A CHARACTERIZATION OF SOLENOIDS

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Suppose \( M \) is a homogeneous continuum and every proper subcontinuum of \( M \) is an arc. Using a theorem of E. G. Effros involving topological transformation groups, we prove that \( M \) is circle-like. This answers in the affirmative a question raised by R. H. Bing. It follows from this result and a theorem of Bing that \( M \) is a solenoid. Hence a continuum is a solenoid if and only if it is homogeneous and all of its proper subcontinua are arcs. The group \( G \) of homeomorphisms of \( M \) onto \( M \) with the topology of uniform convergence has an unusual property. For each point \( w \) of \( M \), let \( G_w \) be the isotropy subgroup of \( w \) in \( G \). Although \( G_w \) is not a normal subgroup of \( G \), it follows from Effros’ theorem and Theorem 2 of this paper that the coset space \( G/G_w \) is a solenoid homeomorphic to \( M \) and, therefore, a topological group.

1. Introduction. Let \( \mathcal{S} \) be the class of all homogeneous continua \( M \) such that every proper subcontinuum of \( M \) is an arc. It is known that every solenoid belongs to \( \mathcal{S} \). It is also known that every circle-like element of \( \mathcal{S} \) is a solenoid. In fact, in 1960 R. H. Bing [4, Theorem 9, p. 228] proved that each homogeneous circle-like continuum that contains an arc is a solenoid. At that time Bing [4, p. 219] asked whether every element of \( \mathcal{S} \) is a solenoid. In this paper we answer Bing’s question in the affirmative by proving that every element of \( \mathcal{S} \) is circle-like.

2. Definitions and related results. We call a nondegenerate compact connected metric space a continuum.

A chain is a finite sequence \( L_1, L_2, \cdots, L_n \) of open sets such that \( L_i \cap L_j \neq \emptyset \) if and only if \( |i - j| \leq 1 \). If \( L_1 \) also intersects \( L_n \), the sequence is called a circular chain. Each \( L_i \) is called a link. A chain (circular chain) is called an \( \epsilon \)-chain (\( \epsilon \)-circular chain) if each of its links has diameter less than \( \epsilon \). A continuum is said to be arc-like (circle-like) if for each \( \epsilon > 0 \), it can be covered by an \( \epsilon \)-chain (\( \epsilon \)-circular chain).

A space is homogeneous if for each pair \( p, q \) of its points there exists a homeomorphism of the space onto itself that takes \( p \) to \( q \). Bing [2] [3] proved that a continuum is a pseudo-arc if and only if it is homogeneous and arc-like. L. Fearnley [9] and J. T. Rogers, Jr. [20] independently showed that every homogeneous, hereditarily indecomposable, circle-like
continuum is a pseudo-arc [11]. However, there are many topologically different homogeneous circle-like continua that have decomposable subcontinua [24] [25].

Let \( n_1, n_2, \cdots \) be a sequence of positive integers. For each positive integer \( i \), let \( G_i \) be the unit circle \( \{ z \in \mathbb{R}^2 : |z| = 1 \} \), and let \( f_i \) be the map of \( G_{i+1} \) onto \( G_i \) defined by \( f_i(z) = z^{n_i} \). The inverse limit space of the sequence \( \{G_i, f_i\} \) is called a solenoid. Since each \( G_i \) is a topological group and each \( f_i \) is a homomorphism, every solenoid is a topological group [13, Theorem 6.14, p. 56] and therefore homogeneous. Each solenoid is circle-like since it is an inverse limit of circles with surjective bonding maps [17, Lemma 1, p. 147].

A solenoid can be described as the intersection of a sequence of solid tori \( M_1, M_2, \cdots \) such that \( M_{i+1} \) runs smoothly around inside \( M_i \) exactly \( n_i \) times longitudinally without folding back and \( M_i \) has cross diameter of less than \( i^{-1} \). The sequence \( n_1, n_2, \cdots \) determines the topology of the solenoid. If it is 1, 1, \cdots after some place, the solenoid is a simple closed curve. If it is 2, 2, \cdots, the solenoid is the dyadic solenoid defined by D. van Dantzig [7] and L. Vietoris [23]. Other properties involving the sequence \( n_1, n_2, \cdots \) are given in [4, p. 210]. From this description we see that every proper subcontinuum of a solenoid is an arc.

