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ON A THEOREM OF DELAUNAY AND SOME RELATED RESULTS

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Dedicated to the memory of Professor T. S. Motzkin

Delaunay has proved that if $\epsilon = ap\phi^2 + bp\phi + c$ is a unit in the ring $Z[\theta]$, where $\theta^3 - P\theta^2 + Q\theta - R = 0$, p is an odd prime, $\phi = p^t\theta$, $t \geqq 0$ and $p \nmid a$, then no power ϵ^m (m positive) can be a binorm, i.e. $\epsilon^m = u + v\theta$ is impossible for m a positive integer. Hemer has pointed out that in the above situation, $\epsilon^m = u + v\theta$ is also impossible for m a negative integer.

In this paper the above result is extended as follows.

THEOREM 1. *If $\epsilon = a\theta^2 + b\theta + c$ is a unit in $Z[\theta]$, where $\theta^3 = d\theta^2 + e\theta + f$ and $p^\alpha \| a$, $p^\beta \| b$, p being a prime, then $\epsilon^n = u + v\theta$ is impossible for $n \neq 0$ in the following cases:*

(i) *When $1 \leqq \alpha \leqq \beta$ and p is odd,*
(ii) *When $2 \leqq \alpha \leqq \beta$ and $p = 2$,*
(iii) *When $\beta \leqq \alpha < 2\beta$ and p is odd,*
(iv) *When $\beta \leqq \alpha < 2\beta - 1$ and $p = 2$.*

As an application of this and some other similar theorems, all integer solutions of the equation $y^2 = x^3 + 113$ are determined.

First we prove two simple lemmas.

LEMMA 2. *If $p^\alpha \| \binom{n}{p^q}$ then $p^\alpha \mid \binom{n}{i}$, where the prime p satisfies $p^q < i < p^{q+1}$ and $p^{\alpha-1}$ a $\binom{n}{p^{q+1}}$. Furthermore if $p \mid n$ and $p \nmid i$ then $p^{\alpha+1} \mid \binom{n}{i}$.*

Proof. Let $i = p^q + r$. Then $0 < r < p^{q+1} - p^q$. Hence

$$\binom{n}{i} = \binom{n}{p^q}\binom{n-p^q}{r}\frac{r!}{\prod_{j=1}^{r}(p^q + j)}.$$

Since $\Pi_{j=1}^r (p^q + j)/r!$ is an integer not divisible by p and $p^\alpha \| \binom{n}{p^q}$, we have $p^\alpha \mid \binom{n}{i}$.

If $p \mid n$ and $p \nmid i$ then $p \nmid r$ for $i = p^q + r$. Then

$$\binom{n-p^q}{r} = \left(\frac{n-p^q}{r}\right)\binom{n-p^q-1}{r-1}$$

is divisible by p. Hence $p^{\alpha+1} \mid \binom{n}{i}$.

Again from

$$\binom{n}{p^{q+1}} = \binom{n}{p^q}\binom{n-p^q}{p^{q+1}-p^q}\frac{s!}{\prod_{j=1}^{s}(p^{q+1}-j)}\binom{p^{q+1}-p^q}{p^{q+1}},$$

where $s = p^{q+1} - p^q - 1$, we see that $p^{\alpha-1} \mid \binom{n}{p^{q+1}}$, and the lemma is proved.

LEMMA 3. *Let $\epsilon = a\theta^2 + b\theta + c$ be a unit in $Z[\theta]$, where $\theta^3 = d\theta^2 + e\theta + f$, and $\epsilon^{-1} = a'\theta^2 + b'\theta + c'$. If $p^\alpha \| a$, $p^\beta \| b$, where p is a prime and $\alpha\beta \neq 0$, then $p^\alpha \| a'$ and $p^\beta \| b'$ in the following cases:*

(i) $\alpha \leqq \beta < 2\alpha$

(ii) $\beta \leqq \alpha < 2\beta$

For $\alpha \leqq \beta$ we have $p^\alpha \| a'$ and $p^\alpha \mid b'$.

