y$  by  $V_3 y = x$  if  $Tx = y$ . Then

$$V = V_1 V_2 V_3 \quad \text{and} \quad \|V_1\| \|V_2\| \|V_3\| \leq K \|V\|.$$



3.3. DEFINITION. A normed space  $E$  is said to be *series immersed* in a normed space  $X$  if the following statement holds:

(SIX) For each absolutely divergent series  $\sum_n x_n$  in  $E$  there is  $T$  in  $L(E, F)$  such that  $\sum_n Tx_n$  diverges absolutely.

We omit the proof of the following statement which is known [8].

3.4. LEMMA. For  $T$  a continuous linear mapping from  $c_0$  into a normed space  $E$  the following statements are equivalent:

- (a)  $T$  is nuclear;
- (b)  $T$  is integral;
- (c)  $T$  is absolutely summing;
- (d)  $\sum_n \|Te_n\| < \infty$ .

3.5. PROPOSITION. For  $E$  and  $X$  arbitrary Banach spaces the following conditions are equivalent to (SIX) and thus [7] implied by (LIX).

(ASX) For every Banach space  $D$  a mapping  $T$  in  $L(D, E)$  is absolutely summing if  $ST$  is absolutely summing for all  $S$  in  $L(E, X)$ .

(ASX)<sub>0</sub> The same statement as (a) with  $D = c_0$ .

*Proof.* (SIX)  $\Rightarrow$  (ASX). Suppose  $T$  in  $L(D, E)$  is not absolutely summing. Then there is a weakly absolutely summable series  $\sum_n x_n$  in  $D$  such that  $\sum_n \|Tx_n\| = \infty$ . By (SIX) there is  $S$  in  $L(E, X)$  such that  $\sum_n \|STx_n\| = \infty$  so that  $ST$  is not absolutely summing.

(ASX)  $\Rightarrow$  (ASX)<sub>0</sub>. Clear.

(ASX)<sub>0</sub>  $\Rightarrow$  (SIX). Suppose  $\sum_n x_n$  is a series in  $E$  with  $\sum_n \|x_n\| = \infty$ . If  $\sum_n x_n$  is not weakly absolutely summable it is easy to find  $T$  in  $L(E, X)$  such that  $\sum_n \|Tx_n\| = \infty$  where  $T$  has rank one. If  $\sum_n x_n$  is weakly absolutely summable define  $T$  from  $c_0$  into  $E$  by

$$T((t_n)) = \sum_n t_n x_n.$$

By 3.4,  $T$  is not absolutely summing because  $\sum_n \|Te_n\| = \sum_n \|x_n\| = \infty$  so by (ASX)<sub>0</sub> there is  $S$  in  $L(E, X)$  such that  $ST$  is not absolutely summing. Consequently by 3.4,

$$\sum_n \|STe_n\| = \sum_n \|Sx_n\| = \infty.$$

3.6. PROPOSITION. For Banach spaces  $E$  and  $X$  the following condition is implied by (LIX) and implies (SIX):

(int X) A mapping  $T$  from a Banach space  $D$  into  $E$  is integral if  $ST$  is integral for all  $S$  in  $L(E, Y)$  as  $Y$  ranges over the closed subspaces of  $X$ .

*Proof.* (LIX)  $\Rightarrow$  (int X). Suppose  $ST$  is integral for all  $S$  in  $L(E, Y)$ . We first show there is  $M > 0$  such that

$$(3.1) \quad \sup\{\| \| ST \| \| : S \in U(E, Y), Y \text{ is a closed subspace of } X\} \leq M.$$

Let  $\mathcal{Y}$  denote the set of all closed subspaces  $Y$  of  $X$ . Let  $Z_1(\mathcal{Y})$  denote the Banach space of all indexed families  $(S_Y)_{Y \in \mathcal{Y}}$  where  $S_Y$  is in  $L(E, Y)$  and  $\sup_Y \|S_Y\| = \|(S_Y)\| < \infty$ . Let  $Z_2(\mathcal{Y})$  denote the Banach space of all indexed families  $(V_Y)_{Y \in \mathcal{Y}}$  where  $V_Y$  is in  $\text{int}(D, Y)$  and  $\sup_Y \| \| V_Y \| \| = \| \| (V_Y) \| \| < \infty$ . The correspondence  $(S_Y)_{Y \in \mathcal{Y}} \rightarrow (S_Y T)_{Y \in \mathcal{Y}}$  determines a linear mapping from  $Z_1(\mathcal{Y})$  into  $Z_2(\mathcal{Y})$  which is continuous by the Closed Graph Theorem. There is thus  $M > 0$  such that

$$\| \| (S_Y T) \| \| \leq M \| \| (S_Y) \| \| \quad (S_Y) \in Z_1(\mathcal{Y})$$

which proves (3.1).