Solenoids appear as invariant sets in the qualitative theory of differential equations. In [21] E. S. Thomas proved that every compact 1-dimensional metric space that is minimal under some flow and contains an almost periodic point is a solenoid.

Every homogeneous plane continuum that contains an arc is a simple closed curve [4] [10] [15]. Hence each planar solenoid is a simple closed curve.

Each of the three known examples of homogeneous plane continua (a circle, a pseudo-arc [2] [18], and a circle of pseudo-arcs [5]) is circle-like. If one could show that every homogeneous plane continuum is circle-like, it would follow that there does not exist a fourth example [6] [12] [14, p. 49] and a long outstanding problem would be solved.

A topological transformation group \((G, M)\) is a topological group \(G\) together with a topological space \(M\) and a continuous mapping \((g, w) \rightarrow gw\) of \(G \times M\) into \(M\) such that \(ew = w\) (\(e\) denotes the identity of \(G\)) and \((gh)w = g(hw)\) for all elements \(g, h\) of \(G\) and \(w\) of \(M\).

For each point \(w\) of \(M\), let \(G_w\) be the isotropy subgroup of \(w\) in \(G\) (that is, the set of all elements \(g\) of \(G\) such that \(gw = w\)). Let \(G/G_w\) be the left coset space with the quotient topology. The mapping \(\varphi_w\) of \(G/G_w\) onto \(Gw\) that sends \(gG_w\) to \(gw\) is one-to-one and continuous. The set \(Gw\) is called the orbit of \(w\).

Assume \(M\) is a continuum and \(G\) is the topological group of homeomorphisms of \(M\) onto \(M\) with the topology of uniform convergence [16, p. 88]. E. G. Effros [8, Theorem 2.1] proved that each
orbit is a set of the type $G_δ$ in $M$ if and only if for each point $w$ of $M$, the mapping $φ_w$ is a homeomorphism.

Suppose $M$ is a homogeneous continuum. Then the orbit of each point of $M$ is $M$, a $G_δ$-set. According to Effros’ theorem, for each point $w$ of $M$, the coset space $G/G_w$ is homeomorphic to $M$. By Theorem 2 of §4, if $M$ has the additional property that all of its proper subcontinua are arcs, then $G/G_w$ is a solenoid and, therefore, a topological group. Note that $G_w$ is not a normal subgroup of $G$.

Throughout this paper $R^2$ is the Cartesian plane. For each real number $r$, we shall denote the horizontal line $y = r$ and the vertical line $x = r$ in $R^2$ by $H(r)$ and $V(r)$ respectively.

Let $P$ and $Q$ be subsets of $R^2$. The set $P$ is said to project horizontally into $Q$ if every horizontal line in $R^2$ that meets $P$ also meets $Q$.

We shall denote the boundary and the closure of a given set $Z$ by $BdZ$ and $ClZ$ respectively.

3. Preliminary results. In this section $M$ is a homogeneous continuum (with metric $ρ$) having only arcs for proper subcontinua.

Let $p$ and $q$ be two points of the same arc component of $M$. The union of all arcs in $M$ that have $p$ as an endpoint and contain $q$ is called a ray starting at $p$.

The following two lemmas are easy to verify.

**Lemma 1.** Each ray is dense in $M$.

**Lemma 2.** If an open subset $Z$ of $M$ is not dense in $M$, then each component of $Z$ is an arc segment with both endpoints in $BdZ$.

Let $ε$ be a positive number. A homeomorphism $h$ of $M$ onto $M$ is called an $ε$-homeomorphism if $ρ(v, h(v)) < ε$ for each point $v$ of $M$.

**Lemma 3.** Suppose $ε$ is a given positive number and $w$ is a point of $M$. Then $w$ belongs to an open subset $W$ of $M$ with the following property. For each pair $p, q$ of points of $W$, there exists an $ε$-homeomorphism $h$ of $M$ onto $M$ such that $h(p) = q$.

