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Suppose M is a homogeneous continuum and every proper subcontinuum of M is an arc. Using a theorem of E. G. Effros involving topological transformation groups, we prove that M is circle-like. This answers in the affirmative a question raised by R. H. Bing. It follows from this result and a theorem of Bing that M is a solenoid. Hence a continuum is a solenoid if and only if it is homogeneous and all of its proper subcontinua are arcs. The group G of homeomorphisms of M onto M with the topology of uniform convergence has an unusual property. For each point w of M, let G_w be the isotropy subgroup of w in G. Although G_w is not a normal subgroup of G, it follows from Effros' theorem and Theorem 2 of this paper that the coset space G/G_w is a solenoid homeomorphic to M and, therefore, a topological group.

1. Introduction.

Let $\mathscr{S}$ be the class of all homogeneous continua M such that every proper subcontinuum of M is an arc. It is known that every solenoid belongs to $\mathscr{S}$. It is also known that every circle-like element of $\mathscr{S}$ is a solenoid. In fact, in 1960 R. H. Bing **[4**, Theorem 9, p. 228] proved that each homogeneous circle-like continuum that contains an arc is a solenoid. At that time Bing **[4**, p. 219] asked whether every element of $\mathscr{S}$ is a solenoid. In this paper we answer Bing's question in the affirmative by proving that every element of $\mathscr{S}$ is circle-like.

2. Definitions and related results.

We call a nondegenerate compact connected metric space a *continuum*.

A *chain* is a finite sequence $L_1, L_2, \cdots, L_n$ of open sets such that $L_i \cap L_j \neq \emptyset$ if and only if $|i-j| \leqq 1$. If L_1 also intersects L_n, the sequence is called a *circular chain*. Each L_i is called a *link*. A chain (circular chain) is called an *ϵ-chain* (*ϵ-circular chain*) if each of its links has diameter less than ϵ. A continuum is said to be *arc-like* (*circle-like*) if for each $\epsilon > 0$, it can be covered by an ϵ-chain (ϵ-circular chain).

A space is *homogeneous* if for each pair p, q of its points there exists a homeomorphism of the space onto itself that takes p to q. Bing **[2] [3]** proved that a continuum is a pseudo-arc if and only if it is homogeneous and arc-like. L. Fearnley **[9]** and J. T. Rogers, Jr. **[20]** independently showed that every homogeneous, hereditarily indecomposable, circle-like continuum is a pseudo-arc **[11]**. However, there are many topologically different homogeneous circle-like continua that have decomposable subcontinua **[24]** **[25]**.

Let $n_1, n_2, \cdots$ be a sequence of positive integers. For each positive integer i, let G_i be the unit circle $\{z \in R^2 : |z| = 1\}$, and let f_i be the map of G_{i+1} onto G_i defined by $f_i(z) = z^{n_i}$. The inverse limit space of the sequence $\{G_i, f_i\}$ is called a *solenoid.* Since each G_i is a topological group and each f_i is a homomorphism, every solenoid is a topological group [**13**, Theorem 6.14, p. 56] and therefore homogeneous. Each solenoid is circle-like since it is an inverse limit of circles with surjective bonding maps [**17**, Lemma 1, p. 147].

A solenoid can be described as the intersection of a sequence of solid tori $M_1, M_2, \cdots$ such that M_{i+1} runs smoothly around inside M_i exactly n_i times longitudinally without folding back and M_i has cross diameter of less than i^{-1}. The sequence $n_1, n_2, \cdots$ determines the topology of the solenoid. If it is $1, 1, \cdots$ after some place, the solenoid is a simple closed curve. If it is $2, 2, \cdots$, the solenoid is the dyadic solenoid defined by D. van Dantzig **[7]** and L. Vietoris **[23]**. Other properties involving the sequence $n_1, n_2, \cdots$ are given in [**4**, p. 210]. From this description we see that every proper subcontinuum of a solenoid is an arc.

Solenoids appear as invariant sets in the qualitative theory of differential equations. In **[21]** E. S. Thomas proved that every compact 1-dimensional metric space that is minimal under some flow and contains an almost periodic point is a solenoid.

Every homogeneous plane continuum that contains an arc is a simple closed curve **[4]** **[10]** **[15]**. Hence each planar solenoid is a simple closed curve.

