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**In view of weak topology, this paper studies the conditions for $C$- and $C^*$-embedded subsets of $k$-spaces to be closed. For example, we have the following:**

**A $C$-embedded subset $S$ is closed in a space $X$, if $S$ is paracompact and $X$ is a $k$-space. A $C^*$-embedded subset $S$ is closed in a space $X$, if (1) $X$ is a $k$-space in which every point is a $G_\delta$-set; (2) $S$ is normal, or an $F_\sigma$-set of $X$, and $X$ is a sequential space; or (3) $S$ is subparacompact, or an $F_\sigma$-set of $X$, and $X$ is a $k$-space which is hereditarily normal, or hereditarily countably paracompact.**

**0. Introduction.** As is well known, in a normal space every closed subset is $C$-embedded. But the converse is not valid. Indeed, a noncompact and countably compact subset $S$ of a compact space $\nu S$ $(=\beta S)$ is $C$-embedded, but it is not closed in $\nu S$, where $\nu S$ is the Hewitt realcompactification of $S$.

Thus the following question may be arised: *Under what conditions is $S$ closed in $X$ when $S$ is $C$- or $C^*$-embedded in it?*

Concerning this, we shall consider the case that $X$ is a $k$-space.

We shall recall the standard notions of $C$-, $C^*$-embeddings; $k$-spaces. A subspace $S$ of $X$ is $C$- (resp. $C^*$-) *embedded* in $X$, if every function in $C(S)$ (resp. $C^*(S)$) has a continuous extension over $X$. Clearly, every $C$-embedded subset is $C^*$-embedded.

A space $X$ is a $k$-space (resp. *sequential space* **[9]**), if $F \subset X$ is closed whenever $F \cap C$ is closed for each compact (resp. compact metric) subset $C$ of $X$. Clearly, sequential spaces are $k$-spaces.

$k$-spaces (resp. sequential spaces) are precisely the quotient images of locally compact (resp. metric) spaces. This is essentially due to **[5]** (resp. **[9]**).

All spaces considered in this paper will be completely regular Hausdorff.

## 1. Conditions for $C$-embedded subsets to be closed.
We begin by recording the main definitions.

DEFINITION 1.1. A space $X$ is called well-separated in the sense of K. Morita **[19]** (= *ss-discrete* in the sense of T. Isiwata **[14]**), if each countably infinite, discrete closed subset of $X$ contains a $C$-embedded, infinite subset of $X$. If we replace "$C$-embedded" by "$C^*$-embedded", then we call such a space *weakly well-separated.* Clearly, the class of weakly well-separated spaces contain the well-separated spaces and also contain the spaces satisfying the condition (C): *Each countable subset is $C^*$-embedded.*

$P$-spaces, or more generally $F$-spaces satisfy the condition (C), [**10**; 14 N].

Normal spaces, countably paracompact spaces [**19**; Proposition 2.1], and realcompact spaces (more generally, topologically complete spaces [**14**; Theorem 2.9] are well-separated. Recall that a space is *topologically complete,* if it is complete with respect to its finest uniformity.

DEFINITION 1.2. A space $X$ is called *isocompact* [**3**], if each closed countably compact subset of $X$ is compact.

Semi-stratifiable spaces [**6**; Corollary 4.5], subparacompact spaces [**4**; Theorem 3.5], and topologically complete spaces [**8**; Lemma 3.1] are isocompact.

DEFINITION 1.3. Let $\mathscr{C}$ be a covering of a space $X$. Then $X$ is said to have the *weak topology* with respect to $\mathscr{C}$, if $F \subset X$ is closed whenever $F \cap C$ is closed in $C$ for each $C \in \mathscr{C}$. (cf. [**7**; p. 131]).

The following is an analogy of T. Isiwata [**13**; Theorem 1.3]. Recall that a subset $S$ of $X$ is *relatively pseudocompact,* if each $f \in C(X)$ is bounded on $S$.

