A COMMUTATIVITY THEOREM FOR NON-ASSOCIATIVE ALGEBRAS OVER A PRINCIPAL IDEAL DOMAIN

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Let $A$ be an algebra (not necessarily associative) over a principal ideal domain $R$ such that for all $a, b \in A$, there exist $\alpha, \beta \in R$ such that $(\alpha, \beta) = 1$ and $\alpha ab = \beta ba$. It is shown that $A$ is commutative.

Throughout this paper $N$ will denote the set of natural numbers and $Z^+$ the set of positive integers. $A$ will denote an algebra with identity 1 over a Principal Ideal Domain $R$. If $a, b \in A$ then $[a, b] = ab - ba$. If $\alpha, \beta \in R$, then $(\alpha, \beta)$ denotes the greatest common divisor of $\alpha$ and $\beta$. If $a \in A$, then the order of $a$, $o(a)$ is the generator of the ideal $I = \{a \mid a \in R, \alpha a = 0\}$ of $R$.$ o(a)$ is unique up to associates. As a generalization of concepts in [1], [2], [3], [4], [5] we consider the following:

(*) For all $\alpha, b \in A$, there exist $\alpha, \beta \in R$ such that $(\alpha, \beta) = 1$ and $\alpha ab = \beta ba$.

We will show that if $A$ satisfies (*), then $A$ is commutative. This generalizes [3; Theorem 3.5].

Lemma 1. Let $p$ be a prime in $R$, $m \in Z^+$ such that $p^m A = (0)$. If $A$ satisfies (*), then $A$ is commutative.

Proof. Let $C$ denote the center of $A$. Let $x \in A$, $o(x) = p^k$, $k \in N$. We prove by induction on $k$ that $x \in C$. If $k = 0$, then $x = 0$. So let $k > 0$. Let $y \in A$. First we show

(1) $[x, y] \neq 0$ implies $[yx, y] = 0$.

If $yx = 0$, this is trivial. So let $yx \neq 0$. Now for some $\alpha_1, \alpha_2 \in R$,

$\alpha_1 xy = \alpha_2 yx, (\alpha_1, \alpha_2) = 1$

(2) $\beta_1(x + 1)y = \beta_2 y(x + 1), (\beta_1, \beta_2) = 1$.

So $\alpha_1 \beta_1(x + 1)y = \alpha_1 \beta_2 y(x + 1)$. Thus substituting the above, we get

(3) $(\alpha_2 \beta_1 - \alpha_1 \beta_2)y x = (\alpha_1 \beta_2 - \alpha_1 \beta_1)y$. 

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We claim that \((\alpha_2\beta_1 - \alpha_1\beta_2)yx \neq 0\). For otherwise \((\alpha_1\beta_2 - \alpha_1\beta_1)y = 0\). Since \(y \neq 0\), we get \(p \mid \alpha_1\beta_2 - \alpha_1\beta_1\).

Also \((\alpha_1\beta_2 - \alpha_1\beta_1)yx = 0\). Since \((\alpha_2\beta_1 - \alpha_1\beta_2)yx = 0\), we get \((\alpha_2 - \alpha_1)\beta_1yx = 0\). Since \(yx \neq 0\), \(p \mid \beta_1(\alpha_2 - \alpha_1)\). So

\[
p \mid \alpha_1(\beta_2 - \beta_1), p \mid \beta_1(\alpha_2 - \alpha_1).
\]

**Case 1.** \(p \nmid \alpha_1\). Then since \(\alpha_1(\beta_2 - \beta_1)y = 0\), we get \((\beta_2 - \beta_1)y = 0\). So by (2), \(\beta_1[x, y] = 0 = \beta_2[x, y]\). Since \([x, y] \neq 0\), we get \(p \mid \beta_1, p \mid \beta_2\), contradicting (2).

**Case 2.** \(p \mid \alpha_1\). Then \(p \nmid \alpha_2\) and so \(p \nmid \alpha_2 - \alpha_1\). Thus \(p \mid \beta_1\). So \(p \nmid \beta_2, p \nmid \beta_2 - \beta_1\). Since \(\alpha_1(\beta_2 - \beta_1)y = 0\) we get \(\alpha_1y = 0\). So \(\alpha_1xy = 0\). By (2), \(\alpha_2yx = 0\). Since \(yx \neq 0\), we get \(p \mid \alpha_2\), a contradiction.

Hence by (3)

\[
(\alpha_2\beta_1 - \alpha_1\beta_2)yx \neq 0.
\]

In particular

\[
\alpha_2\beta_1 - \alpha_1\beta_2 \