A CHARACTERIZATION OF SOLENOIDS

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Suppose \( M \) is a homogeneous continuum and every proper subcontinuum of \( M \) is an arc. Using a theorem of E. G. Effros involving topological transformation groups, we prove that \( M \) is circle-like. This answers in the affirmative a question raised by R. H. Bing. It follows from this result and a theorem of Bing that \( M \) is a solenoid. Hence a continuum is a solenoid if and only if it is homogeneous and all of its proper subcontinua are arcs. The group \( G \) of homeomorphisms of \( M \) onto \( M \) with the topology of uniform convergence has an unusual property. For each point \( w \) of \( M \), let \( G_w \) be the isotropy subgroup of \( w \) in \( G \). Although \( G_w \) is not a normal subgroup of \( G \), it follows from Effros' theorem and Theorem 2 of this paper that the coset space \( G/G_w \) is a solenoid homeomorphic to \( M \) and, therefore, a topological group.

1. Introduction. Let \( \mathcal{S} \) be the class of all homogeneous continua \( M \) such that every proper subcontinuum of \( M \) is an arc. It is known that every solenoid belongs to \( \mathcal{S} \). It is also known that every circle-like element of \( \mathcal{S} \) is a solenoid. In fact, in 1960 R. H. Bing [4, Theorem 9, p. 228] proved that each homogeneous circle-like continuum that contains an arc is a solenoid. At that time Bing [4, p. 219] asked whether every element of \( \mathcal{S} \) is a solenoid. In this paper we answer Bing's question in the affirmative by proving that every element of \( \mathcal{S} \) is circle-like.

2. Definitions and related results. We call a nondegenerate compact connected metric space a \textit{continuum}.

A \textit{chain} is a finite sequence \( L_1, L_2, \ldots, L_n \) of open sets such that \( L_i \cap L_j \neq \emptyset \) if and only if \( |i - j| \leq 1 \). If \( L_1 \) also intersects \( L_n \), the sequence is called a \textit{circular chain}. Each \( L_i \) is called a \textit{link}. A chain (circular chain) is called an \( \epsilon \)-\textit{chain} (\( \epsilon \)-\textit{circular chain}) if each of its links has diameter less than \( \epsilon \). A continuum is said to be \textit{arc-like} (circle-like) if for each \( \epsilon > 0 \), it can be covered by an \( \epsilon \)-chain (\( \epsilon \)-circular chain).

A space is \textit{homogeneous} if for each pair \( p, q \) of its points there exists a homeomorphism of the space onto itself that takes \( p \) to \( q \). Bing [2] [3] proved that a continuum is a pseudo-arc if and only if it is homogeneous and arc-like. L. Fearnley [9] and J. T. Rogers, Jr. [20] independently showed that every homogeneous, hereditarily indecomposable, circle-like
continuum is a pseudo-arc [11]. However, there are many topologically different homogeneous circle-like continua that have decomposable subcontinua [24] [25].

Let $n_1, n_2, \cdots$ be a sequence of positive integers. For each positive integer $i$, let $G_i$ be the unit circle $\{z \in \mathbb{R}^2 : |z| = 1\}$, and let $f_i$ be the map of $G_{i+1}$ onto $G_i$ defined by $f_i(z) = z^{n_i}$. The inverse limit space of the sequence $\{G_i, f_i\}$ is called a solenoid. Since each $G_i$ is a topological group and each $f_i$ is a homomorphism, every solenoid is a topological group [13, Theorem 6.14, p. 56] and therefore homogeneous. Each solenoid is circle-like since it is an inverse limit of circles with surjective bonding maps [17, Lemma 1, p. 147].

A solenoid can be described as the intersection of a sequence of solid tori $M_i, M_2, \cdots$ such that $M_{i+1}$ runs smoothly around inside $M_i$ exactly $n_i$ times longitudinally without folding back and $M_i$ has cross diameter of less than $i^{-1}$. The sequence $n_1, n_2, \cdots$ determines the topology of the solenoid. If it is $1, 1, \cdots$ after some place, the solenoid is a simple closed curve. If it is $2, 2, \cdots$, the solenoid is the dyadic solenoid defined by D. van Dantzig [7] and L. Vietoris [23]. Other properties involving the sequence $n_1, n_2, \cdots$ are given in [4, p. 210]. From this description we see that every proper subcontinuum of a solenoid is an arc.

