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**GAUGE GROUPS AND CLASSIFICATION OF BUNDLES WITH
SIMPLE STRUCTURAL GROUP**

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Suppose $\pi_i, i = 1, 2$ are principal K -bundles which are C^r -isomorphic in the sense that there exists a K -equivariant C^r -diffeomorphism $f: \mathcal{P}_1 \rightarrow \mathcal{P}_2$. If h belongs to the gauge group H_2 of \mathcal{P}_2 then $h \circ f$ lies in H_1 and we have a group isomorphism $H_2 \rightarrow H_1$ which is C^∞ . It is the purpose of this paper to investigate the converse in the case where K is a simple Lie group. (If K is abelian the gauge group of every K bundle over X is $C^r(X, K)$ so there is no hope of a converse. However for simple groups the situation is much better).

0. Introduction. Let K be a compact connected Lie group with Lie algebra \mathcal{K} . Let $\pi: \mathcal{P} \rightarrow X$ be a principal K -bundle of class C^∞ where X is a compact, connected C^∞ -manifold.

Throughout this paper r will be a positive integer which is chosen at this time and remains unchanged from here on.

We denote by H the subgroup of $C^r(\mathcal{P}, K)$ consisting of all those h for which $h(pk) = k^{-1}h(p)k$ for all p in \mathcal{P} and $k \in K$. H is naturally isomorphic to the group of all C^r -bundle automorphisms of \mathcal{P} which cover the identity on X [1, 2]. The group H will be called the gauge group of π the terminology being motivated by current usage in theoretical physics. $C^r(\mathcal{P}, K)$ is a Banach Lie group and H is a sub-manifold and so H is a Banach Lie group [2]. The Lie algebra of H can be identified as $\mathcal{H} = \{h: \mathcal{P} \rightarrow \mathcal{K} \mid h \text{ is } C^r \text{ and } h(pk) = Ad(k^{-1})h(p) \text{ for } p \in \mathcal{P}, k \in K\}$.

The bracket in \mathcal{H} and the exponential map $\exp: \mathcal{H} \rightarrow H$ are the natural pointwise operations.

1. Ideals in \mathcal{H} . Suppose $\mathcal{I} \subset \mathcal{H}$ is an ideal. For $p \in \mathcal{P}$ $e_p: \mathcal{H} \rightarrow \mathcal{K}$ is defined by $e_p(h) = h(p)$ for $h \in \mathcal{H}$. e_p is a Lie algebra epimorphism so $e_p(\mathcal{I})$ is an ideal in \mathcal{K} .

LEMMA 1.1. If $p \in \mathcal{P}$ and $k \in K$ then $e_p(\mathcal{I}) = e_{pk}(\mathcal{I})$.

Proof. $e_{pk}(h) = h(pk) = Ad(k^{-1})h(p) = Ad(k^{-1})e_p(h)$. Thus $e_{pk}(\mathcal{I}) = Ad(k^{-1})e_p(\mathcal{I})$. But $e_p(\mathcal{I})$ is an ideal in \mathcal{K} so $Ad(k^{-1})e_p(\mathcal{I}) = e_p(\mathcal{I})$.

DEFINITION 1.2. If $x \in X$ let $\mathcal{H}_x = e_p(\mathcal{I})$ where $p \in \pi^{-1}(x)$.

DEFINITION 1.3. If \mathcal{I} is an ideal in \mathcal{H} we say \mathcal{I} has property s if $[\mathcal{I}, \mathcal{H}] = \mathcal{I}$.

We recall that $[\mathcal{I}, \mathcal{H}]$ is the Lie subalgebra of \mathcal{H} generated by all elements of the form $[a, b]$ where $a \in \mathcal{I}, b \in \mathcal{H}$. $[\mathcal{I}, \mathcal{H}]$ consists exactly of all finite sums $\sum_i [a_i, b_i], a_i \in \mathcal{I}, b_i \in \mathcal{H}$.

