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ANTI-COMMUTATIVE ALGEBRAS AND HOMOGENEOUS SPACES WITH MULTIPLICATIONS

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As a generalization of certain results for Lie groups it is shown that an n-dimensional H-space (M, μ) with identity e has a coordinate system at e in which μ can be represented by a function $F: R^n \times R^n \to R^n$ which is analytic at $(0,0)$ and that the second derivative of F induces a bilinear anti-commutative multiplication α on R^n. In this way an algebra (R^n, α) analogous to the Lie algebra of a Lie group is obtained and all such algebras are shown to be isomorphic. If $M = G/H$ is a reductive homogeneous space, then these results generalize the Lie group–Lie algebra correspondence and the algebra (R^n, α) induces a G-invariant connection on G/H. Relative to this connection it is shown that an automorphism of $(G/H, \mu)$ is an affine map and induces an algebra automorphism of (R^n, α). Also the connection is irreducible if $(G/H, \mu)$ has no proper invariant subsystems (the analog of normal subgroups). In the case where G/H has a Riemannian structure, it may happen that there are no local isometries among the coordinate maps which give rise to anti-commutative multiplications on R^n.

1. Multiplications and change of coordinates. Let M be an n-dimensional real, analytic manifold and let $\mu: M \times M \to M$ be an analytic function such that $\mu(e, e) = e$ for some $e \in M$. In this case μ is called a *multiplication* on M and we denote this *multiplicative structure* by (M, μ). In the examples we consider, e is a two-sided identity element; that is, (M, μ) is an H-space (for other examples see **[6]**). In particular we will consider Lie groups and Moufang loops **[1, 8]**.

For the multiplicative structure (M, μ) let (U, ϕ) be a coordinate system at $e \in M$ where U is a neighborhood of e and $\phi: U \to R^n$ is the coordinate map. Assume that $\phi(e) = 0$ in R^n and let $\phi^{-1}: U_0 \to M$ denote the local inverse function of ϕ defined on a neighborhood U_0 of 0. For $D \subset U_0$ a suitable neighborhood of $0 \in R^n$ we can represent μ in the coordinate system $(\phi^{-1}(D), \phi|_{\phi^{-1}(D)})$ as $\mu(\phi^{-1}X, \phi^{-1}Y) = \phi^{-1}F(X, Y)$ for $X, Y \in D$ where $F: D \times D \to U_0$ is analytic at $(0,0) \in D \times D$ and defines a "local multiplicative structure" (U_0, F).

Let $\theta = (0,0)$; then since F is analytic we can form the kth derivative $F^k = F^k(\theta)$, which is a symmetric k-multilinear form on R^n and, using the notation $F^kZ^{(k)} = F^k(Z, Z, \cdots, Z)$, with $Z = (X, Y)$, we can write

$$F(X, Y) = F(\theta) + F^1(X, Y) + \tfrac{1}{2}F^2(X, Y)^{(2)} + \sum_{k=3}^{\infty} \frac{1}{k!} F^k(X, Y)^{(k)}.$$

Since $\mu(e, e) = e$, we obtain $F(0,0) = 0$. Using the linearity of F^1 on $R^n \times R^n$, it follows that

$$F^1(X, Y) = F^1((X,0) + (0, Y)) = PX + QY$$

where

$$PX = F^1(X,0)$$

and

$$QY = F^1(0, Y).$$

Similarly, using the bilinearity of F^2, we have

$$F^2(X, Y)^{(2)} = F^2((X, Y), (X, Y)) = F^2(X,0)^{(2)} + 2F^2((X,0), (0, Y)) + F^2(0, Y)^{(2)}.$$

Next we assume that (M, μ) is an H-space (or more generally a local H-space) with e the two-sided identity element. Then since $\mu(x, e) = x$, it follows that $F(X,0) = X$ for all $X \in R^n$ sufficiently near 0, which implies

$$PX = X \quad \text{and} \quad F^k(X,0)^{(k)} = 0 \quad \text{for} \quad k = 2, 3, \cdots.$$

Similarly $\mu(e, x) = x$ implies

$$QX = X \quad \text{and} \quad F^k(0, X)^{(k)} = 0 \quad \text{for} \quad k = 2, 3, \cdots.$$

Thus the Taylor's series representing μ has the form

$$F(X, Y) = X + Y + \alpha(X, Y) + \cdots$$

where $\alpha(X, Y) = F^2((X,0), (0, Y))$ defines a bilinear function $\alpha: R^n \times R^n \to R^n$. That is, R^n with the multiplication α becomes a nonassociative algebra which we denote by (R^n, α).