THEOREM 1.4. *Let $X$ have the weak topology with respect to a covering $\mathscr{C}$ consisting of relatively pseudocompact subsets of $X$. Let $S$ be well-separated and isocompact. If $S$ is $C$-embedded in $X$, then $S$ is closed in $X$.*

*Proof.* Since $X$ has the weak topology with respect to $\mathscr{C}$, it has also the weak topology with respect to $\{\bar{C};\ C \in \mathscr{C}\}$. Thus, to prove the theorem, we need show only that $S \cap \bar{C}$ is closed for each $C \in \mathscr{C}$. Suppose that, for some $K \in \mathscr{C}$, $S \cap \bar{K}$ is not countably compact. Then there is a countably infinite, discrete closed subset of $S \cap \bar{K}$, and hence of $S$. Since $S$ is well-separated, there is an infinite subset $D = \{d_n;\ n = 1, 2, \cdots\}$ of $\bar{K}$, and $D$ is $C$-embedded in $S$. Thus, $D$ is $C$-embedded in $X$. Define $f \in C(D)$ as $f(d_n) = n$. Then $f$ has an extension $g \in C(X)$. But $K$ is relatively pseudocompact, so that $g \mid K$ is bounded. Hence $g \mid \bar{K}$ is also bounded. Thus $f$ is bounded. This is a contradiction. Hence, $S \cap \bar{C}$ is a closed and countably compact subset of $S$ for each $C \in \mathscr{C}$. Since $S$ is isocompact, for each $C \in \mathscr{C}$, $S \cap \bar{C}$ is compact, hence is closed. This implies that $S$ is closed in $X$.

DEFINITION 1.5. As a generalization of $k$-spaces. J. Nagata **[22]** introduced the notion of quasi-$k$-spaces and characterized such spaces as being precisely the quotient images of $M$-spaces. A space $X$ is a *quasi-$k$-space*, if $F \subset X$ is closed whenever $F \cap C$ is closed in $C$ for every countably compact (not necessarily closed) subset $C$ of $X$.

Since each countably compact subset is relatively pseudocompact, by Theorem 1.4, we have

PROPOSITION 1.6. *Let $S$ be $C$-embedded in a quasi-$k$-space $X$. If $S$ is well-separated and isocompact, then $S$ is closed in $X$.*

As a generalization of $M$-spaces **[17]**, T. Isiwata introduced the notion of $M'$-spaces. (For the definition, see p. 358 in **[12]**).

COROLLARY 1.7. *Let $S$ be $C$-embedded in an $M'$-space $X$. If $S$ is well-separated and isocompact, then $S$ is a paracompact $M$-space, and it is closed in $X$.*

*Proof.* By [**18**; Theorem 4.4], the completion $\mu X$ of $X$ with respect to its finest uniformity is a paracompact $M$-space. Since a paracompact $M$-space is a $k$-space **[22]**, $\mu X$ is a $k$-space. While, $X \subset \mu X \subset \nu X$ **[18]**. Then $S$ is $C$-embedded in $\mu X$. Thus, by Proposition 1.6, $S$ is closed in $\mu X$. Hence $S$ is a paracompact $M$-space and is closed in $X$. That completes the proof.

For $S \subset X \subset \nu S$, $S$ is a dense and $C$-embedded subset of $X$. Then, by Proposition 1.6 we have a following modification of [**13**; Theorem 1.3].

PROPOSITION 1.8. *Let $S \subset X \subset \nu S$, and let $X$ be a quasi-$k$-space. If $S$ is well-separated and isocompact, then $S = X$.*

## 2. Conditions for $C^*$-embedded subsets to be closed.

First, we shall consider the closedness of $C^*$-embeddings in sequential spaces.

The following Lemma is a modification of [**10**; 9 N].

LEMMA 2.1. *Let $\{x\}$ be a nonisolated, zero-set in $X$.* (1) *Then $S = X - \{x\}$ is not $C$-embedded in $X$.* (2) *Moreover, when there is a sequence in $S$ which converges to the point $x$, $S$ is not $C^*$-embedded in $X$.*

*Proof.* (1) is easily proved as in [**10**; Theorem 1.18], so we shall prove (2). There is $f \in C(X)$ with $Z(f) = \{x\}$. Let $\{x_n;\ n = 1, 2, \cdots\}$ be a sequence in $S$ which converges to $x$, and let $f(x_n) = a_n$. Then we can assume that, if $n \neq m$ then $a_n \neq a_m$. Let $A_1 = \{a_{2n};\ n = 1, 2, \cdots\}$, $A_2 = \{a_{2n+1};\ n = 1, 2, \cdots\}$. Then $A_1$ and $A_2$ are disjoint and closed in $\mathbf{R} -$

$\{0\}$. Thus there is $h \in C^*(\mathbf{R}-\{0\})$ such that $h(A_1)=1$, $h(A_2)=-1$. Suppose that $S$ is $C^*$-embedded in $X$. Then the composition $h \cdot g \in C^*(S)$, where $g = f \mid S$, has an extension $F \in C(X)$. But $F(x)=\{1, -1\}$. This is a contradiction. Thus $S$ is not $C^*$-embedded in $X$.