We denote by $\mathcal{F}(X)$ the algebra of C^r , real valued functions on X . \mathcal{H} is a module over $\mathcal{F}(X)$ for if $f \in \mathcal{F}(X)$ and $h \in \mathcal{H}$ define $fh: \mathcal{P} \rightarrow \mathcal{H}$ by $(fh)(p) = f(\pi(p))h(p)$. One easily sees fh lies in \mathcal{H} so we have a module.

LEMMA 1.4. *If the ideal $\mathcal{I} \subset \mathcal{H}$ has property s then \mathcal{I} is a $\mathcal{F}(X)$ -submodule of \mathcal{H} .*

Proof. Let $h \in \mathcal{I}, \phi \in \mathcal{F}(X)$. We show $\phi h \in \mathcal{I}$. \mathcal{I} has property s so we may write $h = \sum_i [h_i, f_i]$ where $h_i \in \mathcal{I}$ and $f_i \in \mathcal{H}$. Then $\phi h = \sum_i \phi [h_i, f_i] = \sum_i [h_i, \phi f_i] \in \mathcal{I}$ where we used the pointwise nature of the bracket to get the last equation.

LEMMA 1.5. *If \mathcal{H}_1 and \mathcal{H}_2 correspond to bundles π_1 and π_2 and $\psi: \mathcal{H}_1 \rightarrow \mathcal{H}_2$ is a Lie algebra isomorphism then if \mathcal{I} has property s in \mathcal{H}_1 then $\psi(\mathcal{I})$ has property s in \mathcal{H}_2 .*

Before proving the final lemma of this section we make a preliminary construction. Suppose U is open in X and ξ is a section of π over U . Suppose $h \in \mathcal{H}$ and h has support in $\pi^{-1}(U)$. Define $\bar{h}: X \rightarrow \mathcal{H}$ by,

$$\bar{h}(x) = \begin{cases} h(\xi(x)) & x \in U \\ 0 & x \notin U. \end{cases}$$

$\bar{h} \in C^r(X, \mathcal{H})$ has support in U . Conversely if we start with $\bar{h}: X \rightarrow \mathcal{H}$ having support in U we can define $h \in \mathcal{H}$ as follows. There is a unique C^∞ -map $\theta: \pi^{-1}(U) \rightarrow \mathcal{H}$ such that $\xi(\pi(p))\theta(p) = p$ for $p \in \pi^{-1}(U)$. We define

$$h(p) = \begin{cases} Ad(\theta(p)^{-1})\bar{h}(\pi(p)) & p \in \pi^{-1}(U) \\ 0 & p \notin \pi^{-1}(U). \end{cases}$$

It is easily checked that $h \in \mathcal{H}$.

If $x_0 \in X$ we have:

$$H_{x_0} = \{f \in H \mid f(p) = e \text{ for all } p \in \pi^{-1}(x_0)\}.$$

$$\mathcal{H}_{x_0} = \{h \in \mathcal{H} \mid h(p) = 0 \text{ for all } p \in \pi^{-1}(x_0)\}.$$

LEMMA 1.6. *Assume \mathcal{H} is semisimple. Then \mathcal{H}_{x_0} has property s.*

Proof. Let $(\phi_i)_i$ be a finite partition of unity on X subordinate to an open cover $(U_i)_i$ such that π is trivial over each U_i . Then if $h \in \mathcal{H}_{x_0}$ we have $h = \sum_i \phi_i h$ and each $\phi_i h \in \mathcal{H}_{x_0}$. Therefore the problem is reduced to proving the following: If $U \subset X$ is open such that π has a local section ξ defined on U and if $h \in \mathcal{H}_{x_0}$ has support in $\pi^{-1}(U)$ then h can be written as $h = \sum_\nu [g_\nu, \phi_\nu]$ where $g_\nu \in \mathcal{H}_{x_0}$, $\phi_\nu \in \mathcal{H}$.

