Pacific Journal of Mathematics

# ON NONSINGULARLY $k$-PRIMITIVE RINGS

ANN K. BOYLE, M. G. DESHPANDE AND EDMUND H. FELLER

Vol. 68, No. 2 April 1977

PACIFIC JOURNAL OF MATHEMATICS
Vol 68, No 2, 1977

# ON NONSINGULARLY $k$-PRIMITIVE RINGS

A. K. BOYLE, M. G. DESHPANDE AND E. H. FELLER

**A ring $R$ is called $k$-primitive if it has a faithful cyclic critical right module $C$ with $|C| = k$. We first show that $k$-primitive rings with Krull dimension have many properties in common with prime rings. For the case where $R$ is a $PWD$ with a faithful critical right ideal, we obtain an internal characterization.**

**1. Introduction.** Let $R$ be a ring with Krull dimension. Then $R$ is prime if and only if $R$ has a faithful compressible right $R$-module. In this paper we consider a broader class of rings, those which have a faithful cyclic critical right $R$-module. From **[2]** such a ring is called a *$k$-primitive* ring where $k$ denotes the Krull dimension of the faithful critical.

In the case where the faithful critical is nonsingular, these rings exhibit many of the properties of prime rings. Not all $k$-primitive rings have this additional property as an example in §4 shows. We call a $k$-primitive ring whose faithful critical is nonsingular, a *nonsingularly $k$-primitive* ring. Section 2 is devoted to showing some of the similarities with prime rings.

In §3 we consider piecewise domains ($PWD$) which are $k$-primitive rings. An internal characterization of $PWD$'s with faithful critical right ideal is obtained, which is our main result.

All rings will have identity, and the modules are right unital. The singular submodule of a module $M_R$ is denoted $Z(M)$. If $X$ is a subset of $R$, then ann $X$ or $X^r$ denotes the right annihilator of $X$ in $R$. The Krull dimension of a module $M_R$ is denoted by $|M|$. A certain familiarity with the definitions and basic results concerning Krull dimension is assumed. See **[5]** for reference.

**2. Properties of $k$-primitive rings.** If $R$ is a prime ring with Krull dimension then $R$ is nonsingular and has a faithful critical $C$ such that $|C| = |R|$. These conditions are also true for nonsingularly $k$-primitive rings.

PROPOSITION 2.1. *Let $R$ be a $k$-primitive ring with faithful cyclic critical $C$. Then $Z(R) = 0$ and $|C| = |R|$ if and only if $R$ is nonsingularly $k$-primitive.*

*Proof.* Suppose $Z(C)=0$. This immediately implies $Z(R)=0$. Let $X$ be the collection of annihilators of finite subsets of $C$. By [**4**, Theorem 1.24], $X$ satisfies the descending chain condition. Since $C$ is faithful $\bigcap_{x\in C}(x^r)=0$ and there exists a finite subset such that $\bigcap_{i=1}^{n}(x_i^r)=0$. This implies the existence of an $R$-monomorphism $R\to\Sigma_{i=1}^{n} R/x_i^r\to C^{(n)}$. Thus $|R|\leqq|C^{(n)}|=|C|\leqq|R|$.

Conversely if $Z(R)=0$, then $Z(C)=C$ or $Z(C)=0$. Suppose $Z(C)=C$. Then $C\cong R/K$ where $K$ is a large right ideal. Let $L=\{D\mid D \text{ is a critical right ideal}\}$. Then $S=\Sigma D$, $D\in L$ is a two sided ideal of $R$. Since $R/K$ is faithful, $S\not\subset K$. Hence there exists $D\in L$ such that $D\not\subset K$. Since $K$ is large $D\cap K\neq 0$ and $|R/K|=|D+K/K|=|D/D\cap K|<|D|\leqq|R|$. This contradicts the fact that $|C|=|R|$, and therefore $Z(C)=0$.

Let $C$ be the faithful $k$-critical of a nonsingularly $k$-primitive ring $R$. Then $P=\operatorname{ass} C$ is a prime ideal. In the remainder of this paper $C$ and $P$ will be used in this way.