THEOREM 2.2. *Let $S$ be an $F_\sigma$-set of $X$, or let each point of $X$ be a $G_\delta$-set.*

(1) **[2**; Theorems 12 and 13]. *If $S$ is $C$-embedded in $X$, then $S$ is closed in $X$.*

(2) *Let $X$ be sequential. If $S$ is $C^*$-embedded in $X$, then $S$ is closed in $X$.*

*Proof.* Part (1) is directly from Lemma 2.1(1), so we shall prove (2). Suppose that $S$ is not closed in $X$. Then there is a sequence in $S$ which converges to a point $x_0 \notin S$. Let $Z = S \cup \{x_0\}$. Then the point $x_0$ is not isolated in $Z$, and since $\{x_0\}$ is a $G_\delta$-set of $Z$, it is a zero-set in $Z$. Thus, by Lemma 2.1(2), $S$ is not $C^*$-embedded in $Z$. But $S$ is $C^*$-embedded in $X$, and hence $S$ is $C^*$-embedded in $Z$. This is a contradiction. Thus $S$ is closed in $X$.

Since a $k$-space each of whose point is a $G_\delta$-set is sequential **[15**; Theorem 7.3], we have

COROLLARY 2.3. *Let $S$ be $C^*$-embedded in a first countable space, more generally a $k$-space each of whose point is a $G_\delta$-set. Then $S$ is closed.*

Since each countably compact subset of a sequential space is always closed, then we have

COROLLARY 2.4. *Let $S$ be $C^*$-embedded in a sequential space $X$. If $S$ is a countable union of countably compact subsets, then $S$ is closed in $X$.*

Every $C$-embedded subset of a pseudocompact space is also pseudocompact. On the other hand, a $C^*$-embedded subset of a compact space need not be pseudocompact. But, when the whole space is sequentially compact (that is, each sequence has a convergent sequence), we have

COROLLARY 2.5. *Let $S$ be $C^*$-embedded in a sequentially compact space $X$. Then $S$ is pseudo-compact. Moreover, when $S$ is either weakly-separated, or an $F_\sigma$-set of $X$, then $S$ is sequentially compact.*

*Proof.* Suppose that $S$ is not pseudocompact. Then, by **[10**;

Corollary 1.12], $S$ contains a $C^*$-embedded copy of the positive integers **N**. Since $X$ is sequentially compact, there is a sequence $T$ in **N** which converges to a point $x_0 \not\in \mathbf{N}$. Since **N** is considered as a $C^*$-embedded subset of $X$, it follows that $T$, as a set, is a $C^*$-embedded subset of $T \cup \{x_0\}$. But this is a contradiction to Lemma 2.1(2). Similarly, in case that $S$ is weakly well-separated, $S$ is countably compact. In case that $S$ is an $F_\sigma$-set of $X$, as in the proof of Theorem 2.2(2), $S$ is also countably compact. Thus, in both cases, $S$ is sequentially compact, because each countably compact subset of a sequentially compact space is also sequentially compact.

Second, we shall consider the closedness of $C^*$-embeddings in $k$-spaces.

PROPOSITION 2.6. *Let $S$ be weakly well-separated, and let be $C^*$-embedded in a $k$-space $X$. If $S$ is either isocompact, or an $F_\sigma$-set of $X$* (resp. let $S$ satisfy the condition (C) in Definition 1.1), *then $S$ is closed* (resp. discrete and closed) *in $X$, otherwise $X$ contains a copy $\beta\mathbf{N}$.*

