

# Pacific Journal of Mathematics

**A COMMUTATIVITY THEOREM FOR NON-ASSOCIATIVE  
ALGEBRAS OVER A PRINCIPAL IDEAL DOMAIN**

JIANG LUH AND MOHAN S. PUTCHA

## A COMMUTATIVITY THEOREM FOR NON-ASSOCIATIVE ALGEBRAS OVER A PRINCIPAL IDEAL DOMAIN

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**Let  $A$  be an algebra (not necessarily associative) over a principal ideal domain  $R$  such that for all  $a, b \in A$ , there exist  $\alpha, \beta \in R$  such that  $(\alpha, \beta) = 1$  and  $\alpha ab = \beta ba$ . It is shown that  $A$  is commutative.**

Throughout this paper  $N$  will denote the set of natural numbers and  $Z^+$  the set of positive integers.  $A$  will denote an algebra with identity 1 over a Principal Ideal Domain  $R$ . If  $a, b \in A$  then  $[a, b] = ab - ba$ . If  $\alpha, \beta \in R$ , then  $(\alpha, \beta)$  denotes the greatest common divisor of  $\alpha$  and  $\beta$ . If  $a \in A$ , then the *order* of  $a$ ,  $o(a)$  is the generator of the ideal  $I = \{\alpha \mid a \in R, \alpha a = 0\}$  of  $R$ .  $o(a)$  is unique up to associates. As a generalization of concepts in [1], [2], [3], [4], [5] we consider the following:

(\*) For all  $a, b \in A$ , there exist  $\alpha, \beta \in R$  such that  $(\alpha, \beta) = 1$  and  $\alpha ab = \beta ba$ .

We will show that if  $A$  satisfies (\*), then  $A$  is commutative. This generalizes [3; Theorem 3.5].

LEMMA 1. *Let  $p$  be a prime in  $R$ ,  $m \in Z^+$  such that  $p^m A = (0)$ . If  $A$  satisfies (\*), then  $A$  is commutative.*

*Proof.* Let  $C$  denote the center of  $A$ . Let  $x \in A$ ,  $o(x) = p^k$ ,  $k \in N$ . We prove by induction on  $k$  that  $x \in C$ . If  $k = 0$ , then  $x = 0$ . So let  $k > 0$ . Let  $y \in A$ . First we show

$$(1) \quad [x, y] \neq 0 \text{ implies } [yx, y] = 0.$$

If  $yx = 0$ , this is trivial. So let  $yx \neq 0$ . Now for some  $\alpha_1, \alpha_2 \in R$ ,

$$(2) \quad \begin{aligned} \alpha_1 xy &= \alpha_2 yx, (\alpha_1, \alpha_2) = 1 \\ \beta_1(x+1)y &= \beta_2 y(x+1), (\beta_1, \beta_2) = 1. \end{aligned}$$

So  $\alpha_1 \beta_1(x+1)y = \alpha_1 \beta_2 y(x+1)$ . Thus substituting the above, we get

$$(3) \quad (\alpha_2 \beta_1 - \alpha_1 \beta_2)yx = (\alpha_1 \beta_2 - \alpha_1 \beta_1)y.$$

We claim that  $(\alpha_2\beta_1 - \alpha_1\beta_2)yx \neq 0$ . For otherwise  $(\alpha_1\beta_2 - \alpha_1\beta_1)y = 0$ . Since  $y \neq 0$ , we get  $p \mid \alpha_1\beta_2 - \alpha_1\beta_1$ .

Also  $(\alpha_1\beta_2 - \alpha_1\beta_1)yx = 0$ . Since  $(\alpha_2\beta_1 - \alpha_1\beta_2)yx = 0$ , we get  $(\alpha_2 - \alpha_1)\beta_1yx = 0$ . Since  $yx \neq 0$ ,  $p \mid \beta_1(\alpha_2 - \alpha_1)$ . So

$$p \mid \alpha_1(\beta_2 - \beta_1), p \mid \beta_1(\alpha_2 - \alpha_1).$$

*Case 1.*  $p \nmid \alpha_1$ . Then since  $\alpha_1(\beta_2 - \beta_1)y = 0$ , we get  $(\beta_2 - \beta_1)y = 0$ . So by (2),  $\beta_1[x, y] = 0 = \beta_2[x, y]$ . Since  $[x, y] \neq 0$ , we get  $p \mid \beta_1$ ,  $p \mid \beta_2$ , contradicting (2).