LEMMA 2.2. *If $R$ is a nonsingularly $k$-primitive ring with faithful critical $C$, then $P=\operatorname{ass} C$ is a nonessential minimal prime and $|R|=|R/P|$.*

*Proof.* That $P$ is a nonessential minimal prime is straightforward.

The module $C$ contains a nonzero submodule $C^*$ where ann $C^*=P$. Since $C^*$ is a nonsingular, faithful $R/P$-module, then by 2.1 $|R|=|C^*|=|R/P|$.

PROPOSITION 2.3. *Let $R$ be a nonsingularly $k$-primitive ring with faithful, critical $C$ and let $P=\operatorname{ass} C$. Then*

(1) *$P$ contains all nonessential two sided ideals.*
(2) *$R$ has exactly one nonessential prime ideal, namely $P$.*
(3) *If $R$ is semiprime, then $R$ is prime.*
(4) *Every uniform right ideal which misses $P$ is compressible.*
(5) *Every uniform right ideal is critical and subisomorphic to $C$.*

*Proof.* (1) If $H$ is not essential, there exists a right ideal $I$ such that $I\cap H=0$. Then $IH=0$ which implies $H\subset \operatorname{ass} C=P$ by [**2**, Proposition 3.2].

(2) This follows from (1).

(3) If $0=P_1\cap\cdots\cap P_n$ is an irredundant intersection of minimal primes, then $P_i$ is not large and hence $P_i=P$ by (2) for each $i$. Thus $P=0$.

(4) If $U$ is uniform and $U\cap P=0$, then $U\cong U+P/P\subseteq R/P$. Since uniform right ideals of a prime ring are compressible, the result follows.

(5) Let $U$ be a uniform right ideal. Then $CU \neq 0$. Thus there exists $x \in C$ such that $xU \neq 0$. Since $Z(C) = 0$, $U \cong xU \subset C$ and $U$ is critical.

A ring $R$ with Krull dimension is called *very smooth* if every right ideal of $R$ has the same Krull dimension. In a prime ring any two critical right ideals are subisomorphic and thus the ring is very smooth.

PROPOSITION 2.4. *Let $R$ be a nonsingularly $k$-primitive ring with faithful critical $C$.*

(1) *If $D$ is a critical right ideal, then* ass $D = P$.
(2) *$R$ is a very smooth ring.*
(3) *If $D$ is a critical right ideal then $D \cap P = 0$ or $D \subseteq P$.*
(4) *If $R$ is not prime and if $C \subset R$, then $C \subset P$.*

*Proof.* (1) Let $P_0 = ass\ D$. Then $D$ has a submodule $D^*$ such that ann $D^* = P_0$. Since $Z(D^*) = 0$, $P_0$ is not essential and hence by 2.3(2), $P_0 = P = ass\ C$.

(2) It suffices to show that if $D$ is critical, $|D| = |R|$. By (1) $D$ has a submodule $D^*$ which is nonsingular and faithful as an $R/P$-module and hence $|D| = |D^*| = |R/P| = |R|$.

(3) If $D \cap P \neq 0$ then $D/D \cap P \cong D + P/P \subset R/P$. Since $R/P$ is very smooth and $|R/P| = |D|$ then $|D/D \cap P| < |D|$ implies $D = D \cap P$.

(4) If $C \subset R$ and $C \cap P = 0$, then $CP = 0$ contradicting the faithfulness of $C$. Thus $C \cap P \neq 0$. Therefore by (3) $C \subset P$.

PROPOSITION 2.5. *Let $R$ be a nonsingularly $k$-primitive ring. The compressible right ideals of $R$ are subisomorphic.*

*Proof.* Since $P$ = ass $C$ is not large, there exists a critical right ideal $D \neq 0$ such that $D \cap P = 0$. By 2.3 (4) $D$ is compressible. Let $K \neq 0$ be any compressible right ideal. Then $KD \neq 0$ since $D \not\subset P$ = ass $C$. Thus there exists $a \in K$ and a monomorphism $D \to aD \subseteq K$. Since $K$ is compressible, $K$ is subisomorphic to $aD$ and hence to $D$. Since being subisomorphic is a transitive property, any two compressible right ideals are subisomorphic.