For example, let G be an n-dimensional Lie group with Lie algebra g and identify g and R^n as vector spaces. Then as above the Lie group multiplication μ induces the bilinear multiplication α on g relative to some coordinate system (U, ϕ) at $e \in G$. Denoting this algebra by

(g, α), we will show for $\phi^1 = \phi^1(e)$, the differential of ϕ at e, that the original multiplication $[X, Y]$ in g satisfies

$$\phi^1[X, Y] = \alpha(\phi^1 X, \phi^1 Y) - \alpha(\phi^1 Y, \phi^1 X).$$

Thus the Lie algebra g is isomorphic to the algebra $(g, \alpha)^-$ which is the vector space g with multiplication $\alpha(X, Y) - \alpha(Y, X)$; consequently the algebra (g, α) is Lie admissible **[9]**. The proof of the above formula is contained in Remark 3 below. However, if a canonical coordinate system is used, the Taylor's series representing μ is given by the Campbell–Hausdorff formula $X + Y + \frac{1}{2}[X, Y] + \cdots$; see **[8]**. So relative to a canonical coordinate system the nonassociative algebra induced on g by μ has bilinear multiplication $\frac{1}{2}[X, Y]$. In particular, in the case of a Lie group there always exists a coordinate system in which the nonassociative algebra induced on g by μ is anti-commutative. We will now prove that this is true in general for analytic H-spaces (or more generally, local analytic H-spaces).

Let (M, μ) be an analytic H-space with identity element e and with coordinate system (U, ϕ) at e. As before, represent μ by

$$\mu(\phi^{-1}X, \phi^{-1}Y) = \phi^{-1}F(X, Y) \qquad (*)$$

where ϕ^{-1} is the local inverse of ϕ and $F(X, Y) = X + Y + \alpha(X, Y) + \cdots$. Now for a suitable neighborhood W of $0 \in R^n$ we define a function ψ: $W \to R^n$, analytic at 0 in R^n, by the formula

$$\psi(X) = X - \tfrac{1}{2}\alpha(X, X).$$

Then since $(D\psi)(0) = I$, the inverse function theorem implies there is a neighborhood V of 0 in R^n so that (V, ψ) is a coordinate system at 0 in R^n and $\psi(0) = 0$.

Next for X, Y near 0 in R^n, define the function K by

$$K(X, Y) = \psi F(\psi^{-1}X, \psi^{-1}Y).$$

Using $(*)$, we see that the R^n-valued function $z = \psi \circ \phi$ restricted to a suitable neighborhood U' of e gives a coordinate system in which μ is represented by

$$\begin{aligned}\mu(z^{-1}X, z^{-1}Y) &= \mu(\phi^{-1} \circ \psi^{-1}X, \phi^{-1} \circ \psi^{-1}Y)\\ &= \phi^{-1}F(\psi^{-1}X, \psi^{-1}Y)\\ &= \phi^{-1} \circ \psi^{-1}K(X, Y)\\ &= z^{-1}K(X, Y)\end{aligned}$$

for X, Y near 0 in R^n. As in the previous consideration of F, note that K has the Taylor's series

$$K(X, Y) = X + Y + \beta(X, Y) + \cdots$$

where $\beta: R^n \times R^n \to R^n$ is the bilinear term. Using the equation $\psi F(X, Y) = K(\psi X, \psi Y)$ and the series for F, K and ψ, we observe that up to degree two the approximations are

$$\begin{aligned}\psi F(X, Y) &= F(X, Y) - \tfrac{1}{2}\alpha(F(X, Y), F(X, Y)) + \cdots \\ &= X + Y - \tfrac{1}{2}\alpha(X, X) - \tfrac{1}{2}\alpha(Y, Y) + \tfrac{1}{2}[\alpha(X, Y) - \alpha(Y, X)] + \cdots\end{aligned}$$

and

$$\begin{aligned}K(\psi X, \psi Y) &= \psi X + \psi Y + \beta(\psi X, \psi Y) + \cdots \\ &= X + Y - \tfrac{1}{2}\alpha(X, X) - \tfrac{1}{2}\alpha(Y, Y) + \beta(X, Y) + \cdots.\end{aligned}$$

From this we see

$$\beta(X, Y) = \tfrac{1}{2}[\alpha(X, Y) - \alpha(Y, X)]. \tag{1}$$

Thus $\beta(X, Y) = -\beta(Y, X)$ and the algebra (R^n, β) induced by μ relative to the coordinate system (U', z) is anti-commutative.