Proof. Since $(a\theta^2 + b\theta + c)(a'\theta^2 + b'\theta + c') = 1$, we have,

(1) $$aa'd^2 + ab'd + a'bd + aa'e + ac' + ca' + bb' = 0,$$

(2) $$aa'f + aa'de + ab'e + a'be + bc' + b'c = 0,$$

and

(3) $$aa'df + ab'f + a'bf + cc' = 1.$$

From (3) it follows that $p \nmid c'$.

Case (i). From (1) we have $ca' \equiv 0 \pmod{p^\alpha}$ as $\alpha \leqq \beta$. Since $p \nmid c$ we get $a' \equiv 0 \pmod{p^\alpha}$. From (2) we obtain $b'c \equiv 0 \pmod{p^\alpha}$ for $\alpha \leqq \beta$, whence $b' \equiv 0 \pmod{p^\alpha}$. If $\beta < 2\alpha$, then (2) gives $b'c \equiv 0 \pmod{p^\beta}$, or $b' \equiv 0 \pmod{p^\beta}$. If $p^{\alpha+1} \mid a'$, then from (1) we have $ac' \equiv 0 \pmod{p^{\alpha+1}}$. Since $p \nmid c'$ we get $a \equiv 0 \pmod{p^{\alpha+1}}$, a contradiction. Hence $p^\alpha \| a'$. Similarly if $p^{\beta+1} \mid b'$, then from (2) we get $bc' \equiv 0 \pmod{p^{\beta+1}}$ when $\beta < 2\alpha$. Again we arrive at a contradiction since $p \nmid c'$ and $p^\beta \| b$. Hence $p^\beta \| b'$.

Case (ii). Since $\beta \leqq \alpha$, (2) yields $b'c \equiv 0 \pmod{p^\beta}$. Then we have $b' \equiv 0 \pmod{p^\beta}$ for $p \nmid c$. Using $\alpha < 2\beta$, we get $a'(bd + c) \equiv 0 \pmod{p^\alpha}$ from (1). Then $a' \equiv 0 \pmod{p^\alpha}$ as $p \nmid (bd + c)$. If $b' \equiv$

$0 \pmod{p^{\beta+1}}$, then from (2) we see that $bc' \equiv 0 \pmod{p^{\beta+1}}$, a contradiction. Hence $p^\beta \| b'$. If $a' \equiv 0 \pmod{p^{\alpha+1}}$ we have from (1) $ac' + bb' \equiv 0 \pmod{p^{\alpha+1}}$. We get a contradiction for $\alpha < 2\beta$. Hence $p^\alpha \| a'$.

Proof of Theorem 1. Let $n > 0$. Case (i) and (ii). Let $1 \leqq \alpha \leqq \beta$. Since ϵ is a unit, $p \nmid c$. Moreover $\epsilon = a\theta^2 + b\theta + c = p^\alpha(r\theta^2 + s\theta) + c$ where $p \nmid r$. Let $(r\theta^2 + s\theta)^i = a_i\theta^2 + b_i\theta + c_i$, with a_i, b_i and c_i rational integers. Then

$$\epsilon^n = (a\theta^2 + b\theta + c)^n = [c + p^\alpha(r\theta^2 + s\theta)]^n = c^n + \binom{n}{1} c^{n-1}p^\alpha(r\theta^2 + s\theta)$$

$$+ \binom{n}{2} c^{n-2}p^{2\alpha}(a_2\theta^2 + b_2\theta + c_2) + \cdots + p^{n\alpha}(a_n\theta^2 + b_n\theta + c_n) = u + v\theta.$$

Comparing the coefficients of θ^2, we have

$$nc^{n-1}p^\alpha r + \binom{n}{2} c^{n-2}p^{2\alpha}a_2 + \cdots + p^{n\alpha}a_n = 0. \tag{4}$$

If p is an odd prime, we see using Lemma 2 that the first term of (4) is divisible by a lower power of p than the others. If $p = 2$ and $\alpha \geqq 2$ the same conclusion holds. Hence (4) can never be satisfied. So ϵ^n can never be of the form $u + v\theta$ in these cases.