*Proof.* Suppose that $S$ is not closed in $X$. Then there is a compact subset $C$ of $X$ such that $S \cap C$ is not closed in $X$. Thus, since $S$ is isocompact or an $F_\sigma$-set of $X$, it follows that the closed subset $S \cap C$ of $S$ is not countably compact. Then there is a countably infinite, discrete closed subset $K$ of $S \cap C$. Since $S$ is weakly well-separated, $K$ contains an infinite subset $D$ which is $C^*$-embedded in $S$. Hence $D$ is $C^*$-embedded in $X$. This implies that $D$ is $C^*$-embedded in $C$, so that $\beta D = c1_{\beta C}D$. Clearly $\beta\mathbf{N} = \beta D$ and $\mathrm{cl}_{\beta C}D = \bar{D}^C \subset X$. Then $X$ contains a copy of $\beta\mathbf{N}$. The proof is similar for the case that $S$ satisfies condition (C). Indeed, if $S \cap K$ is infinite for some compact subset $K$ of $X$, then $X$ contains a copy of $\beta\mathbf{N}$. If $S \cap C$ is finite for each compact subset $C$ of $X$, then $S$ is closed in $X$. Thus, $S$ has the weak topology with respect to $\{S \cap C;\ C \text{ is compact in } X\}$. But each $S \cap C$ is finite, so $S$ is discrete.

COROLLARY 2.7. *Let $S$ be $C^*$-embedded in a countably compact $k$-space $X$. Suppose that $X$ does not contain a copy of $\beta\mathbf{N}$. Then $S$ is pseudocompact. If $S$ is weakly well-separated, then it is countably compact.*

*Proof.* Suppose that $X$ is not pseudocompact. Then, by [**10**; Corollary 1.21], $S$ contains a $C^*$-embedded copy of **N**. Thus **N** is considered as a $C^*$-embedded subset of $X$. By Proposition 2.6, **N** is closed in $X$. But, since $X$ is countably compact, this is a contradiction. In case $S$ is weakly well-separated, similarly, $S$ is countably compact.

DEFINITION 2.8. (**[15]**, **[16]**). A space $X$ is called *determined by countable subsets* ( $= X$ has *the countable tightness* in the sense of A. V. Arhangel'skiĭ **[1]**), if it has the following property: If $A \subset X$ and if $\bar{C} \subset A$ for every countable $C \subset A$, then $A$ is closed in $X$.

Sequential spaces, and hereditarily separable spaces are determined by countable subsets **[15**; Lemma 8.3]. If $X$ is determined by countable subsets, so is every subspace and every quotient space **[15**; Lemma 8.4].

LEMMA 2.9. *Let $X$ be the product of $|A|$ copies of positive integers* **N**, *where $|A|$ is the cardinality of the set $A$. If $X$ has any of the following properties, then $|A| \leqq \aleph_0$.*

(1) *Normality* (2) *Countable paracompactness* (3) *Each compact subset of $X$ is hereditarily isocompact* (4) *Each compact subset of $X$ is determined by countable subsets.*

*Proof.* In case (1), (2), and (3), it is proved by **[23**; Theorem 3], **[21**; Lemma 2.6], and **[24**; Lemma 2.1] respectively. So we prove only case (4). Let $K$ be the product of $|A|$ copies of $\{1,2\}$. Then, by the hypothesis the compact subset $K$ of $X$ is determined by countable subsets. Suppose, $|A| > \aleph_0$. Let $D = \{(x_\alpha; \alpha \in A); x_\alpha = 1$ or $2$, and for all but a countable number of points $x_\alpha = 1\}$. Pick a point $p \in K - D$, and let $P = \{p_\alpha; \alpha \in A\}$. Then, since $p \in \bar{D}^K$, by **[15**; Proposition 8.5], $p \in \bar{C}^K$ for some countable $C \subset D$. Let $C = \{c_n; n = 1, 2, \cdots\}$ and $c_n = (c_{\alpha n}; \alpha \in A)$. Then there is a sequence $\{A_1, A_2, \cdots\}$ of countable subsets of $A$ such that, if $\alpha \notin A_n$, $c_{\alpha n} = 1$. Let $A_0 = \{\alpha; p_\alpha = 2\}$. Then $|A_0| > \aleph_0$. Thus there is $\alpha_0 \in A_0 - \bigcup_{n=1}^\infty A_n$. Let $V = \{2\} \times \Pi_{\alpha \neq \alpha_0} D_\alpha$ ($D_\alpha = \{1,2\}$). Then $V$ is an open neighborhood of $p$ in $K$, and is disjoint from the set $C$. This is a contradiction. Thus $|A| \leqq \aleph_0$.

The following Lemma follows from the proof of **[20**; Theorem 1], so we shall omit the proof.

LEMMA 2.10. *Let (P) be some topological property, and let* (P) *be hereditary with respect to closed subsets. Let $X = Z^\omega$ be a union of countably many closed subsets $F_n$. If all $F_n$ have the property* (P), *then $Z$ has the property* (P).