Solenoids appear as invariant sets in the qualitative theory of differential equations. In [21] E. S. Thomas proved that every compact 1-dimensional metric space that is minimal under some flow and contains an almost periodic point is a solenoid.

Every homogeneous plane continuum that contains an arc is a simple closed curve [4] [10] [15]. Hence each planar solenoid is a simple closed curve.

Each of the three known examples of homogeneous plane continua (a circle, a pseudo-arc [2] [18], and a circle of pseudo-arcs [5]) is circle-like. If one could show that every homogeneous plane continuum is circle-like, it would follow that there does not exist a fourth example [6] [12] [14, p. 49] and a long outstanding problem would be solved.

A topological transformation group $(G, M)$ is a topological group $G$ together with a topological space $M$ and a continuous mapping $(g, w) \rightarrow gw$ of $G \times M$ into $M$ such that $ew = w$ ($e$ denotes the identity of $G$) and $(gh)w = g(hw)$ for all elements $g$, $h$ of $G$ and $w$ of $M$.

For each point $w$ of $M$, let $G_w$ be the isotropy subgroup of $w$ in $G$ (that is, the set of all elements $g$ of $G$ such that $gw = w$). Let $G/G_w$ be the left coset space with the quotient topology. The mapping $\varphi_w$ of $G/G_w$ onto $Gw$ that sends $gG_w$ to $gw$ is one-to-one and continuous. The set $Gw$ is called the orbit of $w$.

Assume $M$ is a continuum and $G$ is the topological group of homeomorphisms of $M$ onto $M$ with the topology of uniform convergence [16, p. 88]. E. G. Effros [8, Theorem 2.1] proved that each
orbit is a set of the type $G_\delta$ in $M$ if and only if for each point $w$ of $M$, the mapping $\varphi_w$ is a homeomorphism.

Suppose $M$ is a homogeneous continuum. Then the orbit of each point of $M$ is $M$, a $G_\delta$-set. According to Effros' theorem, for each point $w$ of $M$, the coset space $G/G_w$ is homeomorphic to $M$. By Theorem 2 of §4, if $M$ has the additional property that all of its proper subcontinua are arcs, then $G/G_w$ is a solenoid and, therefore, a topological group. Note that $G_w$ is not a normal subgroup of $G$.

Throughout this paper $R^2$ is the Cartesian plane. For each real number $r$, we shall denote the horizontal line $y = r$ and the vertical line $x = r$ in $R^2$ by $H(r)$ and $V(r)$ respectively.

Let $P$ and $Q$ be subsets of $R^2$. The set $P$ is said to project horizontally into $Q$ if every horizontal line in $R^2$ that meets $P$ also meets $Q$.

We shall denote the boundary and the closure of a given set $Z$ by $\text{Bd} Z$ and $\text{Cl} Z$ respectively.

3. Preliminary results. In this section $M$ is a homogeneous continuum (with metric $\rho$) having only arcs for proper subcontinua.

Let $p$ and $q$ be two points of the same arc component of $M$. The union of all arcs in $M$ that have $p$ as an endpoint and contain $q$ is called a ray starting at $p$.

The following two lemmas are easy to verify.

**Lemma 1.** Each ray is dense in $M$.

**Lemma 2.** If an open subset $Z$ of $M$ is not dense in $M$, then each component of $Z$ is an arc segment with both endpoints in $\text{Bd} Z$.

Let $\epsilon$ be a positive number. A homeomorphism $h$ of $M$ onto $M$ is called an $\epsilon$-homeomorphism if $\rho(v, h(v)) < \epsilon$ for each point $v$ of $M$.

**Lemma 3.** Suppose $\epsilon$ is a given positive number and $w$ is a point of $M$. Then $w$ belongs to an open subset $W$ of $M$ with the following property. For each pair $p, q$ of points of $W$, there exists an $\epsilon$-homeomorphism $h$ of $M$ onto $M$ such that $h(p) = q$.