Let $\bar{h}: X \rightarrow \mathcal{H}$ correspond to h using the section ξ as above. Let $(E_i)_i$ be a basis for \mathcal{H} . Write $\bar{h} = \sum_i \bar{h}^i E_i$ where \bar{h}^i are real valued. Since \mathcal{H} is semisimple we may write $E_i = \sum_j [F_{ij}, G_{ij}]$ where F_{ij}, G_{ij} are in \mathcal{H} . Therefore $h = \sum_{i,j} \bar{h}^i [F_{ij}, G_{ij}] = \sum_{i,j} [\bar{h}^i F_{ij}, G_{ij}] = \sum_\nu [\bar{g}_\nu, \bar{\phi}_\nu]$ where \bar{g}_ν and $\bar{\phi}_\nu: X \rightarrow \mathcal{H}$ are C^r with $\bar{g}_\nu(x_0) = 0$. We can easily arrange that \bar{g}_ν and $\bar{\phi}_\nu$ have support in U . Then let g_ν, ϕ_ν be the corresponding functions on \mathcal{P} . Then if $p \in \mathcal{P}$ with $\pi(p) = x$ we have,

$$h(p) = Ad(\theta(p)^{-1})\bar{h}(x) = Ad(\theta(p)^{-1})\left(\sum_\nu [\bar{g}_\nu(x), \bar{\phi}_\nu(x)]\right)$$

$$= \sum_\nu [Ad(\theta(p)^{-1})\bar{g}_\nu(x), Ad(\theta(p)^{-1})\bar{\phi}_\nu(x)]$$

$$= \sum_\nu [g_\nu(p), \phi_\nu(p)] = \left(\sum_\nu [g_\nu, \phi_\nu]\right)(p).$$

2. A classification theorem. In this section, in addition to the assumptions made in the introduction, we assume K is a simple Lie group with trivial center. We first make some observations.

Given a principal K -bundle $\pi: \mathcal{P} \rightarrow X$ we construct the associated fiber bundle $\mathcal{A} \rightarrow X$ with fiber \mathcal{H} where K acts on \mathcal{H} via the adjoint representation of K . Each $p \in \mathcal{P}$ with $\pi(p) = x$ gives a linear isomorphism $\phi_p: \mathcal{H} \rightarrow \mathcal{A}_x$. Since $Ad: K \rightarrow Lis(\mathcal{H})$ actually takes values in $Aut(\mathcal{H})$ we see \mathcal{A} is a bundle of Lie algebras. Therefore $\Gamma'(\mathcal{A})$, the space of C^r -sections of \mathcal{A} , is a Lie algebra with pointwise bracket. There is a natural isomorphism $\mathcal{H} \rightarrow \Gamma'(\mathcal{A})$ given by $h \rightarrow \tilde{h}$ where $\tilde{h}(x) = \phi_p(h(p))$ for each $x \in X$ where $p \in \pi^{-1}(x)$ [3]. This isomorphism is an isomorphism of $\mathcal{F}(X)$ -modules and is a homeomorphism with respect to the C^r -topologies.

Now suppose $\pi_i: \mathcal{P}_i \rightarrow X$ are principal K -bundles, $i = 1, 2$, with gauge groups H_i and \mathcal{H}_i the Lie algebra of H_i . For $x_0 \in X$ the ideal \mathcal{H}_{x_0}

is closed. Let $\psi: H_1 \rightarrow H_2$ be a C^1 -group isomorphism. There is an induced Lie algebra isomorphism $\psi_*: \mathcal{H}_1 \rightarrow \mathcal{H}_2$ given by

$$\psi_*(h)(p) = \left. \frac{d}{dt} \right|_{t=0} [\psi(\exp(th))](p)$$

ψ_* is a topological isomorphism and so for each $x_0 \in X$ $\psi_*(\mathcal{H}_{1x_0})$ is a closed ideal having property s in \mathcal{H}_2 . If we write $\mathcal{I} = \psi_*(\mathcal{H}_{1x_0})$ and refer to the discussion of section 1 we have ideals $\mathcal{H}_x \subset \mathcal{H}$ for each $x \in X$. There are apparently two possible cases.

Case 1. $\mathcal{H}_x = \mathcal{H}$ for all $x \in X$.