Cases (iii) *and* (iv). Now $\epsilon = p^\beta(r\theta^2 + s\theta) + c$, where $p^{\alpha-\beta} \| r$. Then the coefficient of θ^2 in $\epsilon^n = [c + p^\beta(r\theta^2 + s\theta)]^n$ is

$$nc^{n-1}p^\beta r + \binom{n}{2} c^{n-2}p^{2\beta}a_2 + \cdots + p^{n\beta}a_n, \tag{5}$$

where $(r\theta^2 + s\theta)^i = a_i\theta^2 + b_i\theta + c_i$ with a_i, b_i and c_i rational integers. Again using Lemma 2 and the fact that $\alpha < 2\beta$, we see that the first term of (5) is divisible by a lower power of p than the others if p is an odd prime.

In case $p = 2$ and $\alpha < 2\beta - 1$ the same conclusion holds. Hence (5) can never be zero, i.e. $\epsilon^n = u + v\theta$ is impossible. This proves the theorem for $n > 0$.

We next consider $\epsilon^n = u + v$ for $n < 0$.

Let $n = -m$ and $\epsilon^{-1} = a'\theta^2 + b'\theta + c'$. Then we have $\epsilon^n = (\epsilon^{-1})^m = (a'\theta^2 + b'\theta + c')^m$ where $m > 0$. From Lemma 3, we see that $p^\alpha \| a'$, $p^\alpha | b'$ for $\alpha \leqq \beta$, and $p^\alpha \| a'$, $p^\beta \| b'$ for $\beta \leqq \alpha < 2\beta - 1$, $\alpha \leqq \beta < 2\alpha$ and $\beta \leqq \alpha < 2\beta$. Hence $(a'\theta^2 + b'\theta + c')^m = u + v\theta$ is impossible for $m > 0$. Combining these results we see that $\epsilon^n = u + v\theta$ is impossible for $n \neq 0$, and the theorem is proved.

We note that if the conditions of Theorem 1 are not fulfilled, then $\epsilon^n = u + v\theta$ is possible for $n > 3$; examples are given in [**2**, page 417].

Very often the following theorem is useful.

THEOREM 4. *Let $\epsilon = a_1\theta^2 + b_1\theta + c_1$ be a unit in $Z[\theta]$, where $\theta^3 - p_1\theta - q_1 = 0$. If $p_1 \equiv 0 \pmod 3$, then*

$$\epsilon^n = u + v\theta \tag{6}$$

is impossible for $n \neq 0$ provided $a_1 \not\equiv 0 \pmod 3$, $b_1^2 + 2a_1c_1 \not\equiv 0 \pmod 3$, and $b_1^2c_1 + a_1c_1^2 + a_1^2b_1q_1 \not\equiv 0 \pmod 3$.

Proof. Let $\epsilon^n = a_n\theta^2 + b_n\theta + c_n$. Then we have

$$a_{n+1} = a_n(a_1p_1 + c_1) + b_nb_1 + c_na_1,$$

$$b_{n+1} = a_n(a_1q_1 + b_1p_1) + b_n(c_1 + a_1p_1) + c_nb_1,$$

and

$$c_{n+1} = a_nb_1q_1 + b_na_1q_1 + c_nc_1.$$

Hence we get $a_2 = a_1^2p_1 + b_1^2 + 2a_1c_1$, $b_2 = a_1^2q_1 + 2b_1c_1 + 2a_1b_1p_1$, and $c_2 = c_1^2 + 2a_1b_1q_1$. Then $a_3 = a_1^3p_1^2 + 3a_1b_1^2p_1 + 3a_1^2c_1p_1 + 3b_1^2c_1 + 3a_1c_1^2 + 3a_1^2b_1q_1$, $b_3 = 2a_1^3p_1q_1 + 3a_1b_1^2q_1 + 3a_1^2c_1q_1 + 3a_1^2b_1p_1^2 + b_1^3p_1 + 6a_1b_1c_1p_1 + 3b_1c_1^2$, and $c_3 = 3a_1^2b_1p_1q_1 + b_1^3q_1 + 6a_1b_1c_1q_1 + a_1^3q_1^2 + c_1^3$. Suppose $p_1 \equiv 0 \pmod 3$. Then $a_3 \equiv 0 \pmod 3$, $b_3 \equiv 0 \pmod 3$, and $c_3 \equiv b_1q_1 + a_1q_1^2 + c_1 \pmod 3$.