**Proof.** Define $G$, $G_w$, and $\varphi_w$ as in §2. Since $M$ is homogeneous, the orbit of each point of $M$ is $M$. Therefore $\varphi_w$ is a homeomorphism of $G/G_w$ onto $M$ [8, Theorem 2.1].

Let $\pi_w$ be the natural open mapping of $G$ onto $G/G_w$ that sends $g$ to $gG_w$. Define $T_w$ to be the mapping of $G$ onto $M$ that sends $g$ to $g(w)$. Since $T_w = \varphi_w \pi_w$, it follows that $T_w$ is an open mapping [22, Theorem 3.1]. Note that the following diagram commutes.
Let \( U \) be the open subset of \( G \) consisting of all \( \epsilon/2 \)-homeomorphisms of \( M \) onto \( M \). Define \( W \) to be the open set \( T_\epsilon[U] \).

Since the identity \( e \) belongs to \( U \) and \( T_\epsilon(e) = w \), the set \( W \) contains \( w \).

Assume \( p \) and \( q \) are points of \( W \). Let \( f \) and \( g \) be elements of \( U \) such that \( T_\epsilon(f) = p \) and \( T_\epsilon(g) = q \). Since \( f(w) = p \) and \( g(w) = q \), the mapping \( h = gf^{-1} \) of \( M \) onto \( M \) is an \( \epsilon \)-homeomorphism with the property that \( h(p) = q \).

For each positive integer \( i \), let \( A_i \) be an arc with endpoints \( p_i \) and \( q_i \). The sequence \( A_1, A_2, \ldots \) is said to be folded if it converges to an arc \( A \) and the sequence \( p_1, q_1, p_2, q_2, \ldots \) converges to an endpoint of \( A \).

**Lemma 4.** (Bing [4, Theorem 6, p. 220]). There does not exist a folded sequence of arcs in \( M \).

Lemma 4 follows from a simple argument (shorter than Bing’s) involving Lemma 3 and the fact that \( M \) does not contain a triod.

A chain \( L_1, L_2, \ldots, L_n \) in \( M \) is said to be free if \( \text{Cl} L_1 \cap \text{Cl} L_n = \emptyset \) and \( \text{Bd} \cup \{L_i : 1 \leq i \leq n\} \) is a subset of \( \text{Cl}(L_1 \cup L_n) \).

**Lemma 5.** (Bing [4, Property 17, p. 219]). Let \( A \) be an arc in \( M \) with endpoints \( p \) and \( q \). For each positive number \( \epsilon \), there exists a free \( \epsilon \)-chain \( L_1, L_2, \ldots, L_n \) in \( M \) covering \( A \) such that \( p \) and \( q \) belong to \( L_1 \) and \( L_n \) respectively.

A continuum is decomposable if it is the union of two proper subcontinua; otherwise, it is indecomposable.

**Lemma 6.** If \( M \) is decomposable, then \( M \) is a simple closed curve.

**Proof.** Since \( M \) is the union of two proper subcontinua (arcs), \( M \) is locally connected. Since \( M \) is homogeneous, it does not have a separating point. Hence \( M \) contains a simple closed curve [19, Theorem 13, p. 91]. It follows that \( M \) is a simple closed curve.
4. Principal results.

**Theorem 1.** If $M$ is a homogeneous continuum and every proper subcontinuum of $M$ is an arc, then $M$ is circle-like.

**Proof.** According to Lemma 6, if $M$ is decomposable, then $M$ is a simple closed curve and therefore circle-like. Hence we assume that $M$ is indecomposable.

By Lemmas 4 and 5, there exists a free chain $L_1, L_2, \ldots, L_\alpha$ ($\alpha > 5$) in $M$ such that $N = \text{Cl} \cup \{L_i : 1 \leq i \leq \alpha\}$ is a proper subset of $M$ and $N - \text{Cl} \cup \{L_i : 3 \leq i \leq (\alpha - 2)\}$ contains every arc in $N$ that has both of its endpoints in $\text{Cl}L_1$ or $\text{Cl}L_\alpha$. (This chain is formed from another free chain by unioning links to make $L_2$ and $L_{\alpha-1}$ sufficiently long and narrow.) Let $B$ be the union of all components of $N$ that meet $\text{Cl}(L_3 \cup L_{\alpha-2})$. By Lemma 2, each component of $B$ is an arc with one endpoint in $\text{Bd}L_1$ and the other endpoint in $\text{Bd}L_\alpha$. Note that $B$ is a closed set. Since $M$ is indecomposable, each component of $B$ is a continuum of condensation.