If  $S$  is a finite rank mapping in  $L(E, D)$  let  $V$  denote the inclusion from  $TS(E)$  into  $E$ . By (Sp  $X$ ),  $V = V_1 V_2 V_3$  with  $V_3$  in  $L(TS(E), E)$ ,  $T_2$  in  $L(E, Y)$  where  $Y$  is a subspace of  $X$ ,  $V_1$  is in  $L(Y, E)$  and  $\| \| V_1 \| \| \| V_2 \| \| \| V_3 \| \| \leq K$ . Thus we have

$$\begin{aligned} |\text{tr } TS| &= |\text{tr}(V_1 V_2 V_3 TS)| \\ &\leq \| \| V_1 \| \| \| V_2 V_3 T \| \| \| S \| \| \\ &\leq \| \| V_1 \| \| M \| \| V_2 \| \| \| V_3 \| \| \| S \| \| \leq MK \| S \| \end{aligned}$$

which shows  $T$  is integral.

(int X)  $\Rightarrow$  (SIX) by Lemma 3.4 and Proposition 3.5.

Notice the connection of the following statement with the results of §2.

3.7. PROPOSITION. *The normed space  $E$  is series immersed in the Banach space  $X$  if and only if the following condition holds:*

(diag X) *There is  $M > 0$  such that for each finite dimensional subspace  $F$  of  $E$  one can find a mapping  $R$  from  $E$  into  $m[X]$  and a mapping  $\Delta$  in  $\text{diag}(X)$  such that*

$$\begin{aligned} \| \Delta R x \| &\leq \| x \| \text{ for } x \text{ in } E \\ M \| \Delta R x \| &\geq \| x \| \text{ for } x \text{ in } F. \end{aligned}$$

*Proof.* (diag X)  $\Rightarrow$  (SIX). If  $\sum_n \|Tx_n\| < \infty$  for all  $T$  in  $L(E, X)$  then there is  $K > 0$  such that  $\sum_n \|Tx_n\| \leq K \|T\|$  for each  $T$  in  $L(E, X)$  by the Uniform Boundedness Principle. For  $k$  a fixed positive integer let  $R$  and  $\Delta$  satisfy (diag X) for the finite dimensional subspace spanned by  $\{x_1, x_2, \dots, x_k\}$ . There is a bounded sequence  $(T_i)$  in  $L(E, X)$ , such that  $Rx = (T_i x)$  and a sequence  $t_i$  ( $\geq 0$ ) in  $l$  such that  $\Delta(y_i) = (t_i y_i)$ . Since  $\|\Delta Rx\| \leq \|x\|$  for  $x$  in  $E$  it follows that  $\sum_i t_i \|T_i\| \leq 1$ . Thus we have

$$\begin{aligned} \sum_{n=1}^k \|x_n\| &\leq M \sum_{n=1}^k \|\Delta Rx_n\| = M \sum_{n=1}^k \sum_i t_i \|T_i x_n\| \\ &\leq M \sum_i \sum_{n=1}^k t_i \|T_i x_n\| \leq MK \sum_i t_i \|T_i\| \\ &\leq MK. \end{aligned}$$

Since  $M$  and  $K$  are independent of  $k$ ,  $\sum_n \|x_n\| < \infty$ .

(SIX)  $\Rightarrow$  (int X). We proceed as in [2]. Let  $\sigma(E)$  consist of all sequences  $(x_n)$  in  $E$  such that  $\sum_n \|Tx_n\| < \infty$  for all  $T$  in  $L(E, X)$ . Then  $\sigma(E)$  is a Banach space with the norm

$$\|(x_n)\|_X = \sup \left\{ \sum_n \|Tx_n\| : \|T\| \leq 1 \right\}.$$

If (SIX) holds  $\sigma(E) = l[E]$  so there is  $M' > 0$  such that

$$\sum_n \|x_n\| \leq (1/2)M' \|(x_n)\|_X$$

for all  $(x_n)$  in  $l[E]$ . From this one concludes that for  $(x_n)$  in  $l[E]$

$$(3.2) \quad \sum_n \|x_n\| \leq (M/2) \sup \left\{ \sum_n \varphi_n(Tx_n) : \varphi_n \in U_X^\circ, T \in U(E, X) \right\}.$$

The topological dual space of  $l[E]$  can be represented by  $m[E']$  with duality given by the bilinear form

$$\langle (x'_j), (x_j) \rangle = \sum_j x'_j(x_j) \quad (x'_j) \in m[E']; \quad (x_j) \in l[E].$$

From (3.2) it follows that the unit ball of  $m[E']$  is contained in the  $w^*$ -closed convex cover of sequences having the form  $(M'/2)(T'\varphi_j)$  where  $\|T'\| \leq 1$  and  $\|\varphi_j\| \leq 1$  for each  $j$ .