From Lemmas 2.9 and 2.10, we have

PROPOSITION 2.11. *Let $X$ be the product of $|A|$ copies of* **N**. *Let $X$ be a union of countably many closed subsets $F_n$. If each $F_n$ has any of the properties of Lemma* 2.9, *then $|A| \leqq \aleph_0$.*

DEFINITION 2.12. As a generalization of sequential spaces, we shall introduce the notion of weakly sequential spaces. A space $X$ is *weakly sequential*, if $F \subset X$ is closed whenever $F \cap C$ is closed in $C$ for each sequentially compact subset $C$ of $X$.

PROPOSITION 2.13. (1) *If a $k$-space is orderable in the sense of* **[25]**, *then it is weakly sequential.*

(2) *Every quotient image of a weakly sequential space is also weakly sequential.*

(3) *Every weakly sequential space is precisely a quotient image of a locally sequentially compact space.*

*Proof.* (1) From **[25**; Corollaries 1.4 and 1.9**]**, each compact and separable subset of an orderable space is first countable, hence is sequentially compact. Then each compact subset of an orderable space is sequentially compact, which implies (1).

(2) & (3) (2) requires only routine verification, and (3) is proved similarly to **[15**; Theorem 6.E.3**]**.

LEMMA 2.14. *Let $S \subset X$ be weakly sequential. If $\beta\mathbf{N} \subset X$, then $S \cap \beta\mathbf{N}$ is discrete.*

*Proof.* Let $S$ have the weak topology with respect to the covering $\mathscr{C}$ consisting of all sequentially compact subsets of $S$. Since $S \cap \beta\mathbf{N}$ is closed in $S$, it has the weak topology with respect to $\{C \cap \beta\mathbf{N};\ C \in \mathscr{C}\}$. But, $C \cap \beta\mathbf{N}$ is finite for each $C \in \mathscr{C}$, because each convergent sequence in $\beta\mathbf{N}$ is finite (for example, see **[10**; 6 O**]**). Thus $S \cap \beta\mathbf{N}$ is discrete.

A discrete and $C^*$-embedded subset of a compact (resp. countable) space need not even be closed. Indeed, we consider a subset $\mathbf{N}$ of $\beta\mathbf{N}$ (resp. of $\mathbf{N}$ together with one point of $\beta\mathbf{N} - \mathbf{N}$). But we have

THEOREM 2.15. *Let $S$ be weakly well-separated, and let be $C^*$-embedded in a $k$-space $X$. For each compact subset $K$ of $X$, let there be a sequence $\{K_n;\ n = 1, 2, \cdots\}$ of compact subsets such that, $K = \bigcup_{n=1}^{\infty} K_n$ and each $K_n$ has any of the following properties.*

(1) *Hereditary normality* (2) *Hereditarily countable paracompactness* (3) *Hereditary isocompactness* (4) *It is determined by countable subsets,* (5) *Weakly sequential space.*

*If $S$ is either isocompact, or an $F_\sigma$-set of $X$* (resp. if $S$ satisfies the condition (C) in Definition 1.1), *then $S$ is closed* (resp. discrete and closed) *in $X$.*

*Proof.* Suppose that $S$ is not closed in $X$. Then, by Proposition

2.6, $X$ contains a copy $K$ of $\beta\mathbf{N}$. By Lemma 2.14, we can assume that each $K_n$ has any of the properties (1)–(4).

Let $P = I^c$ be the product of $2^{\aleph_0}$ copies of the unit interval $I$. Then $P$ has a countable dense subset $D$. Let $f$: $\mathbf{N} \to D$ be a map. Then $f$ is extendable to $g$: $\beta\mathbf{N} \to P$. Since $D$ is dense in $P$ and $\beta\mathbf{N}$ is compact, $g$ is a surjection. Then $P = \bigcup_{n=1}^{\infty} g(K_n)$. Let $g_n = g \mid K_n$ and $C_n = g(K_n)$. Then $g_n$ is a perfect map. Hence, $g_n \mid g_n^{-1}(A)$ is perfect for every subset $A$ of $C_n$. Then, since $K_n$ has any of the properties (1)–(4), $C_n$ has also any of these properties. (As for invariance of these properties, refer to [**26**; Theorem 9], [**11**; Theorem 2.2], [**3**; Theorem 2.6], and [**15**; Lemma 8.4] respectively).