*Case 2.*  $p \mid \alpha_1$ . Then  $p \nmid \alpha_2$  and so  $p \nmid \alpha_2 - \alpha_1$ . Thus  $p \mid \beta_1$ . So  $p \nmid \beta_2$ ,  $p \nmid \beta_2 - \beta_1$ . Since  $\alpha_1(\beta_2 - \beta_1)y = 0$  we get  $\alpha_1y = 0$ . So  $\alpha_1xy = 0$ . By (2),  $\alpha_2yx = 0$ . Since  $yx \neq 0$ , we get  $p \mid \alpha_2$ , a contradiction.

Hence by (3)

$$(\alpha_2\beta_1 - \alpha_1\beta_2)yx \neq 0.$$

In particular

$$\alpha_2\beta_1 - \alpha_1\beta_2 \neq 0.$$

So

$$\alpha_2\beta_1 - \alpha_1\beta_2 = p^t\delta, \quad t \in \mathbb{N}, \quad \delta \in R, \quad (\delta, p) = 1.$$

If  $t \geq k$ , then  $(\alpha_2\beta_1 - \alpha_1\beta_2)yx = 0$ , a contradiction. So  $t < k$ . Hence

$$p^{k-t}(\alpha_1\beta_2 - \alpha_1\beta_1)y = p^{k-t}p^t\delta yx = 0.$$

Let  $o(y) = p^i$ ,  $i \in \mathbb{N}$ . If  $i < k$ , then  $y \in C$ , a contradiction. So  $i \geq k$ . Hence

$$p^k \mid p^t \mid p^{k-t}(\alpha_1\beta_2 - \alpha_1\beta_1).$$

So  $p^t \mid \alpha_2\beta_2 - \alpha_1\beta_1$  and  $\alpha_1\beta_2 - \alpha_1\beta_1 = p^t\gamma$ ,  $\gamma \in R$ . Then  $p^t\delta yx = p^t\gamma y$ . Hence  $p^t(\delta yx - \gamma y) = 0$ . By induction hypothesis,  $\delta yx - \gamma y \in C$ . So  $[\delta yx - \gamma y, y] = 0$ . Thus  $\delta[yx, y] = 0$ . Since  $(\delta, p) = 1$ ,  $[yx, y] = 0$ . This establishes (1).

Now let  $u \in A$  and suppose  $[x, u] \neq 0$ . Then also  $[x, u+1] \neq 0$ . By (1),  $[ux, u] = 0 = [(u+1)x, u]$ . So  $[x, u] = 0$ , a contradiction. So  $x \in C$  and the lemma is proved.

**LEMMA 2.** *Suppose  $A$  satisfies (\*). Let  $a, b \in A$ ,  $o(b) = 0$ . If  $ba = 0$ , then  $ab = 0$ .*

*Proof.* Suppose  $ab \neq 0$ . Then there exist  $\beta_1, \beta_2, \gamma_1, \gamma_2 \in R$  such that

$$(4) \quad \begin{aligned} \beta_1(a+1)b &= \beta_2b(a+1), (\beta_1, \beta_2) = 1, \\ \gamma_1a(b+1) &= \gamma_2b(a+1), (\gamma_1, \gamma_2) = 1. \end{aligned}$$

So

$$(5) \quad \beta_1ab = (\beta_2 - \beta_1)b \quad \text{and} \quad (\gamma_2 - \gamma_1)a = \gamma_1ab.$$

If  $\beta_2 = \beta_1$ , then  $\beta_1, \beta_2$  are units and by (5)  $ab = ba = 0$ , a contradiction. So  $\beta_2 - \beta_1 \neq 0$ . Similarly  $\gamma_2 - \gamma_1 \neq 0$ . Since  $o(b) = 0$ , we get by (5) that  $o(ab) = 0$ . So  $o(a) = 0$ . Hence by (5),  $\beta_1 \neq 0$ ,  $\gamma_1 \neq 0$ . Also by (5)  $[\beta_1ab, b] = 0$ .