REMARKS (1). The anti-commutative algebras induced by multiplications such as μ are unique up to isomorphism and consequently we call such an algebra *the algebra associated with* μ. To see the isomorphism, let (U, z) and $(\bar{U}, w)$ be coordinate systems at e in which μ is represented by $\mu(z^{-1}X, z^{-1}Y) = z^{-1}K(X, Y)$ and by $\mu(w^{-1}X, w^{-1}Y) = w^{-1}\bar{K}(X, Y)$ as above. Let $K(X, Y) = X + Y + \beta(X, Y) + \cdots$ and $\bar{K}(X, Y) = X + Y + \bar{\beta}(X, Y) + \cdots$ with β and $\bar{\beta}$ anti-commutative algebra multiplications on R^n. Next, note that the function $\eta = w \circ z^{-1}$ is analytic at 0 in R^n with a series expansion about 0 given by $\eta(Z) = \eta^1 Z + \frac{1}{2}\eta^2 Z^{(2)} + \cdots$ for Z sufficiently near 0 and that η^1 is nonsingular. From the above formulas for μ, K and $\bar{K}$ we have, for X, Y sufficiently near 0 in R^n, that

$$\begin{aligned}\eta K(X, Y) &= wz^{-1}K(X, Y) \\ &= w\mu(z^{-1}X, z^{-1}Y) \\ &= w\mu(w^{-1}(wz^{-1}X), w^{-1}(wz^{-1}Y)) \\ &= ww^{-1}\bar{K}(wz^{-1}X, wz^{-1}Y) \\ &= \bar{K}(\eta X, \eta Y).\end{aligned}$$

Now expanding η, K, $\bar{K}$ in their series, we obtain the 2nd degree approximations

$$\begin{aligned}\eta K(X, Y) &= \eta^1 K(X, Y) + \tfrac{1}{2}\eta^2 K(X, Y)^{(2)} + \cdots \\ &= \eta^1 X + \eta^1 Y + \eta^1 \beta(X, Y) + \tfrac{1}{2}\eta^2 X^{(2)} \\ &\quad + \tfrac{1}{2}\eta^2 Y^{(2)} + \eta^2(X, Y) + \cdots\end{aligned}$$

and

$$\begin{aligned}\bar{K}(\eta X, \eta Y) &= \eta X + \eta Y + \bar{\beta}(\eta X, \eta Y) + \cdots \\ &= \eta^1 X + \eta^1 Y + \tfrac{1}{2}\eta^2 X^{(2)} + \tfrac{1}{2}\eta^2 Y^{(2)} \\ &\quad + \bar{\beta}(\eta^1 X, \eta^1 Y) + \cdots.\end{aligned}$$

These formulas imply

$$\bar{\beta}(\eta^1 X, \eta^1 Y) - \eta^1 \beta(X, Y) = \eta^2(X, Y).$$

Since β and $\bar{\beta}$ are anti-commutative, the left side of this equation is skew-symmetric while η^2 is symmetric in X and Y. Thus $\eta^2(X, Y) = 0$, which implies η^1 is an isomorphism of the algebras (R^n, β) and $(R^n, \bar{\beta})$.

(2). The following observation will be needed in the next section. From formula (1), $\beta(X, Y) = \frac{1}{2}[\alpha(X, Y) - \alpha(Y, X)]$, we see that an automorphism of (R^n, α) is an automorphism of (R^n, β).

We summarize some of these results as follows:

THEOREM 1. *Let (M, μ) be an analytic H-space with identity element e. Then*

(1) *There exists a coordinate system (U, z) at e so that if μ is represented by $F(X, Y) = X + Y + \alpha(X, Y) + \cdots$, then the algebra (R^n, α) is anti-commutative and is unique up to isomorphism.*

(2) *The differential $\tau^1 = \tau^1(e)$ of an analytic automorphism τ of (M, μ) induces an automorphism of (R^n, α).*

To prove the last statement, let $\tau: M \to M$ be an analytic diffeomorphism with $\tau(e) = e$ and $\tau\mu(x, y) = \mu(\tau x, \tau y)$; that is, τ is an automorphism. Let (U, z) be the coordinate system at e given in Theorem 1 and let $z^{-1}: D \to M$ be a local inverse as before with D a neighborhood of 0 in R^n. Since $\tau(e) = e$ and $z(e) = 0$, we can write $\tau(z^{-1}X) = z^{-1}k(X)$ for X near 0 in R^n, where k is analytic at 0 and $k(0) = 0$. Then for X, Y near 0 in R^n we have

$$\begin{aligned}\tau\mu(z^{-1}X, z^{-1}Y) &= \tau(z^{-1}F(X, Y)) \\ &= z^{-1}(kF(X, Y))\end{aligned}$$

and

$$\mu(\tau(z^{-1}X), \tau(z^{-1}Y)) = \mu(z^{-1}(kX), z^{-1}(kY))$$

$$= z^{-1}(F(kX, kY)).$$

Since τ is an automorphism we obtain

$$k(F(X, Y)) = F(k(X), k(Y)).$$

Let k have the Taylor's series

$$k(X) = k^1(X) + \tfrac{1}{2}k^2X^{(2)} + \cdots$$

where X is near 0 in R^n and $k^m = k^m(0)$ is the mth derivative of k at 0. As in the computations in remark (1), we use the series for F to obtain

$$\alpha(k^1X, k^1Y) - k^1\alpha(X, Y) = k^2(X, Y).$$

Since α is anti-commutative, we see that $k^1\alpha(X, Y) = \alpha(k^1X, k^1Y)$. Because τ is a diffeomorphism, we see that k^1 is nonsingular and therefore k^1 is an automorphism of (R^n, α).