Since $B$ contains no folded sequence of arcs, we can assume that $B$ is the intersection of $M$ and the plane $\mathbb{R}^2$ and that the following conditions are satisfied:

I. A component $C$ of $B$ is $\{(x, y) : 0 \leq x \leq 6$ and $y = 0\}$.

II. Each component of $B - C$ is a horizontal interval above $H(0)$ (the $x$-axis) and below $H(1)$ that crosses both $V(1)$ and $V(5)$.

III. The sets $\text{Cl}(L_1 \cup L_2 \cup L_{\alpha-1} \cup L_\alpha)$ and $\{(x, y) : 1 \leq x \leq 5\}$ are disjoint.

(Bing's theorem [2, Theorem 11], involving sequences of refining covers that induce a homeomorphism, can be used to define this embedding of $B$ in $\mathbb{R}^2$. Each cover of $B$ consists of finitely many free chains that correspond to disjoint straight horizontal chains with rectangular links in $\mathbb{R}^2$.) Note that $B \cap \{(x, y) : 1 < x < 5\}$ is an open subset of $M$.

Let $\rho$ be a metric on $M$ whose restriction to $B$ agrees with the Euclidean metric on $\mathbb{R}^2$ [1, Theorems 4 and 5].

There exists a positive number $d$ less than 1 such that $M \cap H(d) = \emptyset$ and the following condition is satisfied:

**Property 1.** Every arc in $M$ that has its endpoints in $\{(x, y) : x = 3$ and $0 \leq y < d\}$ meets both $\{(x, y) : x = 1$ and $0 \leq y < d\}$ and $\{(x, y) : x = 5$ and $0 \leq y < d\}$.

To see this we assume Property 1 does not hold for any positive number $d$. For each positive integer $i$, let $W_i$ be an open set in
\( M \cap \{(x, y) : 1 < x < 5\} \) that contains \((3,0)\) such that for each pair \(p, q\) of points of \(W\) there exists an \(i^{-1}\)-homeomorphism of \(M\) onto \(M\) that takes \(p\) to \(q\) (Lemma 3). For each \(i\), there exists an arc \(A_i\) in \(M\) with endpoints \(p_i\) and \(q_i\) in \(W_i \cap V(3)\) such that the horizontal interval \(\Gamma_i\) from \(p_i\) to \(V(1)\) is in \(A_i\), if and only if the horizontal interval \(\Delta_i\) from \(q_i\) to \(V(1)\) is in \(A_i\).

For each \(i\), let \(h_i\) be an \(i^{-1}\)-homeomorphism of \(M\) onto \(M\) such that \(h_i(p_i) = q_i\). Since each \(h_i\) maps \(\Gamma_i\) approximately onto \(\Delta_i\), for each \(i\), there exists a point \(a_i\) of \(A_i\) such that \(h_i(a_i) = a_i\).

For each \(i\), let \(B_i\) be the arc in \(A_i\) from \(p_i\) to \(a_i\). Note that for each \(i\), the diameter of \(B_i\) is greater than 1 and \(B_i \cap h_i[B_i]\) consists of the point \(a_i\).

Let \(a\) be a limit point of the sequence \(\{a_i\}\). Assume without loss of generality that \(\{a_i\}\) is a convergent sequence in \(E = \{v \in M : \rho(v, a) < 1/2\}\).

For each \(i\), let \(E_i\) be an arc in \(B_i \cap \text{Cl}\ E\) that goes from a point \(b_i\) of \(\text{Bd}\ E\) to \(a_i\). Assume without loss of generality that \(\{b_i\}\) converges to a point of \(\text{Bd}\ E\) and \(\{E_i\}\) converges to an arc \(F\) in \(\text{Cl}\ E\). Since each \(h_i\) is an \(i^{-1}\)-homeomorphism, \(\{E_i \cup h_i[E_i]\}\) is a folded sequence of arcs converging to \(F\). This contradiction of Lemma 4 completes our argument for Property 1.