**Proof.** Define $G$, $G_w$, and $φ_w$ as in §2. Since $M$ is homogeneous, the orbit of each point of $M$ is $M$. Therefore $φ_w$ is a homeomorphism of $G/G_w$ onto $M$ [8, Theorem 2.1].

Let $π_w$ be the natural open mapping of $G$ onto $G/G_w$ that sends $g$ to $gG_w$. Define $T_w$ to be the mapping of $G$ onto $M$ that sends $g$ to $g(w)$. Since $T_w = φ_wπ_w$, it follows that $T_w$ is an open mapping [22, Theorem 3.1]. Note that the following diagram commutes.
Let $U$ be the open subset of $G$ consisting of all $\varepsilon/2$-homeomorphisms of $M$ onto $M$. Define $W$ to be the open set $T_w[U]$. Since the identity $e$ belongs to $U$ and $T_w(e) = w$, the set $W$ contains $w$.

Assume $p$ and $q$ are points of $W$. Let $f$ and $g$ be elements of $U$ such that $T_w(f) = p$ and $T_w(g) = q$. Since $f(w) = p$ and $g(w) = q$, the mapping $h = gf^{-1}$ of $M$ onto $M$ is an $\varepsilon$-homeomorphism with the property that $h(p) = q$.

For each positive integer $i$, let $A_i$ be an arc with endpoints $p_i$ and $q_i$. The sequence $A_1, A_2, \ldots$ is said to be folded if it converges to an arc $A$ and the sequence $p_1, q_1, p_2, q_2, \cdots$ converges to an endpoint of $A$.

**Lemma 4.** (Bing [4, Theorem 6, p. 220]). *There does not exist a folded sequence of arcs in $M$.*

Lemma 4 follows from a simple argument (shorter than Bing's) involving Lemma 3 and the fact that $M$ does not contain a triod.

A chain $L_1, L_2, \ldots, L_n$ in $M$ is said to be free if $\text{Cl } L_1 \cap \text{Cl } L_n = \emptyset$ and $\text{Bd } \cup \{L_i : 1 \leq i \leq n\}$ is a subset of $\text{Cl } (L_1 \cup L_n)$.

**Lemma 5.** (Bing [4, Property 17, p. 219]). *Let $A$ be an arc in $M$ with endpoints $p$ and $q$. For each positive number $\varepsilon$, there exists a free $\varepsilon$-chain $L_1, L_2, \ldots, L_n$ in $M$ covering $A$ such that $p$ and $q$ belong to $L_1$ and $L_n$ respectively.*

A continuum is decomposable if it is the union of two proper subcontinua; otherwise, it is indecomposable.

**Lemma 6.** *If $M$ is decomposable, then $M$ is a simple closed curve.*

**Proof.** Since $M$ is the union of two proper subcontinua (arcs), $M$ is locally connected. Since $M$ is homogeneous, it does not have a separating point. Hence $M$ contains a simple closed curve [19, Theorem 13, p. 91]. It follows that $M$ is a simple closed curve.
4. Principal results.

**Theorem 1.** If \( M \) is a homogeneous continuum and every proper subcontinuum of \( M \) is an arc, then \( M \) is circle-like.

**Proof.** According to Lemma 6, if \( M \) is decomposable, then \( M \) is a simple closed curve and therefore circle-like. Hence we assume that \( M \) is indecomposable.

By Lemmas 4 and 5, there exists a free chain \( L_1, L_2, \ldots, L_\alpha \) in \( M \) such that \( N = \text{Cl} \cup \{L_i : 1 \leq i \leq \alpha \} \) is a proper subset of \( M \) and \( N - \text{Cl} \cup \{L_i : 3 \leq i \leq \alpha - 2\} \) contains every arc in \( N \) that has both of its endpoints in \( \text{Cl} L_1 \) or \( \text{Cl} L_\alpha \). (This chain is formed from another free chain by unioning links to make \( L_2 \) and \( L_{\alpha-1} \) sufficiently long and narrow.) Let \( B \) be the union of all components of \( N \) that meet \( \text{Cl}(L_3 \cup L_{\alpha-2}) \). By Lemma 2, each component of \( B \) is an arc with one endpoint in \( \text{Bd} L_1 \) and the other endpoint in \( \text{Bd} L_\alpha \). Note that \( B \) is a closed set. Since \( M \) is indecomposable, each component of \( B \) is a continuum of condensation.

