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The following relations between a Banach space E and a Banach space X are, roughly speaking, generalizations of the relation "E is a closed subspace of X."

(LIX) **The finite dimensional subspaces of E are uniformly isomorphic to subspaces of X under isomorphisms which extend to all of E without increase of norm.**

(SpX) **Finite rank mappings from any Banach space into E can be uniformly factored through subspaces of X.**

(ASX) **The continuous linear mappings from E into X distinguish the absolutely summing mappings from any Banach space into E.**

(SIX) **For each absolutely divergent series $\Sigma_n x_n$ in E there is a continuous linear mapping T from E into X such that $\Sigma_n Tx_n$ diverges absolutely.**

Our main result is that these four conditions are equivalent if X contains a subspace isomorphic to $\lambda[X]$ where λ is a normal BK-space. A related result of some interest is that the class of continuous linear mappings which factor through spaces which contain a complemented copy of $\lambda[X]$ forms a Banach operator ideal.

The consideration of the above relations continues the theme begun in [2] and [7]. A similar result to our main result is proven in [7] under a different assumption on the space X—an isometric assumption. We do not know whether the hypothesis on X in the present paper is strictly weaker than that in the previous paper. But in this case it is an isomorphic assumption and easier to verify. For example, it is satisfied by any space with a symmetric basis.

1. Some prerequisites. A. *Sequence spaces.* The space of all sequences of scalars (s_i) (real or complex) with the product topology is denoted by ω. The subspace of ω which contains all sequences which are eventually 0 is denoted by φ. A Banach space λ of sequences is called a *BK-space* if the inclusion from λ into ω is continuous. A space of sequences λ is called normal if whenever (s_i) is in and (t_i) is in m, the BK-space of bounded sequences it follows that $(t_i s_i)$ is also in λ. It is known that if λ is a BK space there is an equivalent norm $\| \ \|$ on λ for which

$$\|(t_i s_i)\| \leqq \sup_i |t_i| \, \|(s_i)\|$$

for (t_i) in m.

If λ is any set of sequences λ^α consists of all sequences (t_i) such that $(t_i s_i)$ is in 1 for each (s_i) in λ. Here 1 denotes the BK-space of all sequences (u_i) such that $\Sigma_i |u_i| < \infty$. If λ is a normal BK-space in which φ is dense then λ^α is isomorphic to the topological dual space of λ by means of the correspondence of f in λ' to $(f(e_n))$ in λ^α. Here e_n denotes the sequence with 1 in the nth place and 0's elsewhere. The closed unit ball U_{λ^α} of λ^α for λ a normal BK space is equal to the set of all (t_i) in ω such that $|\Sigma_i t_i s_i| \leqq 1$ for each (s_i) in φ with $\|(s_i)\|_\lambda \leqq 1$. Therefore, U_{λ^α} is compact in ω. For λ a normal BK space and X any Banach space $\lambda[X]$ denotes the space of all sequences (x_n) in X such that $(\|x_n\|_X)$ is in λ. With the norm

$$\|(x_n)\|_\lambda = \|(\|x_n\|_X)\|_\lambda$$

$\lambda[X]$ is known to be a Banach space. The closed linear span of φ in m consists of all sequences which converge to 0 and is denoted by c_0. References: **[5]**, **[6]**, **[7]**.

B. *Operator ideals.* Let L denote the class of all continuous linear mappings between Banach spaces. For two Banach spaces E and F let $L(E, F)$ denote the space of all continuous linear mappings between E and F. A subclass $\mathscr{A}$ of L is called an *operator ideal* if it is closed under sums and by multiplication on the left and right by members of L where multiplication and addition is restricted to pairs of operators for which these operations are meaningful. An operator ideal $\mathscr{A}$ is called a *Banach operator ideal* if there is a nonnegative correspondence α defined on $\mathscr{A}$ such that

(1) For every pair E, F, $\mathscr{A}(E, F) = (\mathscr{A} \cap L(E, F))$ is a Banach space with norm α.

(2) $\alpha(ST) \leqq \alpha(S)\|T\|$ if $S \in \mathscr{A}(F, G)$, $T \in L(E, F)$.

(3) $\alpha(ST) \leqq \|S\|\alpha(T)$ if $S \in L(F, G)$, $T \in \alpha(E, F)$.