For \(i = 1\) and 2, let

\[ D_i = M \cap \{(x, y) : i \leq x \leq 6 - i \text{ and } 0 \leq y < d\}. \]

Let \(\epsilon\) be a given positive number less than \(\rho(D_2, M - D_1)\). We shall complete this proof by defining an \(\epsilon\)-circular chain that covers \(M\).

By Lemma 1, there exists an arc \(A\) in \(M\) that is irreducible with respect to the property that it contains \(\{(5,0),(6,0)\}\) and intersects \(\{(x, y) : x = 5\text{ and } 0 < y < d\}\). According to Property 1, \(A\) intersects \(\{(x, y) : x = 4\text{ and } 0 < y < d\}\).

Let \(W\) be an open set in \(D_1 - A\) containing \((4,0)\) such that for each pair \(p, q\) of points of \(W\), there exists an \(\epsilon/50\)-homeomorphism of \(M\) onto \(M\) that takes \(p\) to \(q\) (Lemma 3).

Let \(c\) be a number \((0 < c < \epsilon/50)\) such that \(M \cap H(c) = \emptyset\) and \(M \cap \{(x, y) : x = 4\text{ and } 0 \leq y < c\}\) is in \(W\). Since \(W\) and \(A\) are disjoint, \(c\) is less than \(d\).

For \(i = 1\) and 2, let

\[ C_i = M \cap \{(x, y) : i \leq x \leq 6 - i \text{ and } 0 \leq y < c\}. \]

Let \(\delta\) be the minimum of \(\epsilon\) and \(\rho(C_2, M - C_1)\). Let \(U\) be an open subset of \(C_1\) containing \((2,0)\) such that for each point \(q\) of \(U\), there exists a \(\delta\)-homeomorphism of \(M\) onto \(M\) that takes \((2,0)\) to \(q\) (Lemma 3).
Define $S$ to be the ray in $M$ that starts at $(2,0)$ and contains $A$. Let 
\{$s_i$\} be the sequence consisting of all points of $S \cap \{(x,y): x = 3$ and 
$0 \leq y < d\}$ and having the property that for each $i$, the points $s_i$ precedes
$s_{i+1}$ with respect to the linear order on $S$.

Define $T_1$ to be an arc containing $A$ in $S$ that starts at the point
$t_1 = (2,0)$ and ends at a point $t_2$ of $U \cap V(2)$. Let $h$ be a $\delta$-
homeomorphism of $M$ onto $M$ that takes $t_1$ to $t_2$.

We proceed inductively. Assume an arc $T_n$ is defined in $S$ with
endpoints $t_n$ and $t_{n+1}$ in $C_2 \cap V(2)$. Let $y$ be the number such that
$h(t_{n+1})$ belongs to $H(y)$. Define $T_{n+1}$ to be the arc in $S$ with endpoints
$t_{n+1}$ and $t_{n+2} = (2, y)$. Since $h$ is a $\delta$-homeomorphism, $t_{n+2}$ belongs to
$C_2$. Note that since each $T_n$ has diameter greater than 1, the ray $S$ is the
union of \{$T_n; n = 1, 2, \cdots$\}.

Define $\beta$ to be the largest integer such that \{$s_i: 1 \leq i \leq \beta$\} is a subset
of $T_1$. The $\delta$-homeomorphism $h$ maps each $T_n$ approximately onto
$T_{n+1}$. Hence, for each $n$, the arc $T_n$ contains \{$s_i: (n-1)\beta < i \leq \n\beta$\}. Furthermore, $\beta$ has the following property:

\textbf{Property 2.} For each positive integer $i$, the point $s_i$ belongs to $C_2$ if
and only if $s_{i+\beta}$ belongs to $C_2$.

Define $\gamma$ to be the least positive integer that has Property 2. Note
that since $s_2$ does not belong to $C_2$, the integer $\gamma$ is greater than 1.

Let $K$ be \{$s_i: i = j\gamma + 1$ and $j = 0, 1, 2, \cdots$\}, and let $L$ be
$(S \cap D_2 \cap V(3)) - K$.

\textbf{Property 3.} The sets $\text{Cl} K$ and $\text{Cl} L$ are disjoint.

