

Pacific Journal of Mathematics

**S-SPACES IN COUNTABLY COMPACT SPACES USING
OSTASZEWSKI'S METHOD**

JOHN NORMAN GINSBURG

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A method adapted from that used by A. J. Ostaszewski is used to construct S -spaces as subspaces of given spaces. Assuming the set-theoretic principle \diamond , it is shown that every countably compact space containing no nontrivial convergent sequences contains a perfect S -space. As a corollary, assuming \diamond , if X is a countably compact F -space, then X contains a hereditarily extremally disconnected, hereditarily normal, perfect S -space.

1. Introduction. The set-theoretic principle \diamond , due to Jensen [3], has found many interesting applications in topology, particularly the construction of Souslin lines and various S -spaces. The basic technique for constructing S -spaces from \diamond is due to A. J. Ostaszewski [6], and has been modified and applied in constructing other interesting topological spaces, notably in [5] and [8]. Roughly speaking, the method involves constructing a space having desired properties by defining its topology inductively over more and more of the space (and in some cases refining a given topology) using some principle of enumeration.

Here we will show how the method can be used to construct S -spaces as subspaces of given spaces. That is, rather than building up a space by inductively defining its topology, the desired examples will be obtained by working within a given topological space and extracting a subspace.

Our principal topological references are [2], [7] and [10]. For set-theoretic notions we refer to [4].

For the reader's convenience we now recall a few notions from topology which we will employ.

A space X is an S -space if X is regular, hereditarily separable and not Lindelöf.

X is *countably compact* if every countable covering of X by open sets has a finite subcover.

For a completely regular space X , βX denotes the Stone-Čech compactification of X .

A subset A of X is *C^* -embedded* in X if every bounded, continuous real-valued function on A admits a continuous extension to X . A *cozero-set* in X is a set of the form $\{p \in X: f(p) \neq 0\}$ where f is a continuous real-valued function on X . X is an F -space if X is com-

pletely regular and every cozero-set in X is C^* -embedded in X . A completely regular space X is *extremally disconnected* if the closure of every open subset of X is open.

For the basic information on F -spaces and extremally disconnected spaces, the reader is referred to [2] and [10]. We will make use of the following two facts, established in 1.62 and 1.64 of [10].

1.1. If X is σ -compact and locally compact, then $\beta X - X$ is a compact F -space.

1.2. If X is an F -space then every countable subspace of X is C^* -embedded in X .

For the consistency of \diamond with the axioms of set theory the reader is referred to [3]. We will not need a precise statement of \diamond , rather we will use the following consequence of \diamond derived in [6].

1.3. Let $\lim \omega_1$ denote the set of limit ordinals less than ω_1 . Then there is a family $\{S_\gamma : \gamma \in \lim \omega_1\}$ of subsets of ω_1 such that each S_γ is a cofinal subset of γ and such that for every uncountable subset S of ω_1 there is a $\gamma \in \lim \omega_1$ with $S_\gamma \subseteq S$.

It is clear we may assume that each S_γ is a simple ω -sequence increasing to γ in 1.3. This is the form in which we will apply 1.3. (the conclusion of 1.3 is often referred to as “club”; see [7])

2. S -subspaces of countably compact spaces. We now assume the conclusion of 1.3. This assumption will enable us to construct S -spaces in certain countably compact spaces. It is apparently not yet known whether 1.3 is equivalent to \diamond or whether it is strictly weaker. It is known that \diamond is equivalent to the conjunction of 1.3 and the continuum hypothesis, and so this question amounts to whether or not 1.3 implies the continuum hypothesis. (see [7])

All hypothesized spaces are assumed to be infinite.