We argue this cannot occur. Since \mathcal{I} is an ideal with property s \mathcal{I} is an $\mathcal{F}(X)$ -submodule. If $\mathcal{H}_x = \mathcal{H}$ for all x in X we shall show $\mathcal{I} = \mathcal{H}_2$ which is impossible since $\mathcal{H}_{1x_0} \neq \mathcal{H}_1$. To show $\mathcal{I} = \mathcal{H}_2$ we regard \mathcal{I} as a closed $\mathcal{F}(X)$ -submodule of $\Gamma'(\mathcal{A}_2)$. Then for $x \in X$, $v \in \mathcal{A}_{2x}$ there is $h \in \mathcal{I}$ for which $h(x) = v$. One now uses the $\mathcal{F}(X)$ -module structure to show for any $x \in X$ and for any r -jet $\xi \in j_x^r \mathcal{A}_2$ there is an $h \in \mathcal{I}$ for which $j_x^r h = \xi$. Since \mathcal{I} is a closed submodule we conclude $\mathcal{I} = \Gamma'(\mathcal{A}_2)$ by applying a ‘‘global’’ version of a well-known theorem of Whitney. We refer to [5], Corollary 1.6, p. 25.

Case 2. $\mathcal{H}_x = \mathcal{H}$ for some x .

In this case there is some x_1 for which $\mathcal{H}_{x_1} = (0)$ since K is simple. We claim there cannot be an $x_2 \neq x_1$, for which $\mathcal{H}_{x_2} = 0$. For if there were then we would have $\mathcal{I} \subset \mathcal{H}_{2x_1} \cap \mathcal{H}_{2x_2}$. But the codimension of \mathcal{I} in \mathcal{H}_2 equals the codimension of \mathcal{H}_{1x_0} in \mathcal{H}_1 which equals the codimension of \mathcal{H}_{2x_1} in \mathcal{H}_2 so $\mathcal{I} \subset \mathcal{H}_{2x_1} \cap \mathcal{H}_{2x_2}$ is not possible. Therefore in the present case we see there is a unique $x_1 \in X$ for which $\mathcal{I} = \mathcal{H}_{2x_1}$.

Thus we see that a C^1 isomorphism $\psi: H_1 \rightarrow H_2$ gives rise to a bijection $\bar{\psi}: X \rightarrow X$ defined by

$$\psi_*(\mathcal{H}_{1x}) = \mathcal{H}_{2\bar{\psi}(x)}.$$

Now let $h \in \mathcal{H}_1$, $f \in \mathcal{F}(X)$. We have $\bar{\psi}: X \rightarrow X$ and we write $\bar{\psi}_*(f) = f \circ \bar{\psi}^{-1}$.

LEMMA 2.1. $\psi_*(fh) = \bar{\psi}_*(f)\psi_*(h)$.

Proof. Let $p_2 \in \mathcal{P}_{2x}$ let $\lambda = \bar{\psi}_*(f)(x)$. Then

$$\begin{aligned} \psi_*(fh)(p_2) &= \psi_*(fh - \lambda h)(p_2) + \psi_*(\lambda h)(p_2) \\ &= \psi_*((f - \lambda)h)(p_2) + \lambda \psi_*(h)(p_2). \end{aligned}$$

Let $x' = \bar{\psi}^{-1}(x)$ and let $p_1 \in \mathcal{P}_{1x'}$. Then $(f - \lambda)h(p_1) = (f(x') - \lambda)h(p_1) = 0$ by choice of λ . Thus $(f - \lambda)h \in \mathcal{H}_{1x'}$ and so $\psi_*((f - \lambda)h) \in \mathcal{H}_{2x}$ so $\psi_*((f - \lambda)h)(p_2) = 0$. Thus

$$\psi_*(fh)(p_2) = \lambda\psi_*(h)(p_2) = (\bar{\psi}_*(f) \cdot \psi_*(h))(p_2)$$

as desired.