Here $\| \ \|$ denotes the uniform operator. With the uniform operator topology, L is a Banach operator ideal. Let $U(E, F)$ denote the unit ball of $L(E, F)$ with the uniform norm.

The class $\mathscr{F}$ of finite rank mappings between Banach spaces is an operator ideal. Every finite rank mapping T from E to F has a nonunique representation

$$Tx = \sum_{i=1}^{n} x_i'(x) y_i$$

with each x'_i in E' the topological dual space of E and each y_i in F. If $E = F$ the number

$$\operatorname{tr}(T) = \sum_{i=1}^{n} x'_i(y_i)$$

does not depend on the representation of T; it is called the trace of T.

In this paper we shall refer to the following three operator ideals:

(1) The class int of all integral mappings between Banach spaces. A mapping T in $L(E, F)$ is integral if there is $K > 0$ such that

$$|\operatorname{tr}(ST)| \leqq K \|S\| \quad S \in \mathscr{F}(F, E).$$

The norm on $L(E, F)$ is given by

$$\|T\| = \sup\{\operatorname{tr}(ST): S \in \mathscr{F}(F, E), \quad \|S\| \leqq 1\}.$$

(2) The class N of nuclear mappings. A mapping T in $L(E, F)$ is nuclear if $T = \Sigma_{i=1}^{\infty} T_i$ has rank one and $\Sigma_{i=1}^{\infty} \|T_i\| < \infty$. The norm on $N(E, F)$ is given by

$$\|T\|_N = \inf\left\{\sum_{i=1}^{\infty} \|T_i\|: \sum_{i=1}^{\infty} T_i = T, \text{ each } T_i \text{ has rank one}\right\}.$$

(3) The class AS of absolutely summing mappings. A mapping T in $L(E, F)$ is absolutely summing if

$$\sum_n \|Tx_n\|$$

whenever

$$\sum_i |x'(x_n)| < \infty \text{ for all } x' \text{ in } E'.$$

The norm on $AS(E, F)$ is given by

$$\|T\|_{AS} = \sup\left\{\sum_n \|Tx_n\|: \sum_n |x'(x_n)| \leqq 1 \ \forall x' \in U_{E'}\right\}.$$

Here $U_{E'}$ denotes the unit ball in E'.

References: **[1]**, **[4]**, **[8]**.

2. Mappings which factor through X. For X a Banach space let $\langle X\rangle$ denote the class of all continuous linear mappings which factor through X. That is, a mapping T in $L(E, F)$ is in $\langle X\rangle$ if $T = T_1T_2$

where T_1 is in $L(X, F)$ and T_2 is in $L(E, X)$. One can show that $\langle X\rangle$ is an operator ideal if and only if $X \times X$ is isomorphic to a complemented subspace of X.

For X a Banach space and (t_n) a sequence of scalars the sequence $(t_n x_n)$ is in $l[X]$ for all (x_n) in $m[X]$ if and only if (t_n) is in l. We denote by $\operatorname{diag}(X)$ the collection of all such scalar diagonal mappings from $m[X]$ to $l[X]$.

2.1. PROPOSITION. *The smallest Banach operator ideal which contains $\langle X\rangle$ is equal to the class of all T in L which have as a factor a mapping from* $\operatorname{diag}(X)$. *In other words, T in $L(E, F)$ is in this ideal if and only if*

$$T = T_1 \Delta T_2$$

where T_2 is in $L(E, m[X])$, Δ is in $\operatorname{diag}(X)$ *and T_1 is in $L(l[X], F)$.*

Proof. Let $\{X\}$ denote the smallest Banach operator ideal which contains $\langle X\rangle$. We first show that T in $L(E, F)$ is in $\{X\}$ if and only if

$$T = \sum_n S_n V_n \qquad \sum_n \|S_n\| \|V_n\| < \infty \tag{2.1}$$

where each S_n is in $L(X, F)$ and each V_n is in $L(E, X)$. It is a routine task to verify that the class $[X]$ of all such mappings does form a Banach operator ideal with the norm

$$\|T\| = \inf\left\{ \sum_n \|S_n\| \|V_n\| : \sum_n S_n V_n = T\right\},$$

and it is clear that $[X]$ contains $\langle X\rangle$ and hence $\{X\}$.