2.1. THEOREM. *If X is a regular, countably compact Hausdorff space containing no nontrivial convergent sequences, then X contains a perfect S -space.*

Proof. Let $\{S_\gamma : \gamma \in \lim \omega_1\}$ satisfy 1.3 where each S_γ is an ω -sequence increasing to γ . Let X satisfy the hypotheses of the theorem. We inductively select points $(x_\xi : \xi \in \omega_1)$ in X , and open sets $(G_\xi : \xi \in \omega_1)$ in X so that

- (i) for all ξ , $x_\xi \in G_\xi$
- (ii) $\xi < \eta \rightarrow x_\eta \notin G_\xi$
- (iii) for all limit ordinals γ and all $n \in \omega$, $x_{\gamma+n} \in \text{cl}\{x_\xi : \xi \in S_\gamma\}$.

To get the desired sequences $(x_\xi: \xi \in \omega_1)$ and $(G_\xi: \xi \in \omega_1)$ we construct $(x_\xi: \xi < \gamma)$ and $(G_\xi: \xi < \gamma)$ by induction on the limit ordinal γ . To start the construction, we choose a countable discrete subset $(x_n: n \in \omega)$ of X , (X is assumed infinite), and a sequence of open sets $(G_n: n \in \omega)$ in X such that $x_n \in G_n$ and $m \neq n \rightarrow x_m \notin G_n$.¹

Now suppose $\sigma \in \lim \omega_1$ and for every limit ordinal $\gamma < \sigma$ we have chosen the sequences $(x_\xi: \xi < \gamma)$ and $(G_\xi: \xi < \gamma)$ satisfying (i), (ii), and (iii). If σ is a limit of limits, we simply gather together all the x_ξ 's and G_ξ 's previously constructed to form $(x_\xi: \xi < \sigma)$ and $(G_\xi: \xi < \sigma)$, clearly satisfying (i), (ii), and (iii). So we need only consider the case where $\sigma = \gamma + \omega$ for some limit ordinal γ . Thus, having the sequences $(x_\xi: \xi < \gamma)$ and $(G_\xi: \xi < \gamma)$ we must define the points $(x_{\gamma+n}: n \in \omega)$ and the open sets $(G_{\gamma+n}: n \in \omega)$. Consider the infinite set $R_\gamma = \{x_\xi: \xi \in S_\gamma\}$. Since X is countably compact, every countable subset of X has a limit point in X . But since X contains no nontrivial convergent sequences, every countable set has infinitely many (in fact uncountably many) limit points. Thus $\text{cl} R_\gamma - R_\gamma$ is infinite, and so contains a countable discrete subspace $(x_{\gamma+n}: n \in \omega)$. Choose a sequence of open sets $(G_{\gamma+n}: n \in \omega)$ which witnesses this discreteness, that is, with $x_{\gamma+n} \in G_{\gamma+n}$ and such that $m \neq n \rightarrow x_{\gamma+m} \notin G_{\gamma+n}$.

We now check (i), (ii), and (iii) for $(x_\xi: \xi < \gamma + \omega)$ and $(G_\xi: \xi < \gamma + \omega)$. (i) is clear, as is (iii), by virtue of the induction hypothesis and the selection of the points $x_{\gamma+n}$ in $\text{cl} R_\gamma$. To verify (ii), because of the induction hypothesis and the choice of $(x_{\gamma+n}: n \in \omega)$ and $(G_{\gamma+n}: n \in \omega)$, it is sufficient to check the following:

If $\xi < \gamma$ and $n \in \omega$, then $x_{\gamma+n} \notin G_\xi$. But S_γ is an ω -sequence increasing to γ , and so there are at most finitely many ordinals in S_γ which are less than ξ . By property (ii) of the induction hypothesis, this means there are at most finitely many x_η with $\eta \in S_\gamma$ which lie in G_ξ . But $x_{\gamma+n}$ is a limit point of R_γ , so every neighborhood of $x_{\gamma+n}$ contains infinitely many x_η with $\eta \in S_\gamma$. In particular, $x_{\gamma+n} \notin G_\xi$.

This completes the inductive construction, and results in sequences $(x_\xi: \xi \in \omega_1)$ and $(G_\xi: \xi \in \omega_1)$ satisfying (i), (ii), and (iii).