LEMMA 2.2. *The map $\bar{\psi}: X \rightarrow X$ is a C^r -diffeomorphism.*

Proof. We need only show $\bar{\psi}^{-1}$ is C^r . It is enough to show that if $f \in \mathcal{F}(X)$ then $f \circ \bar{\psi}^{-1}$ is C^r . Choose $x_0 \in X$, U a neighborhood of x_0 , \mathcal{P}_2 trivial over U . Then let V be a neighborhood of x_0 with $\bar{V} \subset U$. Let k be a section of \mathcal{A}_2 over U which in the local trivialization has constant principal part. We can then cut k down to get a new section, again called k , defined on all of X and agreeing with the original k on V . Then choose $h \in \Gamma(\mathcal{A}_1)$ such that $\psi_*(h) = k$. (We are identifying \mathcal{H}_i and $\Gamma(\mathcal{A}_i)$.) Now by Lemma we have $\psi_*(fh) = (f \circ \bar{\psi}^{-1})\psi_*(h) = (f \circ \bar{\psi}^{-1})k$. When we view the C^r -section $(f \circ \bar{\psi}^{-1})k$ in our local trivialization we conclude $f \circ \bar{\psi}^{-1}$ is C^r on V . So we conclude $f \circ \bar{\psi}^{-1}$ is C^r and hence $\bar{\psi}^{-1}$ is C^r .

We now define a bundle isomorphism $\tilde{\psi}$ such that the following commutes:

$$\begin{array}{ccc} \mathcal{A}_1 & \xrightarrow{\tilde{\psi}} & \mathcal{A}_2 \\ \downarrow & & \downarrow \\ X & \xrightarrow{\tilde{\psi}} & X \end{array}$$

Let $\alpha_x \in \mathcal{A}_{1x}$. Choose a section $h \in \Gamma(\mathcal{A}_1)$ such that $h(x) = \alpha_x$. Define $\tilde{\psi}(\alpha_x)$ by $\tilde{\psi}(\alpha_x) = \psi_*(h)(\bar{\psi}(x))$. This is independent of the choice of h for if h_1 were another section with $h_1(x) = \alpha_x$ then $h - h_1$ vanishes at x . Hence $\psi_*(h - h_1)$ vanishes at $\bar{\psi}(x)$ so $\psi_*(h)(\bar{\psi}(x)) = \psi_*(h_1)(\bar{\psi}(x))$. It is clear that the diagram commutes and that $\tilde{\psi}$ mapping \mathcal{A}_{1x} to $\mathcal{A}_{2\bar{\psi}(x)}$ is a Lie algebra isomorphism.

LEMMA 2.3. *$\tilde{\psi}$ is C^r .*

Proof. We work locally trivializing \mathcal{A}_1 . Let U be open in X , $V \subset U$ also open, $\gamma: U \times \mathbb{R}^m \rightarrow \mathcal{A}_1|_U$ be a trivialization of \mathcal{A}_1 over U . Using this we see there are C^r -sections $h_1, \dots, h_m \in \Gamma(\mathcal{A}_1)$ such that for each x in the subset V , $h_1(x), \dots, h_m(x)$ give a basis for the fiber over x which corresponds to the standard basis of \mathbb{R}^m under γ . We claim

$\tilde{\psi} \circ \gamma : V \times R^m \rightarrow \mathcal{A}_2$ is given by

$$\tilde{\psi} \circ \gamma (x, \xi^1, \dots, \xi^m) = \sum_{i=1}^m \xi^i \psi_*(h_i)(\bar{\psi}(x)).$$

If so then $\tilde{\psi}$ is C^r . But given ξ^1, \dots, ξ^m choose $f^i \in \mathcal{F}(X)$, $f^i(x) = \xi^i$. Then by Lemma 2.1 we see

$$\begin{aligned} \tilde{\psi}(\gamma(x, \xi^1, \dots, \xi^m)) &= \tilde{\psi}\left(\sum_{i=1}^m \xi^i h_i(x)\right) = \tilde{\psi}\left(\left(\sum_{i=1}^m f^i h_i\right)(x)\right) \\ &= \psi_*\left(\sum_{i=1}^m f^i h_i\right)(\bar{\psi}(x)) \\ &= \sum_{i=1}^m \bar{\psi}_*(f^i)(\bar{\psi}(x)) \psi_*(h_i)(\bar{\psi}(x)) \\ &= \sum_{i=1}^m \xi^i \psi_*(h_i)(\bar{\psi}(x)). \end{aligned}$$

Let $p \in \mathcal{P}_{1x}$. Then $\phi_p^1: \mathcal{H} \rightarrow \mathcal{A}_{1x}$ is a Lie algebra isomorphism. If $q \in \mathcal{A}_{2\bar{\psi}(x)}$ then we have a Lie algebra isomorphism $\phi_q^2: \mathcal{H} \rightarrow \mathcal{A}_{2\bar{\psi}(x)}$. (Note the superscripts tell which bundle is being used).