Since \( B \) contains no folded sequence of arcs, we can assume that \( B \) is the intersection of \( M \) and the plane \( \mathbb{R}^2 \) and that the following conditions are satisfied:

I. A component \( C \) of \( B \) is \( \{(x, y) : 0 \leq x \leq 6 \text{ and } y = 0\} \).

II. Each component of \( B - C \) is a horizontal interval above \( H(0) \) (the \( x \)-axis) and below \( H(1) \) that crosses both \( V(1) \) and \( V(5) \).

III. The sets \( \text{Cl}(L_1 \cup L_2 \cup L_{\alpha-1} \cup L_\alpha) \) and \( \{(x, y) : 1 \leq x \leq 5\} \) are disjoint.

(Bing’s theorem [2, Theorem 11], involving sequences of refining covers that induce a homeomorphism, can be used to define this embedding of \( B \) in \( \mathbb{R}^2 \). Each cover of \( B \) consists of finitely many free chains that correspond to disjoint straight horizontal chains with rectangular links in \( \mathbb{R}^2 \).) Note that \( B \cap \{(x, y) : 1 < x < 5\} \) is an open subset of \( M \).

Let \( \rho \) be a metric on \( M \) whose restriction to \( B \) agrees with the Euclidean metric on \( \mathbb{R}^2 \) [1, Theorems 4 and 5].

There exists a positive number \( d \) less than 1 such that \( M \cap H(d) = \emptyset \) and the following condition is satisfied:

**Property 1.** Every arc in \( M \) that has its endpoints in \( \{(x, y) : x = 3 \text{ and } 0 \leq y < d\} \) meets both \( \{(x, y) : x = 1 \text{ and } 0 \leq y < d\} \) and \( \{(x, y) : x = 5 \text{ and } 0 \leq y < d\} \).

To see this we assume Property 1 does not hold for any positive number \( d \). For each positive integer \( i \), let \( W_i \) be an open set in
$M \cap \{(x, y): 1 < x < 5\}$ that contains $(3,0)$ such that for each pair $p$, $q$ of points of $W_i$, there exists an $i^{-1}$-homeomorphism of $M$ onto $M$ that takes $p$ to $q$ (Lemma 3). For each $i$, there exists an arc $A_i$ in $M$ with endpoints $p_i$ and $q_i$ in $W_i \cap V(3)$ such that the horizontal interval $\Gamma_i$ from $p_i$ to $V(1)$ is in $A_i$ if and only if the horizontal interval $\Delta_i$ from $q_i$ to $V(1)$ is in $A_i$.

For each $i$, let $h_i$ be an $i^{-1}$-homeomorphism of $M$ onto $M$ such that $h_i(p_i) = q_i$. Since each $h_i$ maps $\Gamma_i$ approximately onto $\Delta_i$, for each $i$, there exists a point $a_i$ of $A_i$ such that $h_i(a_i) = a_i$.

For each $i$, let $B_i$ be the arc in $A_i$ from $p_i$ to $q_i$. Note that for each $i$, the diameter of $B_i$ is greater than 1 and $B_i \cap h_i[B_i]$ consists of the point $a_i$.

Let $a$ be a limit point of the sequence $\{a_i\}$. Assume without loss of generality that $\{a_i\}$ is a convergent sequence in $E = \{v \in M: \rho(v, a) < 1/2\}$.

For each $i$, let $E_i$ be an arc in $B_i \cap \text{Cl } E$ that goes from a point $b_i$ of $\text{Bd } E$ to $a_i$. Assume without loss of generality that $\{b_i\}$ converges to a point of $\text{Bd } E$ and $\{E_i\}$ converges to an arc $F$ in $\text{Cl } E$. Since each $h_i$ is an $i^{-1}$-homeomorphism, $\{E_i \cup h_i[E_i]\}$ is a folded sequence of arcs converging to $F$. This contradiction of Lemma 4 completes our argument for Property 1.