We now claim that $Y = \{x_\xi: \xi \in \omega_1\}$ is a perfect S -space. The verification of this is essentially identical with the argument given in [6], so we will be content to sketch that argument here. That Y is not Lindelöf is immediate from (ii) and (i). Any countable subspace of Y is separable, and if $\{x_\xi: \xi \in S\}$ is an uncountable subspace of Y , there is, by 1.3, a $\gamma \in \lim \omega_1$ such that $S_\gamma \subseteq S$. Using (iii) we see that $\{x_\xi: \xi \in S \text{ and } \xi < \gamma\}$ is a countable dense subset of $\{x_\xi: \xi \in S\}$. This proves Y is hereditarily separable. Since $\gamma < \eta \rightarrow x_\eta \in \text{cl}\{x_\xi: \xi \in S_\gamma\}$, the same ar-

¹ The fact that every infinite Hausdorff space contains a countably infinite discrete subspace is well-known and easy to prove. A proof may be found in 0.13 of [2].

gument shows that every closed subset of Y is either countable or co-countable, from which it is immediate that every closed subset of Y is a G_δ in Y , that is, Y is perfect.

2.2. COROLLARY. *If X is a countably compact F -space then X contains a hereditarily extremally disconnected, hereditarily normal, perfect S -space.*

Proof. Using 1.2 it is easy to see there are no nontrivial convergent sequences in an F -space, so the hypotheses of 2.1 apply. We show that the S -space Y obtained in 2.1 is hereditarily extremally disconnected and hereditarily normal under the present assumptions on X . Now, as is well-known, a space is extremally disconnected if and only if each of its open subsets is C^* -embedded (see 1H in [2]), and a space is normal if and only if each of its closed subsets is C^* -embedded (see 3D in [2]). So to verify that Y is normal and extremally disconnected hereditarily, it is sufficient to prove that every subspace of Y is C^* -embedded in Y . So, let $Z \subseteq Y$, and let f be a bounded, continuous real-valued function on Z . Since Y is hereditarily separable, Z contains a countable dense subset D . By 1.2, D is C^* -embedded in X , and so the function $f|_D$ admits a continuous extension F to all of X . Clearly $F|_Y$ is the desired extension of f .

REMARK. 2.3. There is a large number of spaces to which these results can be applied. One class of such spaces is furnished by 1.1. So assuming 1.3 we see for example that $\beta\mathbf{R} - \mathbf{R}$ and $\beta N - N$ contain interesting S -spaces.

REMARK. 2.4. The fact that \diamond implies the existence of S -spaces which are extremally disconnected was previously observed by M. Wage [9]. Wage's construction, like Ostaszewski's original method, involves inductively defining a topology to get the desired example.

One significant difference between the S -spaces obtained in 2.2 and the original S -space described in [6] is countable compactness. The S -space in [6] is, in addition, countably compact, while the S -spaces in 2.2 are never countably compact. If CH is true this follows from the results in [11] which imply that, assuming CH , every countably compact, separable normal F -space is compact, and therefore Lindelöf. If CH is false, we argue as follows: A slight modification of the argument in [1] shows that a countably compact space of cardinality $< c$ is sequentially compact. Since our S -spaces have cardinality ω_1 and contain no convergent sequences, they cannot be countably compact if CH fails either. Thus our S -spaces constructed using 1.3 are not countably compact.

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Received November 29, 1976 and in revised form January 19, 1977.

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WINNIPEG, MANITOBA, CANADA

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The *Pacific Journal of Mathematics* is issued monthly as of January 1966. Regular subscription rate: \$ 72.00 a year (6 Vols., 12 issues). Special rate: \$ 36.00 a year to individual members of supporting institutions.

Subscriptions, orders for back numbers, and changes of address should be sent to *Pacific Journal of Mathematics*, 103 Highland Boulevard, Berkeley, California, 94708.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION
Printed at Jerusalem Academic Press, POB 2390, Jerusalem, Israel.

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