COROLLARY 2.6. *Let $R$ be a nonsingularly $k$-primitive ring with the nonessential prime $P \neq 0$. Then $P$ contains an isomorphic copy of all uniform right ideals.*

**3. $k$-primitive piecewise domains.** Let $R$ be a ring with Krull dimension and suppose that $R$ is a piecewise domain with a faithful critical right ideal. Then $R$ is nonsingularly $k$-primitive and is, in general, not prime. We assume these rings have a faithful critical right ideal. In § 4 we provide an example to show that this need not always be the case. From **[7]** we have

DEFINITION 3.1. A ring $R$ is a *piecewise domain* ($PWD$) with respect to a complete set of orthogonal idempotents $e_1, \cdots, e_n$ if $x \in e_i \mathrm{Re}_k$, $y \in e_k \mathrm{Re}_l$ then $xy = 0$ implies $x = 0$ or $y = 0$.

In [**7**, p. 554] the following criterion is given for $R$ to be a $PWD$. $R$ is a $PWD$ with respect to the complete set of orthogonal idempotents $\{e_i\}$ if and only if every nonzero element of $\mathrm{Hom}_R(e_iR, R)$ is a monomorphism.

PROPOSITION 3.2. *Let $R$ be a ring with Krull dimension. Then $R$ is a PWD with faithful critical right ideal if and only if $R = \Sigma \oplus e_iR$ where $e_iR$ is critical for every $i$ and $e_jR$ is faithful and nonsingular for some $j$. In this case $R$ is nonsingularly $k$-primitive.*

*Proof.* Suppose $R$ is a $PWD$ with faithful critical right ideal. Then $R = \Sigma \oplus e_iR$ where $e_1, \cdots, e_n$ is a complete set of orthogonal idempotents. Since $C$ is faithful, $Ce_iR \neq 0$ for all $i$. Hence for any given $i$ there exists $c \in C$ such that $ce_iR \neq 0$. We can therefore define a homomorphism $\theta$ of $e_iR$ into $C$ using $c$ and by [**7**, p. 554] the mapping $\theta$ is a monomorphism. Thus $e_iR$ is critical.

Now $RC \neq 0$ which implies $e_jRC \neq 0$ for some $j$. By [**8**, Lemma 1], $Z(R) = 0$ and thus the mapping determined by the relation $e_jrC \neq 0$ is a monomorphism. So $C$ is subisomorphic to at least one $e_jR$ and $Z(e_jR) = 0$.

Conversely suppose $R = \Sigma \oplus e_iR$ where $e_iR$ is critical. Since $e_jR$ is faithful and nonsingular by 2.4 we know that $R$ is very smooth. So consider a mapping $f: e_iR \to R$. If $\mathrm{Ker}\ f \neq 0$ then $|f(e_iR)| = |e_iR/\mathrm{Ker}\, f| < |e_iR| = |R|$. Thus necessarily $f = 0$.

The same technique employed in the above proof shows that any two faithful critical right ideals of a $PWD$ are subisomorphic. Furthermore the faithful critical right ideal contains an isomorphic copy of every critical right ideal.

PROPOSITION 3.3. *Let $R$ be a ring with Krull dimension. If $R$ is a PWD with faithful critical right ideal, say $R = \Sigma \oplus e_iR$ where $C = e_1R$ is faithful, then*

(1) *$P = \mathrm{ass}\ C = \Sigma_i\, e_iR$, where $e_iR$ is not compressible.*
(2) *If $Q$ is a prime ideal not equal to $P$ then $|R/Q| < |R|$.*
(3) *If $R$ is not a prime ring, then not all the $e_iR$ are compressible.*

*Proof.* (1) and (3). If $R$ is prime, then the $e_iR$ are all compressible and ass $C = P = 0$. If $R$ is not prime, then $C$ cannot be compressible because it is faithful.