REMARK (3). Modifying the notation of Remark 1, let (U, z) be the coordinate system at $e \in M$ for which μ is represented by $\mu(z^{-1}X, z^{-1}Y) = z^{-1}K(X, Y)$ where $K(X, Y) = X + Y + \alpha(X, Y) + \cdots$ with $\alpha(X, Y) = -\alpha(Y, X)$. Next let $(\bar{U}, w)$ be any other coordinate system at e for which μ is represented by $\mu(w^{-1}X, w^{-1}Y) = w^{-1}\bar{K}(X, Y)$ where $\bar{K}(X, Y) = X + Y + \bar{\beta}(X, Y) + \cdots$ with $\bar{\beta}$ bilinear. Then for $\eta = w \circ z^{-1}$, computations analogous to those in remark 1 yield $\eta K(X, Y) = \bar{K}(\eta X, \eta Y)$ and

$$\bar{\beta}(\eta^1X, \eta^1Y) - \eta^1\alpha(X, Y) = \eta^2(X, Y).$$

Interchanging X and Y in this formula we obtain $\bar{\beta}(\eta^1Y, \eta^1X) - \eta^1\alpha(Y, X) = \eta^2(Y, X)$. Subtracting these formulas and using the fact that η^2 is symmetric, we see that

$$2\eta^1\alpha(X, Y) = \bar{\beta}(\eta^1X, \eta^1Y) - \bar{\beta}(\eta^1Y, \eta^1X).$$

In particular, for a Lie group G with (U, z) a canonical coordinate system, we obtain the results previously mentioned concerning Lie admissible algebras. More generally, the above formula shows the algebra (R^n, α) is isomorphic to the algebra $(R^n, \frac{1}{2}\bar{\beta})^-$ which is the vector space R^n with multiplication $\frac{1}{2}[\bar{\beta}(X, Y) - \bar{\beta}(Y, X)]$.

2. Automorphisms and affine maps of a homogeneous space. We apply the results of §1 to a homogeneous space with multiplication μ to obtain an invariant connection from the anti-commutative algebra associated with μ; see **[6, 8]**. For certain homogeneous spaces we show that an automorphism of the multiplicative structure is an affine map of the corresponding connection.

Let G be a connected Lie group with Lie algebra g and let H be a closed (Lie) subgroup with Lie algebra h. The pair (G, H) or (g, h) is called a *reductive pair* if there exists a subspace m of g such that $g = m + h$ (subspace direct sum) and $(\operatorname{Ad} H)(m) \subset m$; that is, in terms of algebras $[h, m] \subset m$. The corresponding analytic manifold G/H is called a *reductive homogeneous space.* In most of the examples considered in **[6]** G and H are semi-simple with a decomposition $g = m + h$ where $m = h^\perp$ is the orthogonal complement relative to the Killing form of g.

For G/H a reductive homogeneous space with a fixed decomposition $g = m + h$, Nomizu **[3, 2]** established a $1-1$ correspondence between G-invariant affine connections ∇ on G/H and nonassociative algebras (m, α) satisfying $\operatorname{Ad} H \subset \operatorname{Aut}(m, \alpha)$ where $\alpha\colon m \times m \to m$ is the algebra multiplication and $\operatorname{Aut}(m, \alpha)$ is the automorphism group of (m, α). On the algebra level, $\operatorname{Ad} H \subset \operatorname{Aut}(m, \alpha)$ corresponds to $\operatorname{ad} h \subset D(m, \alpha)$, where $D(m, \alpha)$ is the Lie algebra of derivations of the algebra (m, α). For example, if ∇ corresponds to the algebra (m, α), then for all $X \in m$ the one-parameter subgroups $\exp tX$ in G project into geodesics (relative to ∇) in G/H by $\pi\colon G \to G/H$ if an only if $\alpha(X, Y) = -\alpha(Y, X)$. Further, if ∇ has zero torsion, then $\alpha(X, Y) = \frac{1}{2}[X, Y]_m$ where $[X, Y]_m$ is the projection of $[X, Y]$ in g onto m; see **[3, 8]**.