For $i = 1$ and 2, let

$$D_i = M \cap \{(x, y): i \leq x \leq 6 - i \quad \text{and} \quad 0 \leq y < d\}.$$ 

Let $\epsilon$ be a given positive number less than $\rho(D_2, M - D_1)$. We shall complete this proof by defining an $\epsilon$-circular chain that covers $M$.

By Lemma 1, there exists an arc $A$ in $M$ that is irreducible with respect to the property that it contains $\{(5,0), (6,0)\}$ and intersects $\{(x, y): x = 5 \text{ and } 0 < y < d\}$. According to Property 1, $A$ intersects $\{(x, y): x = 4 \text{ and } 0 < y < d\}$.

Let $W$ be an open set in $D_1 - A$ containing $(4,0)$ such that for each pair $p$, $q$ of points of $W$, there exists an $\epsilon/50$-homeomorphism of $M$ onto $M$ that takes $p$ to $q$ (Lemma 3).

Let $c$ be a number $(0 < c < \epsilon/50)$ such that $M \cap H(c) = \emptyset$ and $M \cap \{(x, y): x = 4 \text{ and } 0 \leq y < c\}$ is in $W$. Since $W$ and $A$ are disjoint, $c$ is less than $d$.

For $i = 1$ and 2, let

$$C_i = M \cap \{(x, y): i \leq x \leq 6 - i \quad \text{and} \quad 0 \leq y < c\}.$$ 

Let $\delta$ be the minimum of $\epsilon$ and $\rho(C_2, M - C_1)$. Let $U$ be an open subset of $C_1$ containing $(2,0)$ such that for each point $q$ of $U$, there exists a $\delta$-homeomorphism of $M$ onto $M$ that takes $(2,0)$ to $q$ (Lemma 3).
Define $S$ to be the ray in $M$ that starts at $(2,0)$ and contains $A$. Let 
\{s_i\} be the sequence consisting of all points of $S \cap \{(x,y): x = 3$ and 
$0 \leq y < d\}$ and having the property that for each $i$, the points $s_i$ precedes 
s_{i+1}^\$ with respect to the linear order on $S$.

Define $T_1$ to be an arc containing $A$ in $S$ that starts at the point 
t_1 = (2,0)$ and ends at a point $t_2$ of $U \cap V(2)$. Let $h$ be a $\delta$-
homeomorphism of $M$ onto $M$ that takes $t_1$ to $t_2$.

We proceed inductively. Assume an arc $T_n$ is defined in $S$ with endpoints 
t_n$ and $t_{n+1}$ in $C_2 \cap V(2)$. Let $y$ be the number such that 
h(t_{n+1}) belongs to $H(y)$. Define $T_{n+1}$ to be the arc in $S$ with endpoints 
t_{n+1}$ and $t_{n+2} = (2,y)$. Since $h$ is a $\delta$-homeomorphism, $t_{n+2}$ belongs to 
$C_2$. Note that since each $T_n$ has diameter greater than 1, the ray $S$ is the 
union of \{T_n: n = 1, 2, \ldots \}.

Define $\beta$ to be the largest integer such that \{$\bullet: 1 \leq i \leq \beta$\} is a subset 
of $T_1$. The $\delta$-homeomorphism $h$ maps each $T_n$ approximately onto 
$T_{n+1}$. Hence, for each $n$, the arc $T_n$ contains \{$s_i: (n-1)\beta < i \leq 
n\beta$\}. Furthermore, $\beta$ has the following property:

**Property 2.** For each positive integer $i$, the point $s_i$ belongs to $C_2$ if 
and only if $s_{i+\beta}$ belongs to $C_2$.

Define $\gamma$ to be the least positive integer that has Property 2. Note 
that since $s_2$ does not belong to $C_2$, the integer $\gamma$ is greater than 1.

Let $K$ be \{$s_i = j\gamma + 1$ and $j = 0, 1, 2, \cdots$, and let $L$ be 
$(S \cap D_2 \cap V(3)) - K$.