Now suppose $e_iR$ are compressible for $i = m+1, \cdots, n$ and $e_jR$ are not compressible for $j = 1, \cdots, m$. Then

$$P = RP = \left(\sum_{i=1}^{n} e_iR\right)P \subseteq \sum_{i=1}^{m} e_iRP \subseteq \sum_{i=1}^{m} e_iR.$$

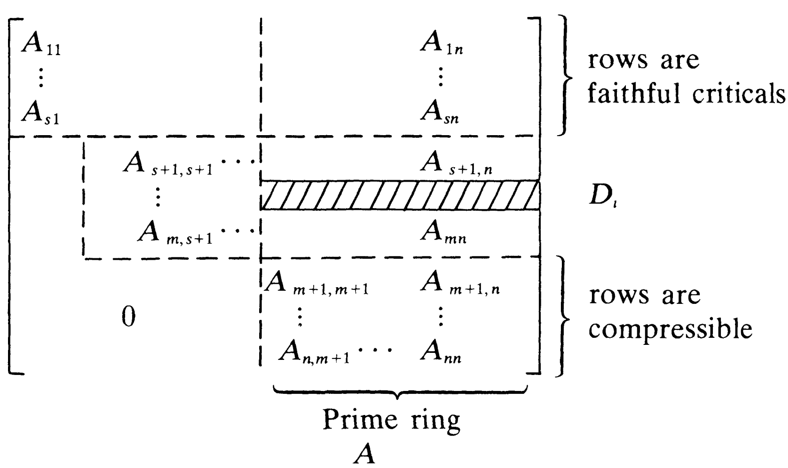
Conversely if $D$ is critical either $D \cap P = 0$ or $D \subset P$ by 2.4(3). Since $e_iR \cap P = 0$ implies by 2.3(4) that $e_iR$ is compressible, necessarily $e_iR \subseteq P$ $1 \leqq i \leqq m$ and hence $P = \Sigma_{i=1}^{m} e_iR$.

(2) If $Q$ is a prime ideal not equal to $P$, then $Q$ is large. Suppose $|R/Q| = |R|$. As in the proof of 2.4(3) if $D$ is critical $D \subseteq Q$. Thus $Q$ contains all critical right ideals and hence $R \subseteq Q$ which is impossible.

In [**8**, Theorem 2] Gordon obtains an internal characterization of prime right Goldie rings which are $PWD$'s. In the following theorem we obtain an internal characterization for a nonsingularly $k$-primitive ring with a faithful critical right ideal which is a $PWD$.

THEOREM 3.4. *Let $R$ be a $PWD$ with Krull dimension, $|R| = k$. Then $R$ is nonsingularly $k$-primitive with faithful critical right ideal if and only if $R \cong (A_{ij})_{n\times n}$ where for some $s$, $m$ where $1 \leqq s < m < n$,*

(1) *$A_{ij} = 0$ for $i > s$ and $j \leqq s$ or $i > m$ and $j \leqq m$.*
(2) *$A_{ij} \neq 0$ for $i \leqq s$ or $j > m$.*
(3) *$A_{ii}$ is a domain for $1 \leqq i \leqq n$.*
(4) *If $j \leqq m$ then $|(A_{ij})_{A_{jj}}| < k$.*
(5) *$A = \{(a_{ij}) \in (A_{ij}) \mid a_{ij} = 0 \ 1 \leqq i \leqq m$ or $1 \leqq j \leqq m\}$ is a prime ring of Krull dimension $k$ and $D_i = \{(a_{kj}) \mid a_{ij} = 0 \ 1 \leqq j \leqq m,\ a_{kj} = 0 \ k \neq i\}$ as a right $A$-module is $k$-critical for all $i$.*



*Proof.* By 3.2 and 3.3 $R = \Sigma_{i=1}^{n} \oplus C_i$ with $C_i$ critical for all $i$ where $C_i$ is faithful for $1 \leqq i \leqq s$ and $C_j$ is compressible for $m+1 \leqq j \leqq n$. Then $R$ is isomorphic to the matrix ring with $(i, j)$ entries in $A_{ij} = \mathrm{Hom}_R(C_j, C_i)$, which are monomorphisms or zero. Since $Z(R) = 0$ by [7], then $Z(C_i) = 0$.