Each of the three known examples of homogeneous plane continua (a circle, a pseudo-arc **[2]** **[18]**, and a circle of pseudo-arcs **[5]**) is circle-like. If one could show that every homogeneous plane continuum is circle-like, it would follow that there does not exist a fourth example **[6]** **[12]** [**14**, p. 49] and a long outstanding problem would be solved.

A *topological transformation group* (G, M) is a topological group G together with a topological space M and a continuous mapping $(g, w) \rightarrow gw$ of $G \times M$ into M such that $ew = w$ (e denotes the identity of G) and $(gh)w = g(hw)$ for all elements g, h of G and w of M.

For each point w of M, let G_w be the isotropy subgroup of w in G (that is, the set of all elements g of G such that $gw = w$). Let G/G_w be the left coset space with the quotient topology. The mapping φ_w of G/G_w onto Gw that sends gG_w to gw is one-to-one and continuous. The set Gw is called the *orbit* of w.

Assume M is a continuum and G is the topological group of homeomorphisms of M onto M with the topology of uniform convergence [**16**, p. 88]. E. G. Effros [**8**, Theorem 2.1] proved that each orbit is a set of the type G_δ in M if and only if for each point w of M, the mapping φ_w is a homeomorphism.

Suppose M is a homogeneous continuum. Then the orbit of each point of M is M, a G_δ-set. According to Effros' theorem, for each point w of M, the coset space G/G_w is homeomorphic to M. By Theorem 2 of §4, if M has the additional property that all of its proper subcontinua are arcs, then G/G_w is a solenoid and, therefore, a topological group. Note that G_w is not a normal subgroup of G.

Throughout this paper R^2 is the Cartesian plane. For each real number r, we shall denote the horizontal line $y = r$ and the vertical line $x = r$ in R^2 by $H(r)$ and $V(r)$ respectively.

Let P and Q be subsets of R^2. The set P is said to *project horizontally* into Q if every horizontal line in R^2 that meets P also meets Q.

We shall denote the boundary and the closure of a given set Z by $\mathrm{Bd}\, Z$ and $\mathrm{Cl}\, Z$ respectively.

3. Preliminary results.

In this section M is a homogeneous continuum (with metric ρ) having only arcs for proper subcontinua.

Let p and q be two points of the same arc component of M. The union of all arcs in M that have p as an endpoint and contain q is called a *ray* starting at p.

The following two lemmas are easy to verify.

LEMMA 1. *Each ray is dense in M.*

LEMMA 2. *If an open subset Z of M is not dense in M, then each component of Z is an arc segment with both endpoints in* $\mathrm{Bd}\, Z$.

Let ϵ be a positive number. A homeomorphism h of M onto M is called an *ϵ-homeomorphism* if $\rho(v, h(v)) < \epsilon$ for each point v of M.

LEMMA 3. *Suppose ϵ is a given positive number and w is a point of M. Then w belongs to an open subset W of M with the following property. For each pair p, q of points of W, there exists an ϵ-homeomorphism h of M onto M such that $h(p) = q$.*

Proof. Define G, G_w, and φ_w as in §2. Since M is homogeneous, the orbit of each point of M is M. Therefore φ_w is a homeomorphism of G/G_w onto M [**8**, Theorem 2.1].

Let π_w be the natural open mapping of G onto G/G_w that sends g to gG_w. Define T_w to be the mapping of G onto M that sends g to $g(w)$. Since $T_w = \varphi_w \pi_w$, it follows that T_w is an open mapping [**22**, Theorem 3.1]. Note that the following diagram commutes.

$$\begin{tikzcd} G \arrow[rr, "T_w"] \arrow[dr, "\pi_w"'] & & M \\ & G/G_w \arrow[ur, "\varphi_w"'] & \end{tikzcd}$$

Let U be the open subset of G consisting of all $\epsilon/2$-homeomorphisms of M onto M. Define W to be the open set $T_w[U]$. Since the identity e belongs to U and $T_w(e) = w$, the set W contains w.

Assume p and q are points of W. Let f and g be elements of U such that $T_w(f) = p$ and $T_w(g) = q$. Since $f(w) = p$ and $g(w) = q$, the mapping $h = gf^{-1}$ of M onto M is an ϵ-homeomorphism with the property that $h(p) = q$.