**Property 3.** The sets $\text{Cl} K$ and $\text{Cl} L$ are disjoint.

To establish Property 3, we assume there is a point $z$ in $\text{Cl} K \cap 
\text{Cl} L$. Let $Z$ be an open subset of $M$ containing $z$ such that for each pair 
p, $q$ of points of $Z$, there exists a $\delta$-homeomorphism of $M$ onto $M$ that 
takes $p$ to $q$ (Lemma 3).

Let $s_i$ and $s_n$ be points of $Z \cap K$ and $Z \cap L$, respectively, and let $f$ 
be a $\delta$-homeomorphism of $M$ onto $M$ such that $f(s_i) = s_n$. Let $\theta$ be the 
smallest positive integer such that $s_{n-\theta}$ belongs to $K$. The existence of $f$ 
implies that $\theta$ has Property 2. Since $\theta$ is less than $\gamma$, this is a 
contradiction and Property 3 is established.

Note that since $M = \text{Cl} S$ (Lemma 1), $\text{Cl}(K \cup L) = D_2 \cap V(3)$.

Let $I$ be the arc in $S$ that goes from $s_i$ to $s_{i+1}$. By an argument 
similar to Bing’s [4, Property 17, p. 219], there exists a free $\epsilon/50$-chain 
P_1, P_2, \cdots, $P_\lambda$ in $M$ covering $I$ such that

(i) $s_1$ and $s_{i+1}$ belong to $P_1$ and $P_\lambda$ respectively,

(ii) $P_1 \cup P_\lambda$ is in $C_2$. 

(iii) each component of $H = \bigcup \{P_i : 1 \leq j \leq \lambda\}$ that meets $\text{Cl} P_1$ also meets $P_1$ and $V(5)$, and

(iv) each component of $H$ that meets $\text{Cl} P_\lambda$ meets $P_\lambda$ and $V(1)$.

From Property 1 we get the following:

**Property 4.** Each component of $H$ meets both $P_1$ and $P_\lambda$.

Let $P_\mu$ be an element of $P_1, P_2, \cdots, P_\lambda$ that contains the point $(4,0)$. Since $W$ intersects each component of $C_2$, there exists a finite sequence $g_1, g_2, \cdots, g_\sigma$ of $\varepsilon/50$-homeomorphisms of $M$ onto $M$ such that $\text{Cl} K$ projects horizontally into $\bigcup \{g_i[P_\mu] : 1 \leq i \leq \sigma\}$. Assume without loss of generality that no proper subsequence of $g_1, g_2, \cdots, g_\sigma$ has this horizontal projection property.

Note that each $g_i[P_\mu]$ is a subset of $D_1$.

From Properties 1 and 4 we get the following:

**Property 5.** For each $i (1 \leq i \leq \sigma)$, if $T$ is a component of $g_i[H]$, then $T \cap g_i[\text{Cl} P_\mu]$ is a nonempty set that projects horizontally to a point of $D_2 \cap V(3)$.

For each $i (1 \leq i \leq \sigma)$, let $X_i$ be the set consisting of all points in $g_i[P_\mu]$ that project horizontally into $\text{Cl} K$, and let $Y_i$ be the union of all components of $g_i[H]$ that meet $X_i$.

For each $i (1 \leq i \leq \sigma)$, the set $Y_i$ is open in $M$. To see this assume that for some $i$, a point $u$ of $Y_i$ is in $\text{Cl}(M - Y_i)$. According to Property 3, $u$ does not belong to $g_i[P_\mu]$. By Property 5, there exists a sequence $\{J_n\}$ of arcs in $g_i[H]$ that meet $g_i[P_\mu]$ such that the limit superior $J$ of $\{J_n\}$ is an arc in $g_i[H]$ that contains $u$ and for each $n$, the set $J_n \cap g_i[P_\mu]$ projects horizontally to a point of $\text{Cl} L$. It follows that $J \cap g_i[\text{Cl} P_\mu]$ is a nonempty set that projects horizontally to a point of $\text{Cl} L$. Since $J$ is in the $u$-component of $Y_n$, this is a contradiction of Property 5. Hence $Y_i$ is an open subset of $M$.