Since ϵ^3 is a unit, $c_3 \not\equiv 0 \pmod 3$ as $a_3 \equiv b_3 \equiv 0 \pmod 3$. Hence we have $c_3 \equiv 1$ or $2 \pmod 3$.

Suppose $n \equiv 1 \pmod 3$, and put $n = 1 + 3m$ in (6). We get

$$\epsilon \cdot (\epsilon^3)^m = u + v\theta,$$

or

$$(a_1\theta^2 + b_1\theta + c_1)(\pm 1)^m \equiv u + v\theta \pmod 3.$$

This congruence is impossible unless $a_1 \equiv 0 \pmod 3$. Hence if $a_1 \not\equiv 0 \pmod 3$, then $n \not\equiv 1 \pmod 3$. Suppose $n \equiv 2 \pmod 3$, and let $n = 2 + 3m$. Then (6) gives

$$(a_2\theta^2 + b_2\theta + c_2)(\pm 1)^m \equiv u + v\theta \pmod 3.$$

This is impossible unless $a_2 \equiv 0 \pmod 3$, i.e. $b_1^2 + 2a_1c_1 \equiv 0$

(mod 3). Hence if $b_1^2+2a_1c_1 \not\equiv 0 \pmod 3$, then $n \equiv 2 \pmod 3$ is impossible. Finally suppose $n = 3m$ in (6). Then we get

$$(a_3\theta^2 + b_3\theta + c_3)^m = u + v\theta. \tag{7}$$

Now $a_3 \equiv b_3 \equiv 0 \pmod 3$, and $a_3 \equiv 3b_1^2c_1 + 3a_1c_1^2 + 3a_1^2b_1q_1 \pmod 9$. If $b_1^2c_1 + a_1c_1^2 + a_1^2b_1q_1 \not\equiv 0 \pmod 3$, then $a_3 \not\equiv 0 \pmod 9$ and hence by Theorem 1, (7) is impossible for m an integer, positive or negative.

Therefore $n = 0$ is the only solution to (6).

LEMMA 5 (Delaunay [2, page 385]). *If $b\theta + c$, where $b \neq 0, \pm 1$, is a positive unit of $Z[\theta]$ where $\theta^3 - P\theta^2 + Q\theta - R = 0$, then no power > 1 of $b\theta + c$ can be a binomial unit.* (In other words all the positive powers of the positive unit $b\theta + c$ are of the form $L\theta^2 + M\theta + N$, where $L \neq 0$).

We prove two theorems which are useful when $b = \pm 1$.

THEOREM 6. *Let $\epsilon = \pm\theta + c$ be a unit in $Z[\theta]$, where $\theta^3 - P\theta^2 + Q\theta - R = 0$. If $\theta^3 \equiv 0 \pmod{p^2}$, where p is a prime, then $p \nmid c$ and $\epsilon^n = u + v\theta$ is impossible for $n > 1$.*

Proof. We have $(\epsilon - c)^3 \equiv 0 \pmod{p^2}$. If $p \mid c$ then $\epsilon^3 \equiv 0 \pmod p$ where $p^3 \mid N(\epsilon^3) = \pm 1$. Hence $p \nmid c$. Let $\epsilon^n = u + v\theta$, $n > 1$. Then

$$(c \pm \theta)^n = c^n + \binom{n}{1} c^{n-1}(\pm\theta) + \binom{n}{2} c^{n-2}\theta^2 + \binom{n}{3} c^{n-3}(\pm\theta)^3 + \cdots + (\pm\theta)^n = u + v\theta.$$

Let $\theta^n = r_n\theta^2 + s_n\theta + t_n$. Then

$$\binom{n}{2} c^{n-2} + \binom{n}{3} c^{n-3}(\pm r_3) + \cdots + (\pm r_n) = 0. \tag{8}$$

As $\theta^3 \equiv 0 \pmod{p^2}$, we have $r_i \equiv 0 \pmod{p^{2[i/3]}}$. Since $p \nmid c$, $p \mid \binom{n}{2}$. Suppose $p^k \| \binom{n}{2}$. If $p = 2$ then $2^k \| \binom{n}{2}$. If $p \neq 2$ then $p^k \| \binom{n}{2}, \binom{n}{3} \cdots \binom{n}{p-1}$ and $p^{k-1} \| \binom{n}{p}$. Using Lemma 2, we see that each term of (8) except the first is divisible by at least p^{k+1}. Hence $p^{k+1} \mid \binom{n}{2}$, a contradiction.