For each positive integer i, let A_i be an arc with endpoints p_i and q_i. The sequence $A_1, A_2, \cdots$ is said to be *folded* if it converges to an arc A and the sequence $p_1, q_1, p_2, q_2, \cdots$ converges to an endpoint of A.

LEMMA 4. (Bing [**4**, Theorem 6, p. 220]). *There does not exist a folded sequence of arcs in M.*

Lemma 4 follows from a simple argument (shorter than Bing's) involving Lemma 3 and the fact that M does not contain a triod.

A chain $L_1, L_2, \cdots, L_n$ in M is said to be *free* if $\mathrm{Cl}\, L_1 \cap \mathrm{Cl}\, L_n = \emptyset$ and $\mathrm{Bd} \cup \{L_i : 1 \leqq i \leqq n\}$ is a subset of $\mathrm{Cl}(L_1 \cup L_n)$.

LEMMA 5. (Bing [**4**, Property 17, p. 219]). *Let A be an arc in M with endpoints p and q. For each positive number ϵ, there exists a free ϵ-chain $L_1, L_2, \cdots, L_n$ in M covering A such that p and q belong to L_1 and L_n respectively.*

A continuum is *decomposable* if it is the union of two proper subcontinua; otherwise, it is *indecomposable.*

LEMMA 6. *If M is decomposable, then M is a simple closed curve.*

Proof. Since M is the union of two proper subcontinua (arcs), M is locally connected. Since M is homogeneous, it does not have a separating point. Hence M contains a simple closed curve [**19**, Theorem 13, p. 91]. It follows that M is a simple closed curve.

4. Principal results.

THEOREM 1. *If M is a homogeneous continuum and every proper subcontinuum of M is an arc, then M is circle-like.*

Proof. According to Lemma 6, if M is decomposable, then M is a simple closed curve and therefore circle-like. Hence we assume that M is indecomposable.

By Lemmas 4 and 5, there exists a free chain $L_1, L_2, \cdots, L_\alpha$ ($\alpha > 5$) in M such that $N = \mathrm{Cl} \cup \{L_i : 1 \leqq i \leqq \alpha\}$ is a proper subset of M and $N - \mathrm{Cl} \cup \{L_i : 3 \leqq i \leqq \alpha - 2\}$ contains every arc in N that has both of its endpoints in $\mathrm{Cl}\, L_1$ or $\mathrm{Cl}\, L_\alpha$. (This chain is formed from another free chain by unioning links to make L_2 and $L_{\alpha-1}$ sufficiently long and narrow.) Let B be the union of all components of N that meet $\mathrm{Cl}(L_3 \cup L_{\alpha-2})$. By Lemma 2, each component of B is an arc with one endpoint in $\mathrm{Bd}\, L_1$ and the other endpoint in $\mathrm{Bd}\, L_\alpha$. Note that B is a closed set. Since M is indecomposable, each component of B is a continuum of condensation.

Since B contains no folded sequence of arcs, we can assume that B is the intersection of M and the plane R^2 and that the following conditions are satisfied:

I. A component C of B is $\{(x, y) : 0 \leqq x \leqq 6 \text{ and } y = 0\}$.

II. Each component of $B - C$ is a horizontal interval above $H(0)$ (the x-axis) and below $H(1)$ that crosses both $V(1)$ and $V(5)$.

III. The sets $\mathrm{Cl}(L_1 \cup L_2 \cup L_{\alpha-1} \cup L_\alpha)$ and $\{(x, y) : 1 \leqq x \leqq 5\}$ are disjoint.

(Bing's theorem [2, Theorem 11], involving sequences of refining covers that induce a homeomorphism, can be used to define this embedding of B in R^2. Each cover of B consists of finitely many free chains that correspond to disjoint straight horizontal chains with rectangular links in R^2.) Note that $B \cap \{(x, y) : 1 < x < 5\}$ is an open subset of M.

Let ρ be a metric on M whose restriction to B agrees with the Euclidean metric on R^2 [1, Theorems 4 and 5].

There exists a positive number d less than 1 such that $M \cap H(d) = \varnothing$ and the following condition is satisfied:

Property 1. Every arc in M that has its endpoints in $\{(x, y) : x = 3 \text{ and } 0 \leqq y < d\}$ meets both $\{(x, y) : x = 1 \text{ and } 0 \leqq y < d\}$ and $\{(x, y) : x = 5 \text{ and } 0 \leqq y < d\}$.