For any finite subset  $A = \{x_1, x_2, \dots, x_k\}$  of  $E$  not containing 0, let  $\eta = (x'_1, x'_2, \dots, x'_k, 0, 0, \dots)$  be such that  $x'_n(x_n) = \|x_n\|$  and  $x'_n = 1$  for  $n = 1, 2, \dots, k$ . By the preceding paragraph we can find  $T_1, \dots, T_r$  in  $U(E, X)$ ,  $c_1, \dots, c_r \geq 0$  with  $\sum_{i=1}^r c_i = 1$  and  $(\varphi_{ij})_{j=1}^{\infty}$   $i = 1, 2, \dots, r$  with each  $\varphi_{ij}$  in the unit ball of  $X'$  such that

$$(3.3) \quad \begin{aligned} & \left| \langle \eta - \sum_{i=1}^r c_i (M'/2) (T'_i \varphi_{ij}), x_n e_n \rangle \right| \\ & < (1/2) \min \{ \|x_n\| : n = 1, 2, \dots, k \} \end{aligned}$$

for each  $n = 1, 2, \dots, k$ . Here  $x_n e_n$  is the sequence with  $x_n$  in the  $n$ th place and 0's elsewhere. From (4.2) we see that for each  $n$

$$\left| \|x_n\| - (M/2) \sum_{i=1}^r c_i T'_i \varphi_{in}(x_n) \right| < (1/2) \|x_n\|$$

from which it follows that

$$(3.4) \quad \begin{aligned} M' \sum_{i=1}^r c_i \|T_i x_n\| & \geq M' \sum_{i=1}^r c_i |T'_i \varphi_{in}(x_n)| \\ & > \|x_n\| \text{ for } n = 1, 2, \dots, k. \end{aligned}$$

If  $F$  is a finite dimensional subspace of  $E$  let  $A$  be a  $(2M')^{-1}$ -net for the unit sphere of  $F$ . If  $T_1, T_2, \dots, T_r$  and  $c_1, c_2, \dots, c_r$  satisfy (3.4) define  $R$  from  $E$  into  $m[X]$  by

$$Rx = (T_1 x, T_2 x, \dots, T_r x, 0, 0, \dots)$$

and  $\Delta$  from  $m[X]$  into  $l[X]$  by

$$\Delta(y_j) = (c_1 y_1, \dots, c_r y_r, 0, 0, \dots).$$

Then  $\|\Delta Rx\| \leq \|x\|$  for  $x$  in  $E$  since  $\|T_i\| \leq 1$  for each  $i$  and  $\sum_i |c_i| = 1$ . For  $x$  in  $F$  with  $\|x\| = 1$  there is  $y$  in  $A$  with  $\|x - y\| < (2M')^{-1}$  so that

$$\begin{aligned} \|\Delta Rx\| & \geq (\|\Delta Ry\| - \|\Delta R(y - x)\|) \\ & \geq (M')^{-1} \|y\| - \|y - x\| \geq (2M')^{-1}. \end{aligned}$$

Therefore, the second inequality of (diag X) holds with  $M = (2M')^{-1}$ .

3.8. THEOREM. *Let  $X$  be a Banach space which contains a subspace isomorphic to  $\lambda[X]$  where  $\lambda$  is a normal BK-space. For  $E$  any Banach space the following statements are equivalent:*

$$(SIX), (LIX), (ASX), (\text{int } X).$$

*Proof.* (SIX)  $\Rightarrow$  (LIX). It suffices to prove that  $E$  is locally immersed in  $\lambda[X]$ . Given  $F$  a finite dimensional subspace of  $E$  let  $R$  and  $\Delta$  be determined by (diag X) of Proposition 3.7.

If  $\Delta(y_n) = (t_n y_n)$  for  $(y_n)$  in  $m[X]$  let  $t_n = u_n v_n$  where  $(u_n)$  is in  $\lambda$ ,  $(v_n)$  is in  $\lambda^\alpha$  and

$$\|(u_n)\|_\lambda \leq 1, \quad \|(v_n)\|_{\lambda^*} \leq K$$

where  $K$  depends only on  $\lambda$  (Proposition 2.5). If  $Rx = (R_n x)$  for  $x$  in  $E$  define  $T$  from  $E$  into  $\lambda[X]$  by

$$Tx = (u_n R_n x).$$

Since  $\|(u_n)\|_\lambda \leq 1$

$$\|Tx\|_{\lambda[X]} = \|(\|u_n R_n x\|)\|_\lambda \leq \sup_n \|R_n x\| \|(u_n)\|_\lambda \leq 1.$$

For  $x$  in  $F$

$$\|Tx\|_{\lambda[X]} = \|(\|u_n R_n x\|)\|_\lambda \geq K^{-1} \sum v_n \|u_n R_n x\|$$

since the function defined by  $f((s_n)) = \sum_n v_n s_n$  is in  $\lambda'$  and  $\|f\| \leq K$ . Thus

$$\begin{aligned} \|Tx\|_{\lambda[X]} &\geq K^{-1} \sum v_n \|u_n R_n x\| \\ &= K^{-1} \|(t_n \|R_n x\|)\|_i \\ &\geq MK^{-1} \|x\|. \end{aligned}$$

Therefore,  $E$  is locally immersed in  $X$ .

(LIX)  $\Leftrightarrow$  (ASX) by Proposition 3.5.

(LIX)  $\Rightarrow$  (int X)  $\Rightarrow$  (SIX) by Proposition 3.6.

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