For each $i (1 \leq i \leq \sigma)$ and $j (1 \leq j \leq \lambda)$, let $Q_{ij} = Y_i \cap g_i[P_j]$. It follows from an argument similar to the one given in the preceding paragraph that for each $i$, the set $\text{Cl}(Q_{i1} \cup Q_{i,\lambda})$ contains $\text{Bd} \cup \{Q_{ij} : 1 \leq j \leq \lambda\}$. Hence, for each $i$, the sequence $Q_{i1}, Q_{i2}, \cdots, Q_{i,\lambda}$ is a free chain in $M$.

**Property 6.** For each $i (1 \leq i \leq \sigma)$, the set $Q_{i1} \cup Q_{i,\lambda}$ projects horizontally into $\text{Cl} K$.

Obviously, $Q_{i1}$ projects horizontally into $\text{Cl} K$. Therefore, to establish Property 6, we assume there is a point $t$ of $Q_{i,\lambda}$ that projects horizontally into $\text{Cl} L$. By Property 3, there exists a positive number $\eta$ less than $\epsilon$ such that $Q = \{v \in M : \rho(v, t) < \eta\}$ projects horizontally in $\text{Cl} L$. 
Let $T$ denote the $t$-component of $Y_h$ and let $w$ be a point of $T \cap Q_{i,1}$ (Property 4). Since $g_i$ is an $\epsilon/50$-homeomorphism, $T$ crosses $D_1 \cap V(1)$ exactly $\gamma$ times (Property 1). Since $w$ belongs to $Q_{i,1}$, it projects horizontally into $ClK$.

By Lemma 3, there exists an $\eta$-homeomorphism $g$ of $M$ onto $M$ such that $g(w)$ belongs to $Q_{i,1}$ and projects horizontally into $K$. Since the $g(w)$-component of $Y_i$ is an arc segment in $S$ that crosses $D_1 \cap V(1)$ exactly $\gamma$ times and is mapped approximately onto $T$ by $g^{-1}$, the point $g(t)$ of $Q$ projects horizontally into $K$. This contradiction of the definition of $Q$ completes our argument for Property 6.

Let $\tau$ be an integer ($5 < \tau < \mu$) such that $P_\tau$ contains the point $(3 + \epsilon/10, 0)$. Let $\omega$ be an integer ($\mu < \omega < \lambda - 4$) such that $P_\omega$ contains the point of $V(3-6/10)$ that projects horizontally to $s_{\tau+1}$.

**Property 7.** For each $n (1 \leq n \leq \sigma)$, the set $Q_{i,1} \cup Q_{i,\lambda}$ does not intersect $\cup\{Q_{i,j} : 1 \leq i \leq \sigma$ and $\pi \leq j \leq \omega\}$.

To see this assume there exist integers $i, j, n$ such that $\pi \leq j \leq \omega$ and a point $p$ belongs to $Q_{i,j} \cap (Q_{n,1} \cup Q_{n,\lambda})$. According to Property 6, $\{p\} \cup Q_{i,1} \cup Q_{i,\lambda}$ projects horizontally into $ClK$. By Property 3, there exists a positive number $\chi$ less than $\epsilon$ such that $\{v \in M : \rho(v, p) < \chi\}$ projects horizontally into $ClK$.

Let $P$ be the $p$-component of $Y_i$. Let $Y$ be an arc in $P$ that goes from a point $q$ of $Q_{i,1}$ to $p$. Since $g_i$ and $g_n$ are $\epsilon/50$-homeomorphisms and $\pi \leq j \leq \omega$, the set $Q_{i,1} \cup Q_{i,\lambda}$ and the $p$-component of $P \cap D_1$ are disjoint. Hence $Y$ crosses $D_1 \cap V(1)$ exactly $\iota$ times where $\iota$ is a positive integer less than $\gamma$.

By Lemma 3, there exists a $\chi$-homeomorphism $k$ of $M$ onto $M$ such that $k(q)$ belongs to $Q_{i,1}$ and projects horizontally into $K$. The arc $k[Y]$ crosses $D_1 \cap V(1)$ exactly $\iota$ times. Since $k[Y]$ is in $S$ and $\rho(p, k(p)) < \chi$, the point $k(p)$ projects horizontally into $K$. It follows from the definition of $K$ that $\iota$ is a multiple of $\gamma$, and this is a contradiction. Hence Property 7 holds.