Now $(\phi_q^2)^{-1} \circ \tilde{\psi} \circ \phi_p^1: \mathcal{H} \rightarrow \mathcal{H}$ lies in $\text{Aut}(\mathcal{H})$. Let $\mathcal{E} = \{(p, q) \mid p \in \mathcal{P}_{1x} \text{ and } q \in \mathcal{P}_{2\bar{\psi}(x)} \text{ for some } x \in X\}$. \mathcal{E} is the total space of the fiber product of \mathcal{P}_1 and $\bar{\psi}^* \mathcal{P}_2$. We have a map $\rho: \mathcal{E} \rightarrow \text{Aut}(\mathcal{H})$, $\rho(p, q) = (\phi_q^2)^{-1} \circ \tilde{\psi} \circ \phi_p^1$. ρ is continuous and \mathcal{E} is connected so ρ takes values in one of the connected components of $\text{Aut}(\mathcal{H})$. Since K is a simple group the identity component of $\text{Aut}(\mathcal{H})$ is $\text{Aut}^\circ(\mathcal{H}) = \text{Ad}(K)$. Suppose $\sigma \in \text{Aut}(\mathcal{H})$ and that $\rho(\mathcal{E}) \subset \text{Aut}^\circ(\mathcal{H}) \sigma = \text{Ad}(K) \sigma$. Let $q \in \mathcal{P}_2$, $k \in K$. Then $\phi_{qk}^2 = \phi_q^2 \circ \text{Ad}(k)$. So $\rho(p, qk) = \text{Ad}(k^{-1}) \circ \rho(p, q)$. We conclude that for each $p \in \mathcal{P}_{1x}$ there is a unique $\mu(p)$ in $\mathcal{P}_{2\bar{\psi}(x)}$ for which $\rho(p, \mu(p)) = \sigma$. We then have a map $\mu: \mathcal{P}_1 \rightarrow \mathcal{P}_2$ covering $\bar{\psi}$. K acts freely on the right of both \mathcal{P}_1 and \mathcal{P}_2 . We now show there is an automorphism $\bar{\sigma}$ of K , induced by σ , such that if a new action of K on \mathcal{P}_2 is defined by $q * k = q \bar{\sigma}(k)$, (the right side being the original action) then μ becomes K -equivariant. We have $\sigma \in \text{Aut}(\mathcal{H})$. $\tau \rightarrow \sigma \tau \sigma^{-1}$ is an automorphism of $\text{Aut}(\mathcal{H})$ and hence restricts to an automorphism of $\text{Aut}^\circ(\mathcal{H}) = \text{Ad}(K)$. Using the isomorphism $\text{Ad}: K \rightarrow \text{Ad}(K)$ we see a unique automorphism $\bar{\sigma}$ is induced. $\bar{\sigma}$ satisfies the equation $\text{Ad}(\bar{\sigma}(k)) = \sigma \text{Ad}(k) \sigma^{-1}$. Now we show $\mu(pk) = \mu(p) * k$ for $p \in \mathcal{P}_1$, $k \in K$. We need only show $\rho(pk, \mu(p) * k) = \sigma$. But

$$\begin{aligned} \rho(pk, \mu(p) * k) &= \rho(pk, \mu(p) \bar{\sigma}(k)) = \text{Ad}(\bar{\sigma}(k))^{-1} \circ \rho(p, \mu(p)) \circ \text{Ad}(k) \\ &= \text{Ad}(\bar{\sigma}(k))^{-1} \circ \sigma \circ \text{Ad}(k) = \sigma \text{Ad}(k)^{-1} \sigma^{-1} \sigma \text{Ad}(k) = \sigma \end{aligned}$$

so we are done.