On the other hand, for each S in $L(X, F)$ the correspondence V to SV is a continuous linear mapping from $L(E, X)$ into $\{X\}(E, F)$ so that

$$\sup\{\|SV\|_{\{X\}} : V \in U(E, X)\} < \infty$$

by the Uniform Boundedness Principle. A second application of this principle shows that the set

$$\{SV : V \in U(E, X), \quad S \in U(X, F)\}$$

is bounded in $\{X\}$. Thus if

$$\sum_n \|S_n\| \|V_n\| < \infty, \quad S_n \in L(X, F), \quad V_n \in L(E, X)$$

it follows that

$$\sum_n \|S_n V_n\|_{\{X\}} < \infty.$$

Therefore, every T of the form (2.1) is in $\{X\}$.

If T has the form (3.1) let $t_n = \|S_n\| \|V_n\|$ for each n. Define T_2 in $L(E, m[X])$ by

$$T_2 x = (V_n(x)/\|V_n\|),$$

Δ from $m[X]$ into $l(X)$ by

$$\Delta(u_n) = (t_n u_n)$$

and T_1 from $l[X]$ into F by

$$T_1(u_n) = \sum_n S_n u_n/\|S_n\|.$$

Then $T = T_1 \Delta T_2$ where Δ is in diag(X). On the other hand we can verify that every mapping T of the form $T = T_1 \Delta T_2$ has the form (2.1) by a routine inversion of the above argument.

The following theorem is proven in **[3]**.

2.2. THEOREM. *Let λ be a normal BK-space containing φ. For each sequence (r_n) in l^1 we can find sequences (s_n) in λ°* (the closure of φ in S) *and (t_n) in $\lambda^{\alpha\circ}$ such that $s_n t_n = r_n$ for all n.*

2.3. THEOREM. *If X contains a complemented subspace isomorphic to $\lambda[X]$ for λ a normal BK-space containing φ then $\langle X\rangle$ is a Banach operator ideal.* $(\langle X\rangle = \{X\})$

Proof. Given T in $\{X\}(E, F)$ we show that T factors through X. Since $\lambda[X]$ is complemented in X it suffices to show that T factors through $\lambda[X]$. By Proposition 2.1 there are T_1 in $L(l^1[X], F)$, T_2 in $L(E, m[X])$ and (r_n) in l such that

$$Tx = \sum_n r_n T_1 T_2 x \qquad x \in E.$$

We may assume that $r_n \geqq 0$ for all n. By Theorem 2.2 there is (s_n) in λ and (t_n) in λ^α such that $s_n t_n = r_n$ for all n. Define R_2 from $m[X]$ into $\lambda[X]$ by

$$R_2(u_n) = (s_n u_n)$$

and R_1 from $\lambda[X]$ into $l[X]$ by

$$R_1(v_n) = (t_n v_n).$$

Then $T = T_1 R_1 R_2 T_3$ so T factors through $\lambda[X]$.

2.4. COROLLARY. *If λ is a symmetric BK space then $\langle\lambda[X]\rangle$ is an ideal for every Banach space X.*

Proof. If λ is a symmetric *BK* space it is not hard to show that $\lambda[\lambda[X]]$ is isomorphic to $\lambda[X]$.

The following fact is needed later.

2.5. PROPOSITION. *If λ is a normal BK-space then there is $K > 0$ such that for each (t_n) in l we can find (u_n) in λ, (v_n) in λ^α with $(u_n v_n) = (t_n)$ such that*

$$\|(u_n)\|_\lambda \|(v_n)\|_{\lambda^\alpha} \leqq K \sum_n |t_n|. \tag{2.2}$$

Proof. Let U_1 denote the closed unit ball in $\lambda^{\alpha\alpha}$ and U_2 the closed unit ball in λ^α. Then both U_1 and U_2 are compact in ω so $U_1 U_2$ is compact in ω and thus closed in l. Since $\lambda^{\alpha\alpha}\lambda^\alpha \supset \lambda\lambda^\alpha = l$ it follows that $\bigcup_{n=1}^\infty nU_1U_2 = l$. Using the Baire Category Theorem we can find $r > 0$ such that $rU \subset U_1 U_2$ where U denotes the unit ball of l.