To see this we assume Property 1 does not hold for any positive number d. For each positive integer i, let W_i be an open set in

$M \cap \{(x,y): 1 < x < 5\}$ that contains (3,0) such that for each pair p, q of points of W_i, there exists an i^{-1}-homeomorphism of M onto M that takes p to q (Lemma 3). For each i, there exists an arc A_i in M with endpoints p_i and q_i in $W_i \cap V(3)$ such that the horizontal interval Γ_i from p_i to $V(1)$ is in A_i if and only if the horizontal interval Δ_i from q_i to $V(1)$ is in A_i.

For each i, let h_i be an i^{-1}-homeomorphism of M onto M such that $h_i(p_i) = q_i$. Since each h_i maps Γ_i approximately onto Δ_i, for each i, there exists a point a_i of A_i such that $h_i(a_i) = a_i$.

For each i, let B_i be the arc in A_i from p_i to a_i. Note that for each i, the diameter of B_i is greater than 1 and $B_i \cap h_i[B_i]$ consists of the point a_i.

Let a be a limit point of the sequence $\{a_i\}$. Assume without loss of generality that $\{a_i\}$ is a convergent sequence in $E = \{v \in M : \rho(v,a) < 1/2\}$.

For each i, let E_i be an arc in $B_i \cap \mathrm{Cl}\, E$ that goes from a point b_i of $\mathrm{Bd}\, E$ to a_i. Assume without loss of generality that $\{b_i\}$ converges to a point of $\mathrm{Bd}\, E$ and $\{E_i\}$ converges to an arc F in $\mathrm{Cl}\, E$. Since each h_i is an i^{-1}-homeomorphism, $\{E_i \cup h_i[E_i]\}$ is a folded sequence of arcs converging to F. This contradiction of Lemma 4 completes our argument for Property 1.

For $i = 1$ and 2, let

$$D_i = M \cap \{(x,y): i \leqq x \leqq 6 - i \quad \text{and} \quad 0 \leqq y < d\}.$$

Let ϵ be a given positive number less than $\rho(D_2, M - D_1)$. We shall complete this proof by defining an ϵ-circular chain that covers M.

By Lemma 1, there exists an arc A in M that is irreducible with respect to the property that it contains $\{(5,0),(6,0)\}$ and intersects $\{(x,y): x = 5 \text{ and } 0 < y < d\}$. According to Property 1, A intersects $\{(x,y): x = 4 \text{ and } 0 < y < d\}$.

Let W be an open set in $D_1 - A$ containing (4,0) such that for each pair p, q of points of W, there exists an $\epsilon/50$-homeomorphism of M onto M that takes p to q (Lemma 3).

Let c be a number ($0 < c < \epsilon/50$) such that $M \cap H(c) = \varnothing$ and $M \cap \{(x,y): x = 4 \text{ and } 0 \leqq y < c\}$ is in W. Since W and A are disjoint, c is less than d.

For $i = 1$ and 2, let

$$C_i = M \cap \{(x,y): i \leqq x \leqq 6 - i \quad \text{and} \quad 0 \leqq y < c\}.$$

Let δ be the minimum of ϵ and $\rho(C_2, M - C_1)$. Let U be an open subset of C_1 containing (2,0) such that for each point q of U, there exists a δ-homeomorphism of M onto M that takes (2,0) to q (Lemma 3).

Define S to be the ray in M that starts at (2,0) and contains A. Let $\{s_i\}$ be the sequence consisting of all points of $S \cap \{(x, y): x = 3$ and $0 \leqq y < d\}$ and having the property that for each i, the points s_i precedes s_{i+1} with respect to the linear order on S.

Define T_1 to be an arc containing A in S that starts at the point $t_1 = (2,0)$ and ends at a point t_2 of $U \cap V(2)$. Let h be a δ-homeomorphism of M onto M that takes t_1 to t_2.

We proceed inductively. Assume an arc T_n is defined in S with endpoints t_n and t_{n+1} in $C_2 \cap V(2)$. Let y be the number such that $h(t_{n+1})$ belongs to $H(y)$. Define T_{n+1} to be the arc in S with endpoints t_{n+1} and $t_{n+2} = (2, y)$. Since h is a δ-homeomorphism, t_{n+2} belongs to C_2. Note that since each T_n has diameter greater than 1, the ray S is the union of $\{T_n : n = 1, 2, \cdots\}$.