DEFINITION 2.4. Let $\pi: \mathcal{P} \rightarrow X$ be a principal K -bundle, τ an automorphism of K . The principal K -bundle $\pi^\tau: \mathcal{P}^\tau \rightarrow X$ is defined by introducing the new action $*$: $\mathcal{P} \times K \rightarrow \mathcal{P}$, $p * k = p\tau(k)$. We say π^τ is conjugate to π by τ .

Considering the previous discussion we have now proved

THEOREM 2.5. Under the assumptions made above if $\psi: H_1 \rightarrow H_2$ is a C^1 isomorphism then there is a C^r -diffeomorphism $\bar{\psi}: X \rightarrow X$ and an automorphism $\bar{\sigma}$ of K such that $\pi_1 \cong \bar{\psi}^*(\pi_2^{\bar{\sigma}})$.

REMARK. Of course if $\bar{\sigma}$ is an inner automorphism we get $\pi_2^{\bar{\sigma}} \cong \pi_2$ and $\bar{\sigma}$ can be dropped.

3. Classical groups. We apply the results of §2 to the groups $SO(2n+1)$ $n \geq 1$, $U(n)$ $n \geq 2$, and $SO(2n)$ $n \geq 3$. Since the center of $SO(2n+1)$ is trivial and the automorphism group of its Lie algebra is connected [6, pages 285–6] we get

THEOREM 3.1. Let $\pi_i: \mathcal{P}_i \rightarrow X$ be principal $SO(2n+1)$ bundles with gauge groups H_i , $i = 1, 2$. Suppose $\psi: H_1 \rightarrow H_2$ is a C^1 (local) isomorphism. Then there is a C^r -diffeomorphism $\bar{\psi}: X \rightarrow X$ so that $\pi_1 \cong \bar{\psi}^*(\pi_2)$.

Now let K be $SO(2n)$ $n \geq 3$ or $U(n)$ $n \geq 2$, $\pi_i: \mathcal{P}_i \rightarrow X$ be principal K bundles with gauge groups H_i and $\psi: H_1 \rightarrow H_2$ a C^r local isomorphism. Let Z denote the center of K . Now $\hat{\mathcal{P}}_i = \mathcal{P}_i/Z$ is a principal K/Z bundle over X . Let \hat{H}_i be the gauge group of $\hat{\mathcal{P}}_i$. In both cases ($SO(2n)$ and $U(n)$) one can show that the Lie algebra isomorphism $\psi_*: \mathcal{H}_1 \rightarrow \mathcal{H}_2$ gives Lie algebra isomorphism $\hat{\psi}_*: \hat{\mathcal{H}}_1 \rightarrow \hat{\mathcal{H}}_2$ and also that the center of K/Z is trivial. Thus the results of §2 give a C^r diffeomorphism $\phi: X \rightarrow X$ and an automorphism σ of K/Z so that $\hat{\pi}_1 \cong \phi^*(\hat{\pi}_2^\sigma)$. Note that if σ is an inner automorphism $\hat{\pi}_2^\sigma \cong \hat{\pi}_2$ so that σ can be dropped. The form of σ not inner is given in [6, page 287]. It can be seen that σ lifts to $\sigma: K \rightarrow K$ and that $(\mathcal{P}_i/Z)^\sigma = \mathcal{P}_i^\sigma/Z$. We thus get

THEOREM 3.2. Let K be $SO(2n)$ $n \geq 3$ or $U(n)$ $n \geq 2$, $\pi_i: \mathcal{P}_i \rightarrow X$ be principal K bundles with gauge groups H_i , $i = 1, 2$. Suppose $\psi: H_1 \rightarrow H_2$ is a (local) C^r isomorphism. Then there is a C^r diffeomorphism $\bar{\psi}: X \rightarrow X$ and automorphism $\sigma: K \rightarrow K$, so that $\mathcal{P}_1/Z \cong \bar{\psi}^*(\mathcal{P}_2/Z)^\sigma \cong \bar{\psi}^*(\mathcal{P}_2^\sigma)/Z$ where Z is the center of K .