(1) If $i > s$, and $j \leqq s$, then $\mathrm{Hom}(C_j, C_i) = 0$ since these $C_i$ are not faithful. Similarly, $\mathrm{Hom}(C_j, C_i) = 0$ $i > m$, $j \leqq m$, since these $C_j$ are not compressible, and the $C_i$ are compressible (using the fact that submodules of compressible modules are compressible). This proves (1).

(2) If $C_i$ is faithful, then $C_iC_j \neq 0$ for all $j$ and $\mathrm{Hom}(C_j, C_i) \neq 0$ for all $j$. If $C_j$ is compressible, then $C_iC_j \neq 0$ for any $i$. Hence $\mathrm{Hom}(C_j, C_i) \neq 0$. This proves (2).

(3) $A_{ii}$ is a domain, since $R$ is *PWD* ring.

(4) For $j \leqq m$, let $K_{ij}$ be an $A_{jj}$ submodule of $A_{ij}$. Let $K^*_{ij} = \{(a_{rt}) \mid a_{ij} \in K_{ij}$, and $a_{rt} = 0$ otherwise$\}$. Let $D_i$ be as in the theorem, then $D_i \neq 0$ by (2) and $D_i$ is a right ideal of $R$ because of (1). Let $S_i = K^*_{ij}R + D_i/D_i$. Then $S_i \subset C_i/D_i$. Then the mapping $K_{ij} \to S_i$ is a lattice isomorphism of the lattice of submodules of $A_{ij}$ over $A_{jj}$ into the lattice of submodules of $C_i/D_i$ over $R$. Since $C_i$ is critical and since $D_i \neq 0$, then the Krull dimension of $A_{ij}$ over $A_{jj}$ must be less than $k$.

(5) The first part follows since ass $C = P$ is equal to $C_1 + \cdots + C_m$ by 3.3 and $A \cong R/P$. Now each $D_i$ is a submodule of a critical module, and hence is critical over $R$. But the lattice of modules of $D_i$ over $R$ is the same as the lattice of $D_i$ over $A$, and $D_i$ is critical over $A$, and since the Krull dimension of $D_i$ over $R$ is $k$, then $|D_i|$ over $A$ is also $k$.

Conversely, let $R$ be a *PWD* satisfying these conditions and let $C_i$ denote the $i$th row of the matrix. We will show that $C_i$ is critical, and that $C_i$, $i = 1, 2, \cdots, s$ is faithful. Since $R$ is a *PWD*, and using (2), we have that $C_i$, for $i = 1, 2, \cdots, s$ is faithful. We now show $C_1$ is $k$-critical. Let $M_1$ be a *nonzero* submodule of $C_1$ over $R$. Let $M_{1j} = \{a \in A_{1j} \mid a$ is the $(1, j)$ entry of some member of $M_1\}$. Then $M_{1j} \neq 0$ for $j > m$. Now for $j \leqq m$, $M_{1j}$ is an $A_{jj}$ module. Thus by (4), the Krull dimension of $A_{1j}/M_{1j}$ over $A_{jj}$ is less than $k$. If $N = \Sigma_{i=m+1}^{n} M^*_{1i}$ where $M^*_{1i} = \{(a_{rt}) \mid a_{1i} \in M_{1i}$ and $a_{rt} = 0$ otherwise$\}$, then $N$ is not zero, and by (4), the Krull dimension of $D_1/N$ over $A$ is less than $k$. Thus with each submodule of $C_1/M_1$, we can associate an $A_{jj}$ submodule of $A_{1j}/M_{1j}$, one for each $j \leqq m$, and a factor module $D_1/N$ of $D_1$ as an $A$-module. We construct a lattice isomorphism from the submodules of $C_1/M_1$ into the lattice of submodules of $T = A_{11}/M_{11} \oplus \cdots \oplus A_{1m}/M_{1m} \oplus D_1/N$ over $S = A_{11} \oplus \cdots \oplus A_{mm} \oplus A$, where in $T_S$ we have scalar multiplication defined as coordinate multiplication by elements of the ring $S$. Now the Krull dimension of $A_{1i}/M_{1i}$ is less than $k$ over $A_{ii}$ for $1 \leqq i \leqq m$, and similarly for $D_1/N$ over $A$. Hence the Krull dimension of $T$ over $S$ is less than $k$. Thus $|C_1/M_1| < k$. Since $D_1$ over $A$ has Krull dimension $k$, one can show that $C_1$ has Krull dimension $k$ over $R$ in a similar fashion. Thus $C_1$ is $k$-critical. Since $R = C_1 \oplus \cdots \oplus C_n$ and $C_1$ is faithful, then each $C_i$ is $R$ isomorphic to a submodule of $C_1$, and hence is critical of Krull dimension $k$. Thus $|R| = k$.