Given (t_n) in l and $\epsilon > 0$ let (r_n) in c_0 and (s_n) in l be such that $(r_n s_n) = (t_n)$ for each n and $|r_n| \leqq 1$ for all n and $\Sigma_n |s_n| \leqq \Sigma_n |t_n| + \epsilon$. Let (u'_n) in $\lambda^{\alpha\alpha}$ and (v_n) in λ^α be such that

$$(u'_n v_n) = (s_n) \qquad \|(u'_n)\| \|(v_n)\| \leqq 1/r.$$

For each (w_n) in φ, $(w_n u'_n)$ is in φ. Since c_0 is the closure of φ in m, $(r_n u'_n)$ is in the closure of φ in $\lambda^{\alpha\alpha}$ so $(r_n u'_n)$ is in λ. Since $\lambda^{\alpha\alpha}$ is normal

$$\|(r_n u'_n)\|_{\lambda^{\alpha\alpha}} \leqq C \|(u_n)\|_{\lambda^{\alpha\alpha}}$$

where C depends only on the norm on λ. Thus we have

$$(r_n u'_n v_n) = (r_n s_n) = (t_n)$$

and

$$\begin{aligned}\|(r_n u'_n)\|_\lambda \|(v_n)\|_{\lambda^\alpha} &\leqq C\|(u_n)\|_{\lambda^{\alpha\alpha}}\|(v_n)\|_{\lambda^\alpha} \\ &\leqq C/r \sum_n |s_n| \\ &\leqq C/r\Big(\sum_n |t_n| + \epsilon\Big).\end{aligned}$$

Since this inequality holds for all $\epsilon > 0$, (2.2) holds with $K = C/r$.

3. Local immersion and series immersion.

3.1. DEFINITION. A normed space E is said to be *locally immersed* in a normed space X if the following condition holds:

(LIX) There is a number $K > 0$ such that for each finite dimensional subspace G of E there is a continuous linear mapping T in $U(E, X)$ such that

$$\|Tx\| \geqq K\|x\| \qquad x \in G.$$

3.2. PROPOSITION. *The following property* ("splits through X") *is equivalent to* (LIX).

(SpX) There is $K \geqq 1$ such that each finite rank mapping from a normed space D to E can be factored

$$V = V_1 V_2 V_3; \ \|V_1\|\|V_2\|\|V_3\| \leqq K\|V\|$$

with V_3 in $L(D, E)$, V_2 in $L(E, Y)$ where Y is a closed subspace of X and V_1, in $L(Y, E)$.

Proof. (SpX) $\Rightarrow$ (LIX). Let V denote the inclusion map from G into E, and let V_1, V_2, V_3 satisfy (SpX). If $T = V_2V_3/\|V_2V_3\|$ then $\|T\| = 1$, and for each x in G we have

$$\|x\| = \|V_1V_2V_3x\| \leqq \|V_1\|\|V_2V_3x\|$$

$$\leqq \|V_1\|\|V_2\|\|V_3\|\|Tx\| \leqq K\|Tx\|.$$

(LIX) $\Rightarrow$ (SpX). Let $G = V(D)$, $V_3 = V$ and $V_2 = T$ where T is given by (LIX). Let $Y = T(G)$ and define V_3 on Y by $V_3 y = x$ if $Tx = y$. Then

$$V = V_1V_2V_3 \quad \text{and} \quad \|V_1\|\|V_2\|\|V_3\| \leqq K\|V\|.$$

3.3. DEFINITION. A normed space E is said to be *series immersed* in a normed space X if the following statement holds:

(SIX) For each absolutely divergent series $\Sigma_n x_n$ in E there is T in $L(E, F)$ such that $\Sigma_n Tx_n$ diverges absolutely.

We omit the proof of the following statement which is known **[8]**.

3.4. LEMMA. *For T a continuous linear mapping from c_0 into a normed space E the following statements are equivalent*:

(a) *T is nuclear*;
(b) *T is integral*;
(c) *T is absolutely summing*;
(d) $\Sigma_n \| Te_n \| < \infty$.

3.5. PROPOSITION. *For E and X arbitrary Banach spaces the following conditions are equivalent to* (SIX) *and thus* [7] *implied by* (LIX).