THEOREM 7. *Let $\epsilon = \pm\theta + c_1$ be a unit of the ring $Z[\theta]$, where $\theta^3 - 3P\theta^2 + 3Q\theta - R = 0$. If $c_1 + P \not\equiv 0 \pmod 3$ and $c_1^2 + 2c_1P + Q \not\equiv 0 \pmod 3$, then $\epsilon^n = u + v\theta$ is impossible for $n > 1$.*

Proof. Let $\varepsilon = \theta + c_1$. Then $\theta = \epsilon - c_1$. So from

$$\theta^3 - 3P\theta^2 + 3Q\theta - R = 0,$$

we get

$$(\epsilon - c_1)^3 - 3P(\epsilon - c_1)^2 + 3Q(\epsilon - c_1) - R = 0,$$

or

$$\epsilon^3 = 3(c_1 + P)\epsilon^2 - 3(c^2 + 2c_1P + Q)\epsilon + (c_1^3 + 3c_1^2P + 3c_1Q + R).$$

Now $N(\epsilon) = c_1^3 + 3c_1^2P + 3c_1Q + R = \pm 1$.

For convenience we write $\epsilon^3 = 3r\epsilon^2 - 3s\epsilon \pm 1$. Now by hypothesis $3 \nmid r$ and $3 \nmid s$. Let $\epsilon^n = u + v\theta$. Then $\epsilon^n = u + v(\epsilon - c_1) = u_1 + v_1\epsilon$, say. Suppose $n \equiv 2 \pmod 3$. Then $\epsilon^2(\epsilon^3)^m = u_1 + v_1\epsilon$, where $n = 2 + 3m$. As $\epsilon^3 \equiv \pm 1 \pmod 3$, we have $\pm \epsilon^2 \equiv u_1 + v_1\epsilon \pmod 3$, which is impossible. Let $n \equiv 0 \pmod 3$ and $n \neq 0$. Putting $n = 3m$, we get

$$(3r\epsilon^2 - 3s\epsilon \pm 1)^m = u_1 + v_1\epsilon. \tag{9}$$

But this is impossible by Theorem 1, whether m is a positive or a negative integer, for $3 \nmid r$. Hence if $n \neq 0$, the only possibility is $n \equiv 1 \pmod 3$.

Let $n = 1 + 3m$, where $m > 0$. Then

$$\epsilon(3r\epsilon^2 - 3s\epsilon \pm 1)^m = u_1 + v_1\epsilon,$$

or

$$(3r\epsilon^2 - 3s\epsilon \pm 1)^m = v_1 \pm u_1(\epsilon^2 - 3r\epsilon + 3s).$$

Let $(r\epsilon^2 - s\epsilon)^i = r_i\epsilon^2 + s_i\epsilon + t_i$, where r_i, s_i, t_i are rational integers. Then

$$(\pm 1)^m + \binom{m}{1}(\pm 1)^{m-1}3(r\epsilon^2 - s\epsilon) + \binom{m}{2}(\pm 1)^{m-2}3^2(r_2\epsilon^2 + s_2\epsilon + t_2)$$
$$+ \cdots + 3^m(r_m\epsilon^2 + s_m\epsilon + t_m) = \pm u_1\epsilon^2 \mp 3ru_1\epsilon + (v_1 \pm 3su_1).$$

On equ: ing coefficients of ϵ^2 and ϵ, we obtain

$$(\pm 1)^{m-1}3mr + (\pm 1)^{m-2}3^2\binom{m}{2}r_2 + (\pm 1)^{m-3}3^3\binom{m}{3}r_3 + \cdots + 3^m r_m = \pm u_1, \tag{10}$$

and

$$(11)\quad -(\pm 1)^{m-1}3ms+(\pm 1)^{m-2}3^2\binom{m}{2}s_2+(\pm 1)^{m-3}3^3\binom{m}{3}s_3+\cdots+3^m s_m$$
$$=\mp 3ru_1.$$