**4. Examples and questions.** Using the results of § 3, one can easily construct nonsingularly $k$-primitive rings. Let $I$ a right Noetherian integral domain, where $|I| = k$, and $I[x]$ be the polynomial ring with commuting $x$. We construct three matrix rings of this type

$$R = \begin{bmatrix} I & I & I[x] & I[x] & I[x][y] & I[x][y] \\ I & I & I[x] & I[x] & I[x][y] & I[x][y] \\ 0 & 0 & I[x] & A & I[x][y] & I[x][y] \\ 0 & 0 & B & I[x] & I[x][y] & I[x][y] \\ 0 & 0 & 0 & 0 & I[x][y] & I[x][y] \\ 0 & 0 & 0 & 0 & I[x][y] & I[x][y] \end{bmatrix}$$

where (1) $A = B = 0$, (2) $A = I[x]$, $B = 0$, (3) $A = I[x]$, $B = I[x]$. These are nonsingularly $k$-primitive rings of Krull dimension $k + 2$.

In each of these three rings, with the notation of Theorem 3.4, $C_1$ is the faithful critical right ideal and $s = 2$, $m = 4$. In the case where $A = B = I$, the two nonfaithful, noncompressible modules $C_3$ and $C_4$ are (sub)isomorphic. Clearly, in all cases, $P = C_1 + C_2 + C_3 + C_4$.

One can show, in general, that if $R$ is nonsingularly $k$-primitive, then the complete ring of quotients $Q$ is a simple Artinian ring. If $R$ is *PWD*, then by **[9]**, $R$ has an Artinian Classical quotient ring $Q_{cl}$. However, $Q_{cl}$ is never $k$-primitive unless $R$ is prime. To illustrate this consider for a field $F$ the ring, $R = \begin{bmatrix} F & F[x] \\ 0 & F[x] \end{bmatrix}$. Then $Q(R) = \begin{bmatrix} F(x) & F(x) \\ F(x) & F(x) \end{bmatrix}$ and $Q_{cl}(R) = \begin{bmatrix} F & F(x) \\ 0 & F(x) \end{bmatrix}$.

*Question* 1. In general does $R$ have a classical quotient ring $Q_{cl}$? Does $Q_{cl}$ contain a maximal $k$-primitive subring which contains $R$?

*Question* 2. Does $N(R)$ being prime imply $R$ is prime? (True for *PWD*'s.)

*Question* 3. If $R$ has left Krull dimension, is $R$ a prime ring?

In regard to questions 1 and 2, if our ring $R$ has the regularity condition (See [6]), and if $N$ is prime, then $R$ is a prime ring.

To give an example of a nonsingularly $k$-primitive $PWD$ where the faithful cyclic is not imbeddable in $R$ consider a field $F$ with a derivation ($'$). Let $M = F[x] \oplus F[x]$. Then $M$ is a right $F[x]$ module under $(f, g)h = (fh, gh)$, and $M$ is a left $F$ module under $a(f, g) = (af + a'g, ag)$, $a \in F$. Thus $M$ is a bimodule, and the matrices

$$R = \begin{bmatrix} F & M \\ 0 & F[x] \end{bmatrix} \text{ form a ring.}$$

Let $I = \begin{bmatrix} 0 & N \\ 0 & F[x] \end{bmatrix}$, where $N = \{(0, f(x) \mid f(x) \in F[x]\}$. Then $I$ is not large, and contains no two-sided ideals. In addition $C = R/I$ is critical. In fact, if $C_0 \neq 0$ is a submodule of $C$, then $C/C_0$ is Artinian. One can show that $C$ cannot be embedded in $R$, and $Z(C) = 0$. One can also show that no right ideal of $R$ is faithful and critical. Hence $R$ is a nonsingularly 1-primitive ring without a faithful critical right ideal.