(ASX) For every Banach space D a mapping T in $L(D, E)$ is absolutely summing if ST is absolutely summing for all S in $L(E, X)$.

$(\mathrm{ASX})_0$ The same statement as (a) with $D = c_0$.

Proof. (SIX) $\Rightarrow$ (ASX). Suppose T in $L(D, E)$ is not absolutely summing. Then there is a weakly absolutely summable series $\Sigma_n x_n$ in D such that $\Sigma_n \| Tx_n \| = \infty$. By (SIX) there is S in $L(E, X)$ such that $\Sigma_n \| STx_n \| = \infty$ so that ST is not absolutely summing.

(ASX) $\Rightarrow$ $(\mathrm{ASX})_0$. Clear.

$(\mathrm{ASX})_0 \Rightarrow$ (SIX). Suppose $\Sigma_n x_n$ is a series in E with $\Sigma_n \| x_n \| = \infty$. If $\Sigma_n x_n$ is not weakly absolutely summable it is easy to find T in $L(E, X)$ such that $\Sigma_n \| Tx_n \| = \infty$ where T has rank one. If $\Sigma_n x_n$ is weakly absolutely summable define T from c_0 into E by

$$T((t_n)) = \Sigma_n t_n x_n.$$

By 3.4, T is not absolutely summing because $\Sigma_n \| Te_n \| = \Sigma_n \| x_n \| = \infty$ so by $(\mathrm{ASX})_0$ there is S in $L(E, X)$ such that ST is not absolutely summing. Consequently by 3.4,

$$\sum_n \| STe_n \| = \sum_n \| Sx_n \| = \infty.$$

3.6. PROPOSITION. *For Banach spaces E and X the following condition is implied by* (LIX) *and implies* (SIX):

(int X) A mapping T from a Banach space D into E is integral if ST is integral for all S in $L(E, Y)$ as Y ranges over the closed subspaces of X.

Proof. (LIX) $\Rightarrow$ (int X). Suppose ST is integral for all S in $L(E, Y)$. We first show there is $M > 0$ such that

$$(3.1) \quad \sup\{ |||ST||| : S \in U(E, Y),\ Y \text{ is a closed subspace of } X\} \leqq M.$$

Let $\mathcal{Y}$ denote the set of all closed subspaces Y of X. Let $Z_1(\mathcal{Y})$ denote the Banach space of all indexed families $(S_Y)_{Y\in\mathcal{Y}}$ where S_Y is in $L(E, Y)$ and $\sup_Y \|S_Y\| = \|(S_Y)\| < \infty$. Let $Z_2(\mathcal{Y})$ denote the Banach space of all indexed families $(V_Y)_{Y\in\mathcal{Y}}$ where V_Y is in $\operatorname{int}(D, Y)$ and $\sup_Y |||V_Y||| = |||(V_Y)||| < \infty$. The correspondence $(S_Y)_{Y\in\mathcal{Y}} \to (S_YT)_{Y\in\mathcal{Y}}$ determines a linear mapping from $Z_1(\mathcal{Y})$ into $Z_2(\mathcal{Y})$ which is continuous by the Closed Graph Theorem. There is thus $M > 0$ such that

$$|||(S_YT)||| \leqq M\|(S_Y)\| \qquad (S_Y) \in Z_1(\mathcal{Y})$$

which proves (3.1).

If S is a finite rank mapping in $L(E, D)$ let V denote the inclusion from TS(E) into E. By (SpX), $V = V_1V_2V_3$ with V_3 in $L(TS(E), E)$, T_2 in $L(E, Y)$ where Y is a subspace of X, V_1 is in $L(Y, E)$ and $\|V_1\|\|V_2\|\|V_3\| \leqq K$. Thus we have

$$\begin{aligned} |\operatorname{tr} TS| &= |\operatorname{tr}(V_1V_2V_3TS)| \\ &\leqq \|V_1\|\, |||V_2V_3T|||\, \|S\| \\ &\leqq \|V_1\| M \|V_2\|\|V_3\|\|S\| \leqq MK\|S\| \end{aligned}$$

which shows T is integral.

(int X) $\Rightarrow$ (SIX) by Lemma 3.4 and Proposition 3.5.