Multiplying both sides of (10) by $3r$ and then adding to (11), we obtain

$$(\pm 1)^{m-1}3m(3r^2-s)+(\pm 1)^{m-2}3^2\binom{m}{2}(3r_2r+s_2)$$
$$+(\pm 1)^{m-3}3^3\binom{m}{3}(3r_3r+s_3)+\cdots+3^m(3r_mr+s_m)=0.$$

We see from this that $3|m(3r^2-s)$. As $3\nmid s$, we have $3|m$. Suppose $3^k\|m$. Using Lemma 2, we easily see that all the terms except the first are divisible by 3^{k+2}, while the first is exactly divisible by 3^{k+1}, which is impossible. Hence $m=0$, i.e. $n=1$.

So if n is a nonnegative integer and $\epsilon^n=u+v\theta$, then $n=0$ or $n=1$. The proof for $\epsilon=-\theta+c$, is completely analogous.

THEOREM 8. *If $\epsilon=b_1\theta+c_1$ is a positive unit in $Z[\theta]$, where $\theta^3-P\theta^2+Q\theta-R=0$ with $D(\theta)$ negative and $\neq -23$, then $\epsilon^n=u+v\theta$ implies that $n\geqq 0$.*

To prove this theorem we need the following well-known result.

LEMMA 9 (Nagell **[8]**). *If η is a unit, $D(\eta)<0$, $0<\eta<1$, then $\eta^n=x+y\eta$ implies that $n\geqq 0$, except in the case when $\eta^3+\eta^2-1=0$. In this case $\eta^{-2}=1+\eta$ and $D(\eta)=-23$.*

Proof of Theorem 8. Let $\epsilon=b_1\theta+c_1$ be a positive unit in $Z[\theta]$. Then $0<\epsilon<1$. Since ϵ is contained in $Z[\theta]$, we get $D(\epsilon)=\delta^2\cdot D(\theta)$. Hence $D(\epsilon)<0$ and $\neq -23$.
Let $\epsilon^n=u+\theta$. Since $\epsilon=b_1\theta+c_1$ we have

$$(b_1\theta+c_1)^n=u+v\theta.$$

Then $b_1|v$ when n is a positive integer. In case n is negative, we put $n=-m$ where m is positive. Let $\epsilon^{-1}=a'\theta^2+b'\theta+c'$. Then $\theta^3=P\theta^2-Q\theta+R$ and $\epsilon\epsilon^{-1}=1$ imply

$$(12)\qquad b_1a'P+b_1b'+c_1a'=0,$$

$$(13)\qquad -b_1a'Q+b_1c'+c_1b'=0,$$

and

$$b_1 a' R + c_1 c' = 1. \tag{14}$$

Since $(b_1, c_1) = 1$, $\epsilon = b_1\theta + c_1$ being a unit, we conclude that $b_1 | a'$ and $b_1 | b'$ from (12) and (13) respectively. Then from

$$(b_1\theta + c_1)^n = (a'\theta^2 + b'\theta + c')^m = u + v\theta,$$

we see that $b_1 | v$.

Since $\epsilon = b_1\theta + c_1$, we have $\theta = (\epsilon - c_1)/b_1$, and hence $\epsilon^n = u + v\theta$ can be written as

$$\epsilon^n = u + \frac{v(\epsilon - c_1)}{b_1} = (u - vc_1/b_1) + v\epsilon/b_1 = x + y\epsilon,$$

where x and y are rational integers. Then by Lemma 9, $n \geqq 0$. For binorms in fields of degree higher than three, one can see **[9]**. Recently Bernstein **[1]** has shown that units of the form $\epsilon = 1 + xw + yw^2$, $x, y \in Q$ exist for infinitely many algebraic number fields $Q(w)$ of degree $n \geqq 4$.