Define β to be the largest integer such that $\{s_i : 1 \leqq i \leqq \beta\}$ is a subset of T_1. The δ-homeomorphism h maps each T_n approximately onto T_{n+1}. Hence, for each n, the arc T_n contains $\{s_i : (n-1)\beta < i \leqq n\beta\}$. Furthermore, β has the following property:

Property 2. For each positive integer i, the point s_i belongs to C_2 if and only if $s_{i+\beta}$ belongs to C_2.

Define γ to be the least positive integer that has Property 2. Note that since s_2 does not belong to C_2, the integer γ is greater than 1.

Let K be $\{s_i : i = j\gamma + 1$ and $j = 0, 1, 2, \cdots\}$, and let L be $(S \cap D_2 \cap V(3)) - K$.

Property 3. The sets $\mathrm{Cl}\, K$ and $\mathrm{Cl}\, L$ are disjoint.

To establish Property 3, we assume there is a point z in $\mathrm{Cl}\, K \cap \mathrm{Cl}\, L$. Let Z be an open subset of M containing z such that for each pair p, q of points of Z, there exists a δ-homeomorphism of M onto M that takes p to q (Lemma 3).

Let s_i and s_n be points of $Z \cap K$ and $Z \cap L$, respectively, and let f be a δ-homeomorphism of M onto M such that $f(s_i) = s_n$. Let θ be the smallest positive integer such that $s_{n-\theta}$ belongs to K. The existence of f implies that θ has Property 2. Since θ is less than γ, this is a contradiction and Property 3 is established.

Note that since $M = \mathrm{Cl}\, S$ (Lemma 1), $\mathrm{Cl}(K \cup L) = D_2 \cap V(3)$.

Let I be the arc in S that goes from s_1 to $s_{\gamma+1}$. By an argument similar to Bing's [**4**, Property 17, p. 219], there exists a free $\epsilon/50$-chain $P_1, P_2, \cdots, P_\lambda$ in M covering I such that

(i) s_1 and $s_{\gamma+1}$ belong to P_1 and P_λ respectively,

(ii) $P_1 \cup P_\lambda$ is in C_2,

(iii) each component of $H = \cup \{P_j : 1 \leqq j \leqq \lambda\}$ that meets $\mathrm{Cl}\, P_1$ also meets P_1 and $V(5)$, and

(iv) each component of H that meets $\mathrm{Cl}\, P_\lambda$ meets P_λ and $V(1)$.

From Property 1 we get the following:

Property 4. Each component of H meets both P_1 and P_λ.

Let P_μ be an element of $P_1, P_2, \cdots, P_\lambda$ that contains the point $(4,0)$. Since W intersects each component of C_2, there exists a finite sequence $g_1, g_2, \cdots, g_\sigma$ of $\epsilon/50$-homeomorphisms of M onto M such that $\mathrm{Cl}\, K$ projects horizontally into $\cup \{g_i[P_\mu] : 1 \leqq i \leqq \sigma\}$. Assume without loss of generality that no proper subsequence of $g_1, g_2, \cdots, g_\sigma$ has this horizontal projection property.

Note that each $g_i[P_\mu]$ is a subset of D_1.

From Properties 1 and 4 we get the following:

Property 5. For each i $(1 \leqq i \leqq \sigma)$, if T is a component of $g_i[H]$, then $T \cap g_i[\mathrm{Cl}\, P_\mu]$ is a nonempty set that projects horizontally to a point of $D_2 \cap V(3)$.

For each i $(1 \leqq i \leqq \sigma)$, let X_i be the set consisting of all points in $g_i[P_\mu]$ that project horizontally into $\mathrm{Cl}\, K$, and let Y_i be the union of all components of $g_i[H]$ that meet X_i.