To establish Property 3, we assume there is a point $z$ in $\text{Cl} K \cap
\text{Cl} L$. Let $Z$ be an open subset of $M$ containing $z$ such that for each pair
$p, q$ of points of $Z$, there exists a $\delta$-homeomorphism of $M$ onto $M$ that
takes $p$ to $q$ (Lemma 3).

Let $s_i$ and $s_n$ be points of $Z \cap K$ and $Z \cap L$, respectively, and let $f$
be a $\delta$-homeomorphism of $M$ onto $M$ such that $f(s_i) = s_n$. Let $\theta$ be the
smallest positive integer such that $s_{n+\theta}$ belongs to $K$. The existence of $f$
implies that $\theta$ has Property 2. Since $\theta$ is less than $\gamma$, this is a
contradiction and Property 3 is established.

Note that since $M = \text{Cl} S$ (Lemma 1), $\text{Cl} (K \cup L) = D_2 \cap V(3)$.

Let $I$ be the arc in $S$ that goes from $s_i$ to $s_{i+1}$. By an argument
similar to Bing's [4, Property 17, p. 219], there exists a free $\epsilon/50$-chain
$P_1, P_2, \cdots, P_\lambda$ in $M$ covering $I$ such that

(i) $s_i$ and $s_{i+1}$ belong to $P_i$ and $P_{i+1}$ respectively,

(ii) $P_i \cup P_{i+1}$ is in $C_2$, 

(iii) each component of \( H = \bigcup \{ P_i \colon 1 \leq j \leq \lambda \} \) that meets \( \text{Cl } P_i \) also meets \( P_i \) and \( V(5) \), and

(iv) each component of \( H \) that meets \( \text{Cl } P_i \) meets \( P_i \) and \( V(1) \).

From Property 1 we get the following:

**Property 4.** Each component of \( H \) meets both \( P_i \) and \( P_\lambda \).

Let \( P_\mu \) be an element of \( P_1, P_2, \ldots, P_\lambda \) that contains the point \((4,0)\). Since \( W \) intersects each component of \( C_2 \), there exists a finite sequence \( g_1, g_2, \ldots, g_\sigma \) of \( \epsilon/50 \)-homeomorphisms of \( M \) onto \( M \) such that \( \text{Cl } K \) projects horizontally into \( \bigcup \{ g_i[P_\mu] : 1 \leq i \leq \sigma \} \). Assume without loss of generality that no proper subsequence of \( g_1, g_2, \ldots, g_\sigma \) has this horizontal projection property.

Note that each \( g_i[P_\mu] \) is a subset of \( D_1 \).

From Properties 1 and 4 we get the following:

**Property 5.** For each \( i (1 \leq i \leq \sigma) \), if \( T \) is a component of \( g_i[H] \), then \( T \cap g_i[\text{Cl } P_\mu] \) is a nonempty set that projects horizontally to a point of \( D_2 \cap V(3) \).

For each \( i (1 \leq i \leq \sigma) \), let \( X_i \) be the set consisting of all points in \( g_i[P_\mu] \) that project horizontally into \( \text{Cl } K \), and let \( Y_i \) be the union of all components of \( g_i[H] \) that meet \( X_i \).

For each \( i (1 \leq i \leq \sigma) \), the set \( Y_i \) is open in \( M \). To see this assume that for some \( i \), a point \( u \) of \( Y_i \) is in \( \text{Cl } (M - Y_i) \). According to Property 3, \( u \) does not belong to \( g_i[P_\mu] \). By Property 5, there exists a sequence \( \{ J_n \} \) of arcs in \( g_i[H] \) that meet \( g_i[P_\mu] \) such that the limit superior \( J \) of \( \{ J_n \} \) is an arc in \( g_i[H] \) that contains \( u \) and for each \( n \), the set \( J_n \cap g_i[P_\mu] \) projects horizontally to a point of \( \text{Cl } L \). It follows that \( J \cap g_i[\text{Cl } P_\mu] \) is a nonempty set that projects horizontally to a point of \( \text{Cl } L \). Since \( J \) is in the \( u \)-component of \( Y_n \) this is a contradiction of Property 5. Hence \( Y_i \) is an open subset of \( M \).