So

$$\begin{aligned} (\gamma_2 - \gamma_1)\beta_1ab &= \gamma_1\beta_1(ab)b \\ &= \gamma_1\beta_1b(ab) \\ &= \beta_1(\gamma_2 - \gamma_1)ba \\ &= 0. \end{aligned}$$

So  $o(ab) \neq 0$ , a contradiction. This proves the lemma.

LEMMA 3. *Suppose  $A$  satisfies (\*). Let  $b \in A$ ,  $o(b) = 0$ . Then  $b \in C$ , the center of  $A$ .*

*Proof.* Let  $a \in A$ . There exist  $\alpha_1, \alpha_2, \beta_1, \beta_2 \in R$  such that

$$(6) \quad \begin{aligned} \alpha_1ab &= \alpha_2ba, (\alpha_1, \alpha_2) = 1, \\ \beta_1(a+1)b &= \beta_2b(a+1), (\beta_1, \beta_2) = 1. \end{aligned}$$

Multiplying the second equation by  $\alpha_1$  and substituting the first we obtain

$$b[(\alpha_2\beta_1 - \alpha_1\beta_2)a - (\alpha_1\beta_2 - \alpha_1\beta_1) \cdot 1] = 0.$$

By Lemma 2,

$$[(\alpha_2\beta_1 - \alpha_1\beta_2)a - (\alpha_1\beta_2 - \alpha_1\beta_1) \cdot 1]b = 0.$$

Let  $\mu = \alpha_2\beta_1 - \alpha_1\beta_2$ . Then  $\alpha_1(\beta_2 - \beta_1)b = \mu ab = \mu ba$ . By (6)  $\alpha_1\mu ab = \alpha_2\mu ba = \alpha_2\mu ab$ . So

$$(\alpha_2 - \alpha_1)\alpha_1(\beta_2 - \beta_1)b = 0.$$

Since  $o(b) = 0$ , we obtain by (6) that either  $\alpha_1 = \alpha_2$  is a unit,  $\beta_1 = \beta_2$  is a unit or else  $\alpha_1 = 0$ . The first two cases imply by (6) that  $ab = ba$ . So let  $\alpha_1 = 0$ . Then  $\alpha_2 ba = 0$  and  $\alpha_2$  is a unit by (6). So  $ba = 0$ . By Lemma 2,  $ab = 0$ . Thus in any case  $ab = ba$  and we are done.

**THEOREM 4.** *Suppose  $A$  satisfies (\*). Then  $A$  is commutative.*

*Proof.* Suppose  $A$  is not commutative. We will obtain a contradiction. There exists  $x \in A$  such that  $x \notin C$ , the center of  $A$ . So  $x + 1 \notin C$ . By Lemma 3  $o(x) \neq 0$  and  $o(x + 1) \neq 0$ . Hence  $o(1) \neq 0$ . Let  $o(1) = d \neq 0$ . Then  $d$  is not a unit and hence  $d = p_1^{\alpha_1} \cdots p_r^{\alpha_r}$  for some primes  $p_1, \cdots, p_r \in A$  and some positive integers  $\alpha_1, \cdots, \alpha_r$ . Let  $A_i = \{a \mid a \in A, p_i^{\alpha_i} a = 0\}$ . Then each  $A_i$  is a nonzero subalgebra of  $A$  and  $A = A_1 \oplus \cdots \oplus A_r$ . Being subalgebras of  $A$ , the  $A_i$ 's also satisfy (\*). Being homomorphic images of  $A$ , all the  $A_i$ 's have identity elements. By Lemma 1 each  $A_i$  and hence  $A$  is commutative, a contradiction. This proves the theorem.

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