Notice the connection of the following statement with the results of §2.

3.7. Proposition. *The normed space E is series immersed in the Banach space X if and only if the following condition holds*:

(diag X) *There is $M > 0$ such that for each finite dimensional subspace F of E one can find a mapping R from E into $m[X]$ and a mapping Δ in* diag(X) *such that*

$$\|\Delta Rx\| \leqq \|x\| \text{ for } x \text{ in } E$$

$$M\|\Delta Rx\| \geqq \|x\| \text{ for } x \text{ in } F.$$

Proof. (diag X) $\Rightarrow$ (SIX). If $\Sigma_n \| Tx_n \| < \infty$ for all T in $L(E, X)$ then there is $K > 0$ such that $\Sigma_n \| Tx_n \| \leqq K \| T \|$ for each T in $L(E, X)$ by the Uniform Boundedness Principle. For k a fixed positive integer let R and Δ satisfy (diag X) for the finite dimensional subspace spanned by $\{x_1, x_2, \cdots, x_k\}$. There is a bounded sequence (T_i) in $L(E, X)$, such that $Rx = (T_i x)$ and a sequence t_i $(\geqq 0)$ in l such that $\Delta(y_i) = (t_i y_i)$. Since $\|\Delta Rx\| \leqq \|x\|$ for x in E it follows that $\Sigma_i t_i \|T_i\| \leqq 1$. Thus we have

$$\begin{aligned}\sum_{n=1}^{k} \|x_n\| &\leqq M \sum_{n=1}^{k} \|\Delta R x_n\| = M \sum_{n=1}^{k} \sum_{i} t_i \|T_i x_n\| \\ &\leqq M \sum_{i} \sum_{n=1}^{k} t_i \|T_i x_n\| \leqq MK \sum_{i} t_i \|T_i\| \\ &\leqq MK.\end{aligned}$$

Since M and K are independent of k, $\Sigma_n \|x_n\| < \infty$.

(SIX) $\Rightarrow$ (int X). We proceed as in **[2]**. Let $\sigma(E)$ consist of all sequences (x_n) in E such that $\Sigma_n \| Tx_n \| < \infty$ for all T in $L(E, X)$. Then $\sigma(E)$ is a Banach space with the norm

$$\|(x_n)\|_X = \sup\left\{ \sum_n \| Tx_n \| : \| T \| \leqq 1 \right\}.$$

If (SIX) holds $\sigma(E) = l[E]$ so there is $M' > 0$ such that

$$\sum_n \|x_n\| \leqq (1/2) M' \|(x_n)\|_X$$

for all (x_n) in $l[E]$. From this one concludes that for (x_n) in $l[E]$

$$\sum_n \|x_n\| \leqq (M/2) \sup\left\{ \sum_n \varphi_n(Tx_n) : \varphi_n \in U_X^\circ, T \in U(E, X) \right\}. \tag{3.2}$$

The topological dual space of $l[E]$ can be represented by $m[E']$ with duality given by the bilinear form

$$\langle (x_j'), (x_j) \rangle = \sum_j x_j'(x_j) \qquad (x_j') \in m[E']; \quad (x_j) \in l[E].$$

From (3.2) it follows that the unit ball of $m[E']$ is contained in the w^*-closed convex cover of sequences having the form $(M'/2)(T'\varphi_j)$ where $\|T'\| \leqq 1$ and $\|\varphi_j\| \leqq 1$ for each j.

For any finite subset $A = \{x_1, x_2, \cdots, x_k\}$ of E not containing 0, let $\eta = (x_1', x_2', \cdots, x_k', 0, 0, \cdots)$ be such that $x_n'(x_n) = \|x_n\|$ and $x_n' = 1$ for $n = 1, 2, \cdots, k$. By the preceding paragraph we can find $T_1, \cdots, T_r$ in $U(E, X)$, $c_1, \cdots, c_r \geqq 0$ with $\Sigma_{i=1}^r c_i = 1$ and $(\varphi_{ij})_{j=1}^\infty$ $i = 1, 2, \cdots, r$ with each φ_{ij} in the unit ball of X' such that