In the case where $R$ is $k$-primitive but the faithful critical is not nonsingular, then $R$ may not have the properties established in §2. Consider the following example.

Let $A = F[x, (')][z]$ where $F$ is a field with derivative ($'$) as in [3, p. 55], and $z$ commutes with $x$. Let

$$R = \begin{bmatrix} F & A/xA \\ 0 & A \end{bmatrix}.$$

The first row of $R$ is a faithful cyclic critical $C$ which is not nonsingular. Now $|C| = 1$ and $|R| = 2$. Thus $R$ is not very smooth. In addition, $R$ does not satisfy the regularity condition. Hence $R$ does not have an Àrtinian classical right quotient ring.

## References

1. A. K. Boyle and E. H. Feller, *Smooth Noetherian modules*, Comm. Algebras, **4** (1976), 617–637.

2. M. G. Deshpande and E. H. Feller, *The Krull radical*, Comm. Algebras, **3** (1975), 185–193.

3. N. J. Divinsky, *Ring and Radicals*, University of Toronto Press, Mathematical Expositions No. 14, 1965.

4. K. R. Goodearl, *Singular torsion and the splitting properties*, Memoirs of Amer. Math. Soc. No. 124 (1972).

5. R. Gordon and J. C. Robson, *Krull dimension*, Memoirs of Amer. Math. Soc. No. 133 (1973).

6. R. Gordon, *Gabriel and Krull dimension*, Ring Theory, Lecture notes in Pure and Applied Math. Marcel Dekker, New York, (1974), 241–295.

7. R. Gordon and L. W. Small, *Piecewise Domains*, J. Algebra, **23**, No. 3 (1972), 553–564.
8. R. Gordon, *Semiprime right Goldie rings which are a direct sum of uniform right ideals*, Bull. London Math. Soc., **3** (1971), 277–282.
9. ———, *Classical quotient rings of PWD's*, Proc. Amer. Math. Soc., **36** (1972), 39–46.

Received March 24, 1976 and in revised form November 22, 1976.

UNIVERSITY OF WISCONSIN–MILWAUKEE
MARQUETTE UNIVERSITY
AND
UNIVERSITY OF WISCONSIN–MILWAUKEE
MILWAUKEE, WI 53201

# PACIFIC JOURNAL OF MATHEMATICS



Mathematical papers intended for publication in the *Pacific Journal of Mathematics* should be in typed form or offset-reproduced (not dittoed), double spaced with large margins. Underline Greek letters in red, German in green, and script in blue. The first paragraph or two must be capable of being used separately as a synopsis of the entire paper. Items of the bibliography should not be cited there unless absolutely necessary, in which case they must be identified by author and Journal, rather than by item number. Manuscripts, in duplicate, may be sent to any one of the four editors. Please classify according to the scheme of Math. Reviews, Index to Vol. 39. All other communications should be addressed to the managing editor, or Elaine Barth, University of California, Los Angeles, California, 90024.

100 reprints are provided free for each article, only if page charges have been substantially paid. Additional copies may be obtained at cost in multiples of 50.


The *Pacific Journal of Mathematics* is issued monthly as of January 1966. Regular subscription rate: $72.00 a year (6 Vols., 12 issues). Special rate: $36.00 a year to individual members of supporting institutions.

Subscriptions, orders for back numbers, and changes of address should be sent to Pacific Journal of Mathematics, 103 Highland Boulevard, Berkeley, California, 94708.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION
Printed at Jerusalem Academic Press, POB 2390, Jerusalem, Israel.

# Pacific Journal of Mathematics

Vol. 68, No. 2 April, 1977