Next, let $M = G/H$ be a reductive space and let $(G/H, \mu)$ be an H-space as in §1 with $\bar{e} = eH$ the 2-sided identity; then we obtain an algebra (m, α) from μ relative to the canonical coordinate system obtained from $\pi \circ \exp$. For $u \in H$ let $\tau(u)\colon G/H \to G/H\colon \bar{x} \to \overline{ux}$ and let $\tau(H) = \{\tau(u)\colon u \in H\}$; then in **[6]** it was shown that $\tau(H) \subset \operatorname{Aut}(G/H, \mu)$ implies $\operatorname{Ad} H \subset \operatorname{Aut}(m, \alpha)$ where $\operatorname{Aut}(G/H, \mu)$ is the automorphism group of $(G/H, \mu)$. Thus a multiplicative system $(G/H, \mu)$ with $\tau(H) \subset \operatorname{Aut}(G/H, \mu)$ induces a G-invariant connection on G/H via the algebra (m, α). But from §1, there is a change of coordinates which determines an anti-commutative algebra (m, β) which is unique up to isomorphism and is given by $2\beta(X, Y) = \alpha(X, Y) - \alpha(Y, X)$. By Remark (2), $\operatorname{Ad} H \subset \operatorname{Aut}(m, \alpha)$ implies $\operatorname{Ad} H \subset \operatorname{Aut}(m, \beta)$ and therefore the anti-commutative algebra (m, β) gives rise to a G-invariant connection called the *connection induced by* μ. Many examples are given in [6] and the Moufang Loop S^7 obtained from the Cayley numbers of norm 1 is discussed in [7].

REMARK (4). For a Lie group (G, μ) with associative multiplication μ, the G-invariant connections are given by all the possible nonassociative algebras (g, α). However, these algebras need not arise from a fixed algebra (g, α_0) by using the formulas obtained from a change of coordinates at $e \in G$. For, as in Remark 3, any algebra (g, β) which arises from a change of coordinates at e in G is Lie admissible with $(g, \beta)^-$ isomorphic to the Lie algebra g. But there are many nonassociative algebras (g, α) which are not Lie admissible and consequently cannot be obtained via a change of coordinates.

We will now consider certain H-spaces $(G/H, \mu)$ which have properties analogous to Lie groups and the Moufang loop S^7. Thus we first assume $(G/H, \mu)$ is an analytic loop; that is, the left and right multiplications

$$L(\bar{x})\colon G/H \to G/H\colon \bar{y} \to \mu(\bar{x}, \bar{y}) \text{ and } R(\bar{x})\colon G/H \to G/H\colon \bar{y} \to \mu(\bar{y}, \bar{x})$$

are analytic diffeomorphisms for all $\bar{x} \in G/H$. Next we observe that the set of all diffeomorphisms $L(\bar{x})$ and $R(\bar{y})$ of the loop $(G/H, \mu)$ generates a subgroup Γ of the group of all diffeomorphisms. In particular note that a Lie group G can be represented by the Lie group K generated by all the maps $L(x)$. Also, the Moufang loop S^7 can be represented as a reductive space K/H where $K \subset \Gamma$ is the Lie group generated by the maps $R(x^2)L(x)$ for all $x \in S^7$ and $\tau(H)$ is contained in the automorphism group of S^7; see [7]. Using this notation we have the following definition.

DEFINITION. An analytic loop (M, μ) is called *multiplicatively homogeneous* if in the group Γ generated by all the diffeomorphisms $L(x)$ and $R(y)$ for $x, y \in M$ there exists a Lie group $K \subset \Gamma$ satisfying:

(1) K acts transitively on M, and

(2) K is generated by a set of fixed monomial expressions in the functions $L(x)$ and $R(y)$ for all $x, y \in M$.

We now consider the relationship between automorphisms of a loop (M, μ) and affine maps of a connection ∇ on M which generalizes some well known results on Lie groups and Moufang loops. An *affine map* of a manifold M with connection ∇ is a diffeomorphism $f\colon M \to M$ such that $f'\nabla(X, Y) = \nabla(f'X, f'Y)$ for all vector fields X, Y on M where f' is the differential of f.

THEOREM 2. *Let (M, μ) be a multiplicatively homogeneous analytic loop such that M can be represented as a reductive homogeneous space*

K/H with K as above and $\tau(H) \subset \mathrm{Aut}(K/H, \mu)$. Then an analytic automorphism of $(K/H, \mu)$ is an affine map relative to the invariant connection induced by μ.

Proof. Since (K, H) is a reductive pair we have a Lie algebra decomposition $k = m + h$ and from Theorem 1 the differential $f' = f'(\bar{e})$ of an automorphism $f \in \mathrm{Aut}(K/H, \mu)$ is an automorphism of the algebra (m, β) associated with μ.

Next note that f being an automorphism of $(K/H, \mu)$ implies

$$fL(\bar{x})f^{-1} = L(f\bar{x}) \quad \text{and} \quad fR(\bar{y})f^{-1} = R(f\bar{y})$$

for all $\bar{x}, \bar{y} \in K/H$. Thus if $k = m(L(\bar{x}_1), R(\bar{y}_1), \cdots) \in K$ is a monomial generator expression, we see that $fkf^{-1} = m(L(f\bar{x}_1), R(f\bar{y}_1), \cdots)$ is in K. Consequently

$$f\tau(K)f^{-1} \subset \tau(K)$$

where for any $a \in K$ we have $\tau(a)$: $K/H \to K/H$: $\bar{x} \to \overline{ax}$ and $\tau(K) = \{\tau(a)$: $a \in K\}$. Thus for any $a \in K$, there exists $a' \in K$ such that

$$f\tau(a)f^{-1} = \tau(a')$$

and this implies f locally commutes with K as defined in **[4]**. It is also shown in **[4]** that if ϕ is an analytic diffeomorphism of K/H with $\phi(\bar{e}) = \bar{e}$ such that ϕ locally commutes with K and $\phi' \in \mathrm{Aut}(m, \beta)$, then ϕ is an affine map of K/H relative to the connection given by (m, β). This result, along with the fact that $f' \in \mathrm{Aut}(m, \beta)$, proves f is an affine map.