$$(3.3)\qquad \begin{aligned} &|\langle \eta - \sum_{i=1}^{r} c_i (M'/2)(T_i'\varphi_{ij}), x_n e_n\rangle| \\ &\quad < (1/2)\min\{\|x_n\|: n = 1, 2, \cdots, k\} \end{aligned}$$

for each $n = 1, 2, \cdots, k$. Here $x_n e_n$ is the sequence with x_n in the nth place and 0's elsewhere. From (4.2) we see that for each n

$$\left| \|x_n\| - (M/2)\sum_{i=1}^{r} c_i T_i'\varphi_{in}(x_n) \right| < (1/2)\|x_n\|$$

from which it follows that

$$(3.4)\qquad \begin{aligned} M'\sum_{i=1}^{r} c_i \|T_i x_n\| &\geqq M'\sum_{i=1}^{r} c_i |T_i'\varphi_{in}(x_n)| \\ &> \|x_n\| \text{ for } n = 1, 2, \cdots, k. \end{aligned}$$

If F is a finite dimensional subspace of E let A be a $(2M')^{-1}$-net for the unit sphere of F. If $T_1, T_2, \cdots, T_r$ and $c_1, c_2, \cdots, c_r$ satisfy (3.4) define R from E into $m[X]$ by

$$Rx = (T_1 x, T_2 x, \cdots, T_r x, 0, 0, \cdots)$$

and Δ from $m[X]$ into $l[X]$ by

$$\Delta(y_j) = (c_1 y_1, \cdots, c_r y_r, 0, 0, \cdots).$$

Then $\|\Delta R x\| \leqq \|x\|$ for x in E since $\|T_i\| \leqq 1$ for each i and $\Sigma_i |c_i| = 1$. For x in F with $\|x\| = 1$ there is y in A with $\|x - y\| < (2M')^{-1}$ so that

$$\begin{aligned} \|\Delta R x\| &\geqq (\|\Delta R y\| - \|\Delta R(y - x)\| \\ &\geqq (M')^{-1}\|y\| - \|y - x\| \geqq (2M')^{-1}. \end{aligned}$$

Therefore, the second inequality of (diag X) holds with $M = (2M')^{-1}$.

3.8. THEOREM. *Let X be a Banach space which contains a subspace isomorphic to $\lambda[X]$ where λ is a normal BK-space. For E any Banach space the following statements are equivalent*:

(SIX), (LIX), (ASX), (int X).

Proof. (SIX) $\Rightarrow$ (LIX). It suffices to prove that E is locally immersed in $\lambda[X]$. Given F a finite dimensional subspace of E let R and Δ be determined by (diag X) of Proposition 3.7.

If $\Delta(y_n) = (t_n y_n)$ for (y_n) in $m[X]$ let $t_n = u_n v_n$ where (u_n) is in λ, (v_n) is in λ^α and

$$\|(u_n)\|_\lambda \leqq 1, \qquad \|(v_n)\|_{\lambda^\alpha} \leqq K$$

where K depends only on λ (Proposition 2.5). If $Rx = (R_n x)$ for x in E define T from E into $\lambda[X]$ by

$$Tx = (u_n R_n x).$$

Since $\|(u_n)\|_\lambda \leqq 1$

$$\| Tx \|_{\lambda[X]} = \|(\| u_n R_n x \|)\|_\lambda \leqq \sup_n \| R_n x \| \|(u_n)\|_\lambda \leqq 1.$$

For x in F

$$\| Tx \|_{\lambda[X]} = \|(\| u_n R_n x \|)\|_\lambda \geqq K^{-1} \Sigma\, v_n \| u_n R_n x \|$$

since the function defined by $f((s_n)) = \Sigma_n v_n s_n$ is in λ' and $\|f\| \leqq K$. Thus

$$\begin{aligned} \| Tx \|_{\lambda[X]} &\geqq K^{-1} \Sigma\, v_n \| u_n R_n x \| \\ &= K^{-1} \|(t_n \| R_n x \|)\|_l \\ &\geqq MK^{-1} \| x \|. \end{aligned}$$

Therefore, E is locally immersed in X.

(LIX) $\Leftrightarrow$ (ASX) by Proposition 3.5.

(LIX) $\Rightarrow$ (int X) $\Rightarrow$ (SIX) by Proposition 3.6.

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