On the other hand, the space $P$ contains a copy of the product $Q = \mathbf{N}^c$. Thus, $Q$ is considered as a union of countably many $C_n \cap Q$, and each $C_n \cap Q$ has any of the properties (1)–(4). But this is a contradiction to Proposition 2.11. Hence $S$ is closed in $X$.

PROPOSITION 2.16. *Let $S$ be weakly well-separated, and let be $C^*$-embedded in a $k$-space $X$. Let $X$ be a union of countably many closed subsets $F_n$, and let each $F_n$ be locally of any type of spaces listed below*:

*hereditarily normal, hereditarily countably paracompact,*
*hereditarily subparacompact, hereditarily separable,*
*sequential, quotient space of an orderable $k$-space.*

*Then,* (a) *if $S$ is isocompact, or an $F_\sigma$-set of $X$, then $S$ is closed in $X$. Especially,* (b) *when each $F_n$ is locally a hereditarily subparacompact, or a sequential space, this condition of $S$ is omitted.* (In case that $X$ is itself sequential, this is a modification of [**2**; Theorem 11]).

*Proof.* (a) Let $K$ be a compact subset of $X$. Then $K = \bigcup_{n=1}^{\infty} K_n$, where $K_n = K \cap F_n$. Here, by [**4**; Theorem 3.5] (resp. [**15**; Lemma 8.4]; Proposition 2.13), hereditarily subparacompact spaces (resp. hereditarily separable spaces; quotient spaces of orderable $k$-spaces) are hereditarily isocompact (resp. determined by countable subsets; weakly sequential). Thus, since each $K_n$ is a compact subset of $F_n$, it is easy to check that each $K_n$ has any of the properties (1)–(5) of Theorem 2.15. Hence, by Theorem 2.15, $S$ is closed in $X$.

(b) We shall prove that each compact subset $K$ of $X$ is hereditarily isocompact. In case that $F_n$ is a locally sequential space (in fact, $F_n$ is sequential), $K_n$ ($= K \cap F_n$) is sequential. Thus, since $K_n$ is compact, $K_n$ is hereditarily isocompact, for each countably compact subset of a sequential space is always closed. Consequently, in case (b), each compact subset of $X$ is a union of countably many closed and hereditarily isocompact subsets. By [**3**; Theorem 2.1], this implies that each compact subset of $X$ is hereditarily isocompact. Thus, since $S$ is weakly well-separated, from the proof of Proposition 2.6, $S$ is closed in $X$, otherwise $X$ contains a copy $\beta\mathbf{N}$. But, since each $K_n$ is hereditarily isocompact, from the proof of Theorem 2.15, $X$ does not contain a copy of $\beta\mathbf{N}$. Hence $S$ is closed in $X$. That completes the proof.

Since each closed subset of a normal space is $C^*$-embedded, by Theorem 2.15, we have

PROPOSITION 2.17. *Let $S$ be a subset of a hereditarily normal, $k$-space. Then $S$ is closed in $X$ if and only if it is $C^*$-embedded and an $F_\sigma$-set in $X$.*

For $S \subset X \subset \beta S$, $S$ is a dense and $C^*$-embedded subset of $X$. Then, by Theorem 2.15, we have

PROPOSITION 2.18. *Let $S \subset X \subset \beta S$. Then, under the same assumptions for $S$ and $X$ of Theorem* 2.15, $S = X$.

In view of the proof of Theorem 2.15, by Corollary 2.7, we have a generalization of [**25**; Proposition 3.2].

PROPOSITION 2.19. *Let $S$ be a $C^*$-embedded subset of a countably compact, $k$-space $X$. Let $X$ satisfy the conditions of Theorem* 2.15. *Then $S$ is pseudocompact. When $S$ is weakly well-separated, $S$ is countably compact.*

Finally, we shall consider a subset of the remainder $\beta Y - Y$.

PROPOSITION 2.20. *Let $Y$ be realcompact, and let $X$ be a subset of $\beta Y - Y$. If $X$ is a $k$-space which satisfies the conditions of Theorem* 2.15, *then $X$ is discrete.*

*Proof.* Suppose that $X$ is not discrete. Then there is an infinite compact subset $C$ of $X$. Thus, by [**10**; Theorem 9.11], $C$ contains a copy of $\beta\mathbf{N}$. But, from the proof of Theorem 2.15, this is a contradiction. Hence $X$ is discrete.

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