For each i $(1 \leqq i \leqq \sigma)$, the set Y_i is open in M. To see this assume that for some i, a point u of Y_i is in $\mathrm{Cl}\,(M - Y_i)$. According to Property 3, u does not belong to $g_i[P_\mu]$. By Property 5, there exists a sequence $\{J_n\}$ of arcs in $g_i[H]$ that meet $g_i[P_\mu]$ such that the limit superior J of $\{J_n\}$ is an arc in $g_i[H]$ that contains u and for each n, the set $J_n \cap g_i[P_\mu]$ projects horizontally to a point of $\mathrm{Cl}\, L$. It follows that $J \cap g_i[\mathrm{Cl}\, P_\mu]$ is a nonempty set that projects horizontally to a point of $\mathrm{Cl}\, L$. Since J is in the u-component of Y_i, this is a contradiction of Property 5. Hence Y_i is an open subset of M.

For each i $(1 \leqq i \leqq \sigma)$ and j $(1 \leqq j \leqq \lambda)$, let $Q_{i,j} = Y_i \cap g_i[P_j]$. It follows from an argument similar to the one given in the preceding paragraph that for each i, the set $\mathrm{Cl}\,(Q_{i,1} \cup Q_{i,\lambda})$ contains $\mathrm{Bd} \cup \{Q_{i,j} : 1 \leqq j \leqq \lambda\}$. Hence, for each i, the sequence $Q_{i,1}, Q_{i,2}, \cdots, Q_{i,\lambda}$ is a free chain in M.

Property 6. For each i $(1 \leqq i \leqq \sigma)$, the set $Q_{i,1} \cup Q_{i,\lambda}$ projects horizontally into $\mathrm{Cl}\, K$.

Obviously, $Q_{i,1}$ projects horizontally into $\mathrm{Cl}\, K$. Therefore, to establish Property 6, we assume there is a point t of $Q_{i,\lambda}$ that projects horizontally into $\mathrm{Cl}\, L$. By Property 3, there exists a positive number η less than ϵ such that $Q = \{v \in M : \rho(v, t) < \eta\}$ projects horizontally in $\mathrm{Cl}\, L$.

Let T denote the t-component of Y_i, and let w be a point of $T \cap Q_{\iota,1}$ (Property 4). Since g_i is an $\epsilon/50$-homeomorphism, T crosses $D_1 \cap V(1)$ exactly γ times (Property 1). Since w belongs to $Q_{\iota,1}$, it projects horizontally into $\mathrm{Cl}\, K$.

By Lemma 3, there exists an η-homeomorphism g of M onto M such that $g(w)$ belongs to $Q_{\iota,1}$ and projects horizontally into K. Since the $g(w)$-component of Y_i is an arc segment in S that crosses $D_1 \cap V(1)$ exactly γ times and is mapped approximately onto T by g^{-1}, the point $g(t)$ of Q projects horizontally into K. This contradiction of the definition of Q completes our argument for Property 6.

Let π be an integer $(5 < \pi < \mu)$ such that P_π contains the point $(3 + \epsilon/10, 0)$. Let ω be an integer $(\mu < \omega < \lambda - 4)$ such that P_ω contains the point of $V(3 - \epsilon/10)$ that projects horizontally to $s_{\gamma+1}$.

Property 7. For each n $(1 \leqq n \leqq \sigma)$, the set $Q_{n,1} \cup Q_{n,\lambda}$ does not intersect $\cup\{Q_{i,j} : 1 \leqq i \leqq \sigma \text{ and } \pi \leqq j \leqq \omega\}$.

To see this assume there exist integers i, j, and n such that $\pi \leqq j \leqq \omega$ and a point p belongs to $Q_{i,j} \cap (Q_{n,1} \cup Q_{n,\lambda})$. According to Property 6, $\{p\} \cup Q_{\iota,1} \cup Q_{\iota,\lambda}$ projects horizontally into $\mathrm{Cl}\, K$. By Property 3, there exists a positive number χ less than ϵ such that $\{v \in M : \rho(v, p) < \chi\}$ projects horizontally into $\mathrm{Cl}\, K$.

Let P be the p-component of Y_ι. Let Y be an arc in P that goes from a point q of $Q_{\iota,1}$ to p. Since g_i and g_n are $\epsilon/50$-homeomorphisms and $\pi \leqq j \leqq \omega$, the set $Q_{i,1} \cup Q_{i,\lambda}$ and the p-component of $P \cap D_1$ are disjoint. Hence Y crosses $D_1 \cap V(1)$ exactly ι times where ι is a positive integer less than γ.