Now we solve $y^2 - 113 = x^3$ to show the application of some of the above theorems. The above equation is a special case of the well-known Mordell Equation $y^2 - k = x^3$, which has interested mathematicians for more than three centuries, and has played an important role in the development of number theory. In the range $0 < k \leqq 100$ it is known that $y^2 - k = x^3$, $k = 17$ has the maximum number of solutions. In the range $100 < k \leqq 200$ it is found **[6]** that $y^2 - k = x^3$, $k = 113$ has the maximum number of solutions. The complete solution of this equation is given below.

The fundamental unit of $Q(\sqrt{113})$ is $\eta = 776 + 73\sqrt{113}$, and $h(Q\sqrt{113}) = 1$. 2 splits into two different prime ideals in the field $Q(\sqrt{113})$. Hence by Theorem 5 of Hemer **[4]**, all the integral solutions of $y^2 - 113 = x^3$ can be obtained from the following equations:

$$\pm y + \sqrt{113} = \left(\frac{a + b\sqrt{113}}{2}\right)^3, \quad x = \frac{a^2 - 113b^2}{4},$$

$$\pm y + \sqrt{113} = (776 + 73\sqrt{113})\left(\frac{a + b\sqrt{113}}{2}\right)^3, \quad x = (113b^2 - a^2)/4,$$

$$\frac{1}{2}(\pm y + \sqrt{113}) = \left(\frac{11 + \sqrt{113}}{2}\right)\left(\frac{a + b\sqrt{113}}{2}\right)^3, \quad x = (a^2 - 113b^2)/2,$$

$$\frac{1}{2}(\pm y+\sqrt{113})=\left(\frac{11+\sqrt{113}}{2}\right)(776+73\sqrt{113})\left(\frac{a+b\sqrt{113}}{2}\right)^3,$$
$$x=(113b^2-a^2)/2,$$

$$\frac{1}{2}(\pm y+\sqrt{113})=\left(\frac{11+\sqrt{113}}{2}\right)(776-73\sqrt{113})\left(\frac{a+b\sqrt{113}}{2}\right)^3,$$
$$x=(113b^2-a^2)/2.$$

On equating irrational parts we have respectively

$$3a^2b+113b^3=8, \tag{15}$$

$$73(a^3+3\cdot 113ab^2)+776(3a^2b+113b^3)=8, \tag{16}$$

$$(a^3+3\cdot 113ab^2)+11(3a^2b+113b^3)=8, \tag{17}$$

$$1579(a^3+3\cdot 113ab^2)+16785(3a^2b+113b^3)=8, \tag{18}$$

$$-27(a^3+3\cdot 113ab^2)+287(3a^2b+113b^3)=8. \tag{19}$$

Clearly (15) has no solution in integers. From (16) it is easily seen that a and b are both even. Putting $a=2u_1$, $b=2v_1$ in (16), we obtain

$$73(u_1^3+3\cdot 113u_1v_1^2)+776(3u_1^2v_1+113v_1^3)=1. \tag{20}$$

The substitution $u_1=21u-52v$, $v_1=-2u+5v$ in (20) yields

$$F(u,v)=u^3-33uv^2+76v^3=1. \tag{21}$$

This corresponds to the ring $Z[\theta]$, where $\theta^3-33\theta-76=0$. In this ring the fundamental unit is $\epsilon=4\theta^2-16\theta-71$. By Theorem 1,

$$(4\theta^2-16\theta-71)^n=u+v\theta$$

is only possible for $n=0$. Then $u=1$, $v=0$, and so $a=42$, $b=-4$. Hence $x=11$, $y=\pm 38$.

The substitution $a=u_1-11v_1$, $b=v_1$ in (17) gives

$$u_1^3-24u_1v_1^2+176v_1^3=8. \tag{22}$$

Hence $u_1\equiv 0 \pmod 2$. Putting $u_1=2u$, $v_1=v$ in (22), we get

$$F(u,v)=u^3-6uv^2+22v^3=1. \tag{23}$$

This corresponds to the ring $Z[\theta]$, where $\theta^3 - 6\theta - 22 = 0$; $Z[\theta]$ has fundamental unit $\epsilon = 2\theta - 7$.

Now we consider

$$(2\theta - 7)^n = u + v\theta. \tag{24}$$

By Theorem 8, $n \geqq 0$ and by Lemma 5, $n \leqq 1$. Therefore (24) has only the two solutions $n = 0$, $n = 1$. These solutions correspond to $x = 2$, $y = \pm 11$ and $x = 422$, $y = \pm 8669$ respectively.