For each $i (1 \leq i \leq \sigma)$ and $j (1 \leq j \leq \lambda)$, let $P_{i,j} = Q_{i,j} - Cl \cup \{Y_n : 1 \leq n < i\}$. By Property 7, for each $i$, the subchain of $P_{i,1}, P_{i,2}, \ldots, P_{i,\lambda}$ that has $P_{i,\pi}$ and $P_{i,\omega}$ as end links is free in $M$.

For each $j (1 \leq j \leq \lambda)$, let $U_j = \cup\{P_{i,j} : 1 \leq i \leq \sigma\}$. The subchain $\mathcal{C}$ of $U_1, U_2, \ldots, U_\lambda$ that has $U_{\pi}$ and $U_{\omega}$ as end links is a free $\epsilon/16$-chain in $M$.

Let $D$ be the union of all components of $C_2 \cap \{(x, y) : 3 - \epsilon/5 < x < 3 + \epsilon/5\}$ that meet $ClK$. According to Property 3, $D$ is open in $M$. The diameter of $D$ is less than $\epsilon/2$. Each point of $U_{\pi} \cup U_{\omega}$ is within $\epsilon/5$ of $V(3)$. By Property 6, $U_{\pi} \cup U_{\omega}$ projects horizontally into $ClK$. Hence $U_{\pi} \cup U_{\omega}$ is in $D$.

Let $\tau$ be the largest integer less than $\mu$ such that $U_{\tau}$ intersects
Let $\psi$ be the smallest integer greater than $\mu$ such that $U_\phi$ intersects $D$. For each $j$ ($1 \leq j < \psi - \tau$), let $Z_j = U_{\tau + j}$. Note that $Z_1, Z_2, \ldots, Z_{\psi - \tau - 1}$ is a free $\epsilon$-chain in $M$.

Define $Z_{\psi - \tau}$ to be the union of $D$ and all elements of $D = \{U_j : \pi \leq j \leq \tau \text{ or } \psi \leq j \leq \omega\}$. Since $\text{Cl} K$ projects horizontally into $U_\mu$ and $\text{C}$ is a free chain in $M$, each element of $D$ intersects $D$. Thus $Z_{\psi - \tau}$ is an open set in $M$ of diameter less than $\epsilon$. Note that $Z_{\psi - \tau}$ meets both $Z_1$ and $Z_{\psi - \tau - 1}$.

Since $\text{C}$ is free and $U_\pi \cup U_\omega$ is in $D$, the boundary of $\cup \{Z_j : 1 \leq j < \psi - \tau\}$ is in $Z_{\psi - \tau}$. Since $\text{Cl} K$ projects horizontally into $U_\mu$, the set $Z_1$ contains every boundary point of $Z_{\psi - \tau}$ that is to the right of $V(3)$ in $R^2$.

Furthermore, each point of $\text{Bd} Z_{\psi - \tau}$ that is to the left of $V(3)$ is in $Z_{\psi - \tau - 1}$. To see this let $s$ be such a point. Let $X$ be the arc in $M$ that intersects $V(1)$ and is irreducible between $s$ and $\text{Cl} U_\mu$ (Lemma 1). By Property 1, $X$ does not meet $U_\pi \cup U_\omega$. Since $U_\mu$ is an interior link in the free chain $\text{C}$, the arc $X$ is covered by $\text{C}$ and $s$ belongs to $Z_{\psi - \tau - 1}$.

It follows that $\text{Bd} Z_{\psi - \tau}$ is in $Z_1 \cup Z_{\psi - \tau - 1}$. Therefore $Z_1, Z_2, \ldots, Z_{\psi - \tau}$ is an $\epsilon$-circular chain that covers $M$. Hence $M$ is circle-like.

Since every homogeneous circle-like continuum that contains an arc is a solenoid [4, Theorem 9, p. 228], Theorem 1 implies the following:

**Theorem 2.** A continuum $M$ is a solenoid if and only if $M$ is homogeneous and every proper subcontinuum of $M$ is an arc.

**References**


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