By Lemma 3, there exists a χ-homeomorphism k of M onto M such that $k(q)$ belongs to $Q_{\iota,1}$ and projects horizontally into K. The arc $k[Y]$ crosses $D_1 \cap V(1)$ exactly ι times. Since $k[Y]$ is in S and $\rho(p, k(p)) < \chi$, the point $k(p)$ projects horizontally into K. It follows from the definition of K that ι is a multiple of γ, and this is a contradiction. Hence Property 7 holds.

For each i $(1 \leqq i \leqq \sigma)$ and j $(1 \leqq j \leqq \lambda)$, let $P_{i,j} = Q_{i,j} - \mathrm{Cl} \cup \{Y_n : 1 \leqq n < i\}$. By Property 7, for each i, the subchain of $P_{i,1}, P_{i,2}, \cdots, P_{i,\lambda}$ that has $P_{i,\pi}$ and $P_{i,\omega}$ as end links is free in M.

For each j $(1 \leqq j \leqq \lambda)$, let $U_j = \cup\{P_{i,j} : 1 \leqq i \leqq \sigma\}$. The subchain $\mathscr{C}$ of $U_1, U_2, \cdots, U_\lambda$ that has U_π and U_ω as end links is a free $\epsilon/16$-chain in M.

Let D be the union of all components of $C_2 \cap \{(x, y) : 3 - \epsilon/5 < x < 3 + \epsilon/5\}$ that meet $\mathrm{Cl}\, K$. According to Property 3, D is open in M. The diameter of D is less than $\epsilon/2$. Each point of $U_\pi \cup U_\omega$ is within $\epsilon/5$ of $V(3)$. By Property 6, $U_\pi \cup U_\omega$ projects horizontally into $\mathrm{Cl}\, K$. Hence $U_\pi \cup U_\omega$ is in D.

Let τ be the largest integer less than μ such that U_τ intersects

D. Let ψ be the smallest integer greater than μ such that U_ψ intersects D. For each j $(1 \leqq j < \psi - \tau)$, let $Z_j = U_{\tau+j}$. Note that $Z_1, Z_2, \cdots, Z_{\psi-\tau-1}$ is a free ϵ-chain in M.

Define $Z_{\psi-\tau}$ to be the union of D and all elements of $\mathcal{D} = \{U_j : \pi \leqq j \leqq \tau \text{ or } \psi \leqq j \leqq \omega\}$. Since $\mathrm{Cl}\, K$ projects horizontally into U_μ and $\mathcal{C}$ is a free chain in M, each element of $\mathcal{D}$ intersects D. Thus $Z_{\psi-\tau}$ is an open set in M of diameter less than ϵ. Note that $Z_{\psi-\tau}$ meets both Z_1 and $Z_{\psi-\tau-1}$.

Since $\mathcal{C}$ is free and $U_\pi \cup U_\omega$ is in D, the boundary of $\cup \{Z_j : 1 \leqq j < \psi - \tau\}$ is in $Z_{\psi-\tau}$. Since $\mathrm{Cl}\, K$ projects horizontally into U_μ, the set Z_1 contains every boundary point of $Z_{\psi-\tau}$ that is to the right of $V(3)$ in R^2.

Furthermore, each point of $\mathrm{Bd}\, Z_{\psi-\tau}$ that is to the left of $V(3)$ is in $Z_{\psi-\tau-1}$. To see this let s be such a point. Let X be the arc in M that intersects $V(1)$ and is irreducible between s and $\mathrm{Cl}\, U_\mu$ (Lemma 1). By Property 1, X does not meet $U_\pi \cup U_\omega$. Since U_μ is an interior link in the free chain $\mathcal{C}$, the arc X is covered by $\mathcal{C}$ and s belongs to $Z_{\psi-\tau-1}$.

It follows that $\mathrm{Bd}\, Z_{\psi-\tau}$ is in $Z_1 \cup Z_{\psi-\tau-1}$. Therefore $Z_1, Z_2, \cdots, Z_{\psi-\tau}$ is an ϵ-circular chain that covers M. Hence M is circle-like.

Since every homogeneous circle-like continuum that contains an arc is a solenoid [**4**, Theorem 9, p. 228], Theorem 1 implies the following:

THEOREM 2. *A continuum M is a solenoid if and only if M is homogeneous and every proper subcontinuum of M is an arc.*

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