One can show that \mathcal{P}_1 is a “tensor product” of $\bar{\psi}^*(\mathcal{P}_2^\sigma)$ with a

principal Z -bundle over X . One way to see this is to use the classification for bundles as given in [4]. We state the result in terms of associated vector bundles.

THEOREM 3.3. *Let $\pi_i: \mathcal{P}_i \rightarrow X$ be principal $SO(2n)$ $n \geq 3$ ($U(n)$ $n \geq 2$) bundles with gauge groups H_i , $i = 1, 2$. Let ξ_i be the real (complex) vector bundle associated with \mathcal{P}_i using the usual representation of $SO(2n)(U(n))$. Suppose $\psi: H_1 \rightarrow H_2$ is a (local) C^1 -isomorphism then there is a C^1 diffeomorphism $\bar{\psi}: X \rightarrow X$, σ an automorphism of $SO(2n)(U(n))$, and η a real (complex) line bundle so that ξ_1 is $SO(2n)(U(n))$ isomorphic to $\psi^*(\xi_2^\sigma) \otimes \eta$.*

Final remark. We need not have assumed that \mathcal{P}_1 and \mathcal{P}_2 were bundles over the same manifold X . We could have considered $\pi_1: \mathcal{P}_1 \rightarrow X$ and $\pi_2: \mathcal{P}_2 \rightarrow Y$. If the gauge groups H_1 and H_2 are (locally) C^1 isomorphic we get a C^1 -diffeomorphism $\bar{\psi}: X \rightarrow Y$.

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Pacific Journal of Mathematics

Vol. 68, No. 2

April, 1977

William Allen Adkins, Aldo Andreotti and John Vincent Leahy, <i>An analogue of Oka's theorem for weakly normal complex spaces</i>	297
Ann K. Boyle, M. G. Deshpande and Edmund H. Feller, <i>On nonsingularly k-primitive rings</i>	303
Rolando Basim Chuaqui, <i>Measures invariant under a group of transformations</i>	313
Wendell Dan Curtis and Forrest Miller, <i>Gauge groups and classification of bundles with simple structural group</i>	331
Garret J. Etgen and Willie Taylor, <i>The essential uniqueness of bounded nonoscillatory solutions of certain even order differential equations</i>	339
Paul Ezust, <i>On a representation theory for ideal systems</i>	347
Richard Carl Gilbert, <i>The deficiency index of a third order operator</i>	369
John Norman Ginsburg, <i>S-spaces in countably compact spaces using Ostaszewski's method</i>	393
Basil Gordon and S. P. Mohanty, <i>On a theorem of Delaunay and some related results</i>	399
Douglas Lloyd Grant, <i>Topological groups which satisfy an open mapping theorem</i>	411
Charles Lemuel Hagopian, <i>A characterization of solenoids</i>	425
Kyong Taik Hahn, <i>On completeness of the Bergman metric and its subordinate metrics. II</i>	437
G. Hochschild and David Wheeler Wigner, <i>Abstractly split group extensions</i>	447
Gary S. Itzkowitz, <i>Inner invariant subspaces</i>	455
Jiang Luh and Mohan S. Putcha, <i>A commutativity theorem for non-associative algebras over a principal ideal domain</i>	485
Donald J. Newman and A. R. Reddy, <i>Addendum to: "Rational approximation of e^{-x} on the positive real axis"</i>	489
Akio Osada, <i>On the distribution of a-points of a strongly annular function</i>	491
Jeffrey Lynn Spielman, <i>A characterization of the Gaussian distribution in a Hilbert space</i>	497
Robert Moffatt Stephenson Jr., <i>Symmetrizable-closed spaces</i>	507
Peter George Trotter and Takayuki Tamura, <i>Completely semisimple inverse Δ-semigroups admitting principal series</i>	515
Charles Irvin Vinsonhaler and William Jennings Wickless, <i>Torsion free abelian groups quasi-projective over their endomorphism rings</i>	527
Frank Arvey Wattenberg, <i>Topologies on the set of closed subsets</i>	537
Richard A. Zalik, <i>Integral representation of Tchebycheff systems</i>	553