For each \( i (1 \leq i \leq \sigma) \) and \( j (1 \leq j \leq \lambda) \), let \( Q_{ij} = Y_i \cap g_i[P_j] \). It follows from an argument similar to the one given in the preceding paragraph that for each \( i \), the set \( \text{Cl } (Q_{i1} \cup Q_{i\lambda}) \) contains \( \text{Bd } \cup \{ Q_{ij} : 1 \leq j \leq \lambda \} \). Hence, for each \( i \), the sequence \( Q_{i1}, Q_{i2}, \ldots, Q_{i\lambda} \) is a free chain in \( M \).

**Property 6.** For each \( i (1 \leq i \leq \sigma) \), the set \( Q_{i1} \cup Q_{i\lambda} \) projects horizontally into \( \text{Cl } K \).

Obviously, \( Q_{i1} \) projects horizontally into \( \text{Cl } K \). Therefore, to establish Property 6, we assume there is a point \( t \) of \( Q_{i\lambda} \) that projects horizontally into \( \text{Cl } L \). By Property 3, there exists a positive number \( \eta \) less than \( \epsilon \) such that \( Q = \{ v \in M : \rho(v, t) < \eta \} \) projects horizontally in \( \text{Cl } L \).
Let $T$ denote the $t$-component of $Y_h$ and let $w$ be a point of $T \cap Q_{i1}$ (Property 4). Since $g_i$ is an $\epsilon/50$-homeomorphism, $T$ crosses $D_1 \cap V(1)$ exactly $\gamma$ times (Property 1). Since $w$ belongs to $Q_{i1}$, it projects horizontally into $\Cl K$.

By Lemma 3, there exists an $\eta$-homeomorphism $g$ of $M$ onto $M$ such that $g(w)$ belongs to $Q_{i1}$ and projects horizontally into $K$. Since the $g(w)$-component of $Y_i$ is an arc segment in $S$ that crosses $D_1 \cap V(1)$ exactly $\gamma$ times and is mapped approximately onto $T$ by $g^{-1}$, the point $g(t)$ of $Q$ projects horizontally into $K$. This contradiction of the definition of $Q$ completes our argument for Property 6.

Let $\pi$ be an integer ($5 < \pi < \mu$) such that $P_\pi$ contains the point $(3 + \epsilon/10, 0)$. Let $\omega$ be an integer ($\mu < \omega < \lambda - 4$) such that $P_\omega$ contains the point of $V(3 - \epsilon/10)$ that projects horizontally to $s_{y+1}$.

Property 7. For each $n$ ($1 \leq n \leq \sigma$), the set $Q_{n1} \cup Q_{n\lambda}$ does not intersect $\bigcup \{Q_{i1} : 1 \leq i \leq \sigma \text{ and } \pi \leq j \leq \omega \}$.

To see this assume there exist integers $i$, $j$, and $n$ such that $\pi \leq j \leq \omega$ and a point $p$ belongs to $Q_{i1} \cap (Q_{n1} \cup Q_{n\lambda})$. According to Property 6, $\{p\} \cup Q_{i1} \cup Q_{i\lambda}$ projects horizontally into $\Cl K$. By Property 3, there exists a positive number $\chi$ less than $\epsilon$ such that $\{v \in M : \rho(v, p) < \chi\}$ projects horizontally into $\Cl K$.

Let $P$ be the $p$-component of $Y_i$. Let $Y$ be an arc in $P$ that goes from a point $q$ of $Q_{i1}$ to $p$. Since $g_i$ and $g_n$ are $\epsilon/50$-homeomorphisms and $\pi \leq j \leq \omega$, the set $Q_{i1} \cup Q_{i\lambda}$ and the $p$-component of $P \cap D_1$ are disjoint. Hence $Y$ crosses $D_1 \cap V(1)$ exactly $\iota$ times where $\iota$ is a positive integer less than $\gamma$.

By Lemma 3, there exists a $\chi$-homeomorphism $k$ of $M$ onto $M$ such that $k(q)$ belongs to $Q_{i1}$ and projects horizontally into $K$. The arc $k[Y]$ crosses $D_1 \cap V(1)$ exactly $\iota$ times. Since $k[Y]$ is in $S$ and $\rho(p, k(p)) < \chi$, the point $k(p)$ projects horizontally into $K$. It follows from the definition of $K$ that $\iota$ is a multiple of $\gamma$, and this is a contradiction. Hence Property 7 holds.