REMARK (5). In the above proof the restrictions on K were used to show $f\tau(K)f^{-1} \subset \tau(K)$, which was needed to prove the local commuting property; thus the preceding proof can be generalized to give the following result.

COROLLARY 3. *Let (G, H) be a reductive pair and let $(G/H, \mu)$ be an H-space with identity $\bar{e}$ such that $\tau(H) \subset \mathrm{Aut}(G/H, \mu)$. Let f be an analytic automorphism of $(G/H, \mu)$, so that $f(\bar{e}) = \bar{e}$ and $f\tau(G)f^{-1} \subset \tau(G)$. Then f is an affine map of G/H relative to the connection induced by μ.*

3. Normal subsystems and holonomy reducibility. For an analytic H-space (M, μ) we now define local inverses and show how they can be used to generalize the concept of a normal subgroup of a Lie group. We then observe the relation between these subsystems and the irreducibility of the connection on a reductive space $M = G/H$ induced by μ.

Let the H-space (M, μ) have identity e and, relative to a suitable coordinate system (U, ϕ) at e with $\phi(e) = 0$ in R^n, let μ be represented by

$$F(X, Y) = X + Y + \alpha(X, Y) + \cdots.$$

At $\theta = (0, 0) \in R^n \times R^n$, the partial derivative of F relative to the second variable is given by $(D_2F)(\theta)(0, Y) = Y$ and thus the transformation $I = (D_2F)(\theta)$: $R^n \to R^n$ is nonsingular. Therefore, by the implicit function theorem, there exists an open ball B in R^n with center at $0 \in R^n$ and a uniquely determined analytic map r: $B \to R^n$ such that $r(0) = 0$ and $F(X, r(X)) = 0$ for all $X \in B$. These facts imply that there exists a neighborhood V of e in M and a unique analytic function ρ: $V \to M$ such that $\rho(e) = e$ and $\mu(x, \rho(x)) = e$ for all $x \in V$. Thus (M, μ) has a *local right inverse function* ρ and similarly a local left inverse function.

Now assume that in the coordinate system in which μ is represented by $F(X, Y)$ the algebra (R^n, α) is anti-commutative as in Theorem 1. Then the local right inverse function r has a series expansion

$$r(X) = r^1X + \frac{r^2}{2} X^{(2)} + \cdots$$

for X near 0 and $r^k = r^k(0)$. This gives

$$\begin{aligned} 0 &= F(X, r(X)) \\ &= X + r^1X + \frac{r^2}{2} X^{(2)} + \alpha(X, r^1X) + \cdots, \end{aligned}$$

which implies the approximation

$$\begin{aligned} r(X) &= -X + \alpha(X, X) + \epsilon(3) \\ &= -X + \epsilon(3) \end{aligned}$$

since $\alpha(X, X) = 0$.

DEFINITION. Let the H-space (M, μ) have identity e and local right inverse function ρ. Then a submanifold N of M containing e is called a *locally invariant subsystem* if $\mu(N, N) \subset N$ and there is neighborhood U of e in the domain of ρ such that $\mu(\mu(x, y), \rho(x)) \in N$ whenever $x \in U$ and $y \in N$.

REMARK (6). Let N be a locally invariant subsystem of the H-space (M, μ) and identify the tangent space $T(N, e)$ with a vector subspace $n \subset R^n$. Then (n, α) is an ideal of the algebra (R^n, α) associated with μ. To see this, let μ be represented by $F(X, Y)$ as before; then for $X \in R^n$, $Y \in n$ sufficiently near $0 \in R^n$, the local invariance of N implies that $F(F(X, Y), r(X))$ is in n. Expanding the Taylor's series, we see that

$$F(F(X, Y), r(X)) = F(X, Y) + r(X) + \alpha(F(X, Y), r(X)) + \cdots$$
$$= Y + 2\alpha(X, Y) + \epsilon(3)$$

is in n. Since $Y \in n$, this implies $\alpha(X, Y) \in n$ and also $\alpha(Y, X) = -\alpha(X, Y) \in n$; that is, n is an ideal of (R^n, α).