Substituting $a = -21u_1 + 53v_1$, $b = 2u_1 - 5v_1$ in (18), we get

$$8v_1^3 + 12v_1^2u_1 - 42v_1u_1^2 + 27u_1^3 = 8. \tag{25}$$

We put $u_1 = 2v$, $v_1 = u - v$ in (25), since $u_1 \equiv 0 \pmod 2$. This gives

$$F(u, v) = u^3 - 24uv^2 + 50v^3 = 1. \tag{26}$$

This corresponds to the ring $Z[\theta]$, where $\theta^3 - 24\theta - 50 = 0$, with the fundamental unit $\epsilon = -3\theta^2 + 10\theta + 41$. We see that $\epsilon \equiv 2\theta^2 + 1 \pmod 5$ and $\epsilon^2 \equiv 1 \pmod 5$ while $\epsilon^2 \equiv -5\theta^2 + 5\theta + 6 \pmod{25}$. Hence $\epsilon^2 = a_1\theta^2 + b_1\theta + c_1$ implies that $5 \| a_1$, $5 \| b_1$. Hence, by Theorem 1, $\epsilon^n = u + v\theta$ is impossible for an even integer $n \neq 0$. When n is odd we have

$$2\theta^2 + 1 \equiv u + v\theta \pmod 5.$$

This is impossible. So we have $n = 0$. Then $u = 1$, $v = 0$ and hence $x = 8$, $y = \pm 25$.

The substitution $a = 111u_1 + 10v_1$, $b = 11u_1 + v_1$ in (19) yields

$$v_1^3 - 312v_1u_1^2 - 2128u_1^3 = 8. \tag{27}$$

Since (27) implies $v_1 \equiv 0 \pmod 2$, we put $v_1 = 12u + 10v$, $u_1 = -u - v$ and get

$$F(u, v) = v^3 + 12vu^2 + 14u^3 = 1. \tag{28}$$

The fundamental unit of the ring $Z[\theta]$, where $\theta^3 + 12\theta - 14 = 0$, is $\epsilon = \theta - 1$, satisfying $\epsilon^3 + 3\epsilon^2 + 15\epsilon - 1 = 0$.

Then by Theorems 8 and 6,

$$\epsilon^n = (\theta - 1)^n = v + u\theta$$

has only two solutions, viz. $n = 0$ and 1.

Incidentally, we cannot reach this conclusion by using the standard criterion of Hemer **[4]**, which is as follows:

Let $\epsilon = \pm\theta + c$ be a unit in a cubic ring, and let the odd prime p be a divisor of $N(\epsilon' + \epsilon'')$. Suppose further that $\epsilon^m = a_m\epsilon^2 + b_m\epsilon + c_m$ is the least power of ϵ with $m > 0$ such that $a_m \equiv b_m \equiv 0 \pmod p$. Then $\epsilon^n = u + v\epsilon$ has no even solution except $n = 0$ if $a_m \not\equiv 0 \pmod{p^2}$, and no odd solution except $n = 1$ if $c_{m+2} \not\equiv 0 \pmod{p^2}$.

Now $N(\epsilon' + \epsilon'') = N(-3-\epsilon) = -46$ has only the odd prime divisor $p = 23$. The least exponent m such that $a_m \equiv b_m \equiv 0 \pmod{23}$ is $m = 22$, and $a_m \not\equiv 0 \pmod{23^2}$. But unfortunately $c_{24} \equiv 0 \pmod{23^2}$.

When $n = 0,\ u = 0,\ v = 1;\ a = -11,\ b = -1;\ x = -4,\ y = \pm 7.$

When $n = 1,\ u = 1,\ v = -1;\ a = 20,\ b = 2;\ x = 26,\ y = \pm 133.$

Hence the Diophantine equation $y^2 - 113 = x^3$ has exactly 6 solutions in integers. They are $(x, y) = (11, \pm 38)$, $(8, \pm 25)$, $(2, \pm 11)$, $(-4, \pm 7)$, $(422, \pm 8669)$ and $(26, \pm 133)$.

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