For each $i$ ($1 \leq i \leq \sigma$) and $j$ ($1 \leq j \leq \lambda$), let $P_{ij} = Q_{ij} - \Cl \cup \{Y_n : 1 \leq n < i\}$. By Property 7, for each $i$, the subchain of $P_{i1}, P_{i2}, \cdots, P_{i\lambda}$ that has $P_{i\pi}$ and $P_{i\omega}$ as end links is free in $M$.

For each $j$ ($1 \leq j \leq \lambda$), let $U_j = \bigcup \{P_{ij} : 1 \leq i \leq \sigma\}$. The subchain $\mathcal{C}$ of $U_1, U_2, \cdots, U_\lambda$ that has $U_\pi$ and $U_\omega$ as end links is a free $\epsilon/16$-chain in $M$.

Let $D$ be the union of all components of $C_2 \cap \{(x, y) : 3 - \epsilon/5 < x < 3 + \epsilon/5\}$ that meet $\Cl K$. According to Property 3, $D$ is open in $M$. The diameter of $D$ is less than $\epsilon/2$. Each point of $U_\pi \cup U_\omega$ is within $\epsilon/5$ of $V(3)$. By Property 6, $U_\pi \cup U_\omega$ projects horizontally into $\Cl K$. Hence $U_\pi \cup U_\omega$ is in $D$.

Let $\tau$ be the largest integer less than $\mu$ such that $U_\tau$ intersects
Let $\psi$ be the smallest integer greater than $\mu$ such that $U_\phi$ intersects $D$. For each $j$ ($1 \leq j < \psi - \tau$), let $Z_j = U_{\tau+j}$. Note that $Z_1, Z_2, \cdots, Z_{\psi-\tau-1}$ is a free $\epsilon$-chain in $M$.

Define $Z_{\psi-\tau}$ to be the union of $D$ and all elements of $D = \{U_j : \pi \leq j \leq \tau \text{ or } \psi \leq j \leq \omega\}$. Since $C\overline{K}$ projects horizontally into $U_\mu$ and $C$ is a free chain in $M$, each element of $D$ intersects $D$. Thus $Z_{\psi-\tau}$ is an open set in $M$ of diameter less than $\epsilon$.

Note that $Z_{\psi-\tau}$ meets both $Z_1$ and $Z_{\psi-\tau-1}$.

Since $C$ is free and $U_\sigma \cup U_\omega$ is in $D$, the boundary of $\cup \{Z_j : 1 \leq j < \psi - \tau\}$ is in $Z_{\psi-\tau}$. Since $C\overline{K}$ projects horizontally into $U_\mu$, the set $Z_1$ contains every boundary point of $Z_{\psi-\tau}$ that is to the right of $V(3)$ in $R^2$.

Furthermore, each point of $\text{Bd} Z_{\psi-\tau}$ that is to the left of $V(3)$ is in $Z_{\psi-\tau-1}$. To see this let $s$ be such a point. Let $X$ be the arc in $M$ that intersects $V(1)$ and is irreducible between $s$ and $C\overline{U}_\mu$ (Lemma 1). By Property 1, $X$ does not meet $U_\sigma \cup U_\omega$. Since $U_\mu$ is an interior link in the free chain $C$, the arc $X$ is covered by $C$ and $s$ belongs to $Z_{\psi-\tau-1}$.

It follows that $\text{Bd} Z_{\psi-\tau}$ is in $Z_1 \cup Z_{\psi-\tau-1}$. Therefore $Z_1, Z_2, \cdots, Z_{\psi-\tau}$ is an $\epsilon$-circular chain that covers $M$. Hence $M$ is circle-like.

Since every homogeneous circle-like continuum that contains an arc is a solenoid [4, Theorem 9, p. 228], Theorem 1 implies the following:

**Theorem 2.** A continuum $M$ is a solenoid if and only if $M$ is homogeneous and every proper subcontinuum of $M$ is an arc.

**References**


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