We now let $M = G/H$ be a reductive homogeneous space and consider what a locally invariant subsystem implies about the holonomic properties of the induced connection; see **[2, 3, 4, 5]** for more results on holonomy. For the reductive pair (G, H) with a fixed Lie algebra decomposition $g = m + h$ and $(\mathrm{Ad}\, H)m \subset m$, let the algebra (m, α) determine a G-invariant connection ∇ as before. For $X, Y, Z \in m$ we have the map

$$R(X, Y)\colon m \to m\colon Z \to R(X, Y)Z$$

where

$$R(X, Y)Z = \alpha(X, \alpha(Y, Z)) - \alpha(Y, \alpha(X, Z)) - \alpha(XY, Z) - [h(X, Y), Z]$$

is the curvature of ∇ evaluated at $\bar{e} = eH$ in G/H; recall that $XY = [X, Y]_m$ (resp. $h(X, Y) = [X, Y]_h$) is the projection of $[X, Y]$ in g onto m (resp. h). The *holonomy algebra* of G/H is the Lie algebra of the holonomy group of G/H relative to ∇. From **[2, 3]**, we know that the holonomy algebra is the smallest Lie algebra h^* of endomorphisms of m such that $R(X, Y) \in h^*$ and $[L(X), h^*] \subset h^*$ for all $X, Y \in m$ where $L(X)\colon m \to m\colon Y \to \alpha(X, Y)$. Denote h^* by $\mathrm{hol}(\alpha)$.

REMARK (7). Let $L(m, \alpha)$ be the Lie algebra of endomorphisms generated by the set of all $L(X)$ for $X \in m$ and let $D(m, \alpha)$ be the Lie algebra of derivations of the algebra (m, α) which we now assume to be anti-commutative. Since the mappings $\mathrm{ad}\, U\colon m \to m\colon X \to [UX]$ for $U \in h$ are in $D(m, \alpha)$, we see from the formulas for $\mathrm{hol}(\alpha)$ that $\mathrm{hol}(\alpha) \subset L(m, \alpha) + D(m, \alpha)$ which is a Lie algebra since $[L(m, \alpha), D(m, \alpha)] \subset L(m, \alpha)$. We say that the holonomy group acts *irreducibly* on G/H if $\mathrm{hol}(\alpha)$ acts irreducibly on m. The relation between irreducibility and the algebra (m, α) is as follows: Let n be a proper ideal of the algebra (m, α); then in **[4]** it was shown that there exists a proper ideal n' of (m, α) which is $D(m, \alpha)$-invariant. Thus $\text{hol}(\alpha)n' \subset [L(m, \alpha) + D(m, \alpha)](n') \subset n'$ and therefore the action of $\text{hol}(\alpha)$ is reducible on m if (m, α) has a proper ideal. We use the terminology that a locally invariant subsystem N of M is "*proper*" if its tangent space n is a proper subspace of the tangent space of M. The proof of the following result now follows from remarks (7) and (8).

THEOREM 4. *Let (G/H) be a reductive pair with decomposition $g = m + h$ and let the H-space $(G/H, \mu)$ with identity $\bar{e}$ satisfy $\tau(H) \subset$ $\text{Aut}(G/H, \mu)$. If $(G/H, \mu)$ has a "proper" locally invariant subsystem N, then the algebra (m, α) associated with μ has a proper ideal n' such that $\text{ad}\, h(n') \subset n'$. Thus, in this case, G/H is holonomy reducible relative to the connection induced by μ.*

REMARK (8). Let (M, μ) and (M', μ') be analytic H-spaces and let $\phi\colon M \to M'$ be an analytic homomorphism of M onto M'. Then, as for Lie groups, the kernel of ϕ is a subsystem of (M, μ) which is also a locally invariant subsytem. Thus if ϕ is an analytic homomorphism of $(G/H, \mu)$ such that the kernel of ϕ is a "proper" invariant subsystem, then one obtains a proper ideal of the algebra (m, α) associated with μ. Consequently G/H is holonomy reducible relative to the connection induced by μ. The converse-type statements appear to be false unless further associativity assumptions on μ are assumed.

4. Isometric change of coordinates. In §1 we showed that for an analytic H-space (M, μ) there exists a coordinate system in which the algebra (R^n, α) associated with μ is anti-commutative. However, if further conditions are imposed on the coordinates, then this need not be the case. In particular we shall now consider pseudo-Riemannian connections and coordinates.

Let G/H be a reductive homogeneous space with the usual decomposition $g = m + h$ and let C^* be a pseudo-Riemannian metric **[2, 8]** which induces the G-invariant connection ∇ corresponding to the algebra (m, α). Then C^* is given by a symmetric nondegenerate form C on m such that for all $X, Y, Z \in m$ and $V \in h$ the following conditions are satisfied:

$$C(\alpha(Z, X), Y) + C(X, \alpha(Z, Y)) = 0 \quad \text{and} \tag{1}$$

$$C((\text{ad}\, V)X, Y) + C(X, (\text{ad}\, V)Y) = 0.$$

We denote such an algebra by (m, α, C); see **[5, 10]** for more details. The algebra multiplication α is given uniquely by

$$\alpha(X, Y) = 1/2XY + U(X, Y)$$

where $XY = [X, Y]_m$ as before, and $U(X, Y) = U(Y, X)$ is uniquely determined by

$$2C(U(X, Y), Z) = C(ZX, Y) + C(X, ZY). \tag{2}$$

Now suppose D^* is another pseudo-Riemannian structure on G/H which is given by a symmetric nondegenerate form D on m. A mapping $f\colon m \to m$ with $f(0) = 0$ is a local isometry relative to the structures C and D on m if f is a local diffeomorphism at 0 in m and for $f^1 = f^1(0)$ we have as usual $C(f^1X, f^1Y) = D(X, Y)$. With these formulas we prove the following results about a local isometric change of coordinates for an H-space $(G/H, \mu)$.

THEOREM 5. *Let $M = G/H$ be a reductive homogeneous space with fixed Lie algebra decomposition $g = m + h$ and pseudo-Riemannian structures C^* and D^*. Let the algebras (m, α, C) and (m, β, D) be obtained from the H-space multiplication μ on G/H by coordinate maps ϕ_1 and ϕ_2 as before, and assume these algebras determine G-invariant pseudo-Riemannian connections relative to C^* and D^* respectively. If the change of coordinates map $\phi = \phi_1 \circ \phi_2^{-1}\colon m \to m$ is a local isometry, then the algebras (m, α, C) and (m, β, D) are isomorphic.*

In this case the new algebra is anti-commutative if and only if the original algebra is anti-commutative. Conditions for the algebra (m, α, C) inducing an invariant pseudo-Riemannian connection to be anti-commutative are discussed in **[11]**; roughly the conditions are that the algebra (m, α, C) must be power-associative.

For the proof first note that we have the following diagram:

$$\begin{array}{ccc}
U \times U & \xrightarrow{F} & U \\
\phi_1 \times \phi_1 \uparrow & & \uparrow \phi_1 \\
G/H \times G/H & \xrightarrow[\mu]{} & G/H \\
\phi_2 \times \phi_2 \downarrow & & \downarrow \phi_2 \\
V \times V & \xrightarrow[K]{} & V
\end{array}$$

where U and V are suitable neighborhoods of 0 in m and for $\phi = \phi_1 \circ \phi_2^{-1}$ we have $F(\phi X, \phi Y) = \phi K(X, Y)$ for X, Y near 0 in m. From the Taylor's series expansions of ϕ, F and K we obtain as before

$$\alpha(\phi^1 X, \phi^1 Y) - \phi^1 \beta(X, Y) = \phi^2(X, Y) \tag{3}$$

where $\phi^1 = \phi^1(0)$ and $\phi^2 = \phi^2(0)$. Also using the fact that ϕ is a local isometry, we have

$$C(\phi^1 X, \phi^1 Y) = D(X, Y). \tag{4}$$

Now β satisfies formulas similar to those for α; that is, β is given by

$$\beta(X, Y) = 1/2XY + \bar{U}(X, Y)$$

where $\bar{U}(X, Y) = \bar{U}(Y, X)$ is uniquely determined by

$$2D(\bar{U}(X, Y), Z) = D(ZX, Y) + D(X, ZY). \tag{5}$$

Hence, we see from (3) that

$$1/2\phi^1 X \phi^1 Y + U(\phi^1 X, \phi^1 Y) - 1/2\phi^1(XY) - \phi^1 \bar{U}(X, Y) = \phi^2(X, Y).$$

Since U, $\bar{U}$ and ϕ^2 are symmetric in X and Y,

$$U(\phi^1 X, \phi^1 Y) - \phi^1 \bar{U}(X, Y) = \phi^2(X, Y) \quad \text{and}$$

$$\phi^1(XY) = \phi^1 X \phi^1 Y. \tag{6}$$

Using equations (2), (4), (5) and (6) we see that

$$\begin{aligned} 2C(U(\phi^1 X, \phi^1 Y), \phi^1 Z) &= C(\phi^1 Z \phi^1 X, \phi^1 Y) + C(\phi^1 X, \phi^1 Z \phi^1 Y) \\ &= C(\phi^1(ZX), \phi^1 Y) + C(\phi^1 X, \phi^1(ZY)) \\ &= D(ZX, Y) + D(X, ZY) \\ &= 2D(\bar{U}(X, Y), Z) \\ &= 2C(\phi^1 \bar{U}(X, Y), \phi^1 Z). \end{aligned}$$

Since C is nondegenerate and ϕ^1 is nonsingular, we obtain

$$\phi^1 \bar{U}(X, Y) = U(\phi^1 X, \phi^1 Y).$$

Thus from the formulas for α, β and (6) we see (m, α, C) and (m, β, D) are isomorphic; this proves Theorem 5.

REMARK (9). The above result shows an isometry induces an isomorphism of algebras. However the results in [4] indicate the converse is false in general; the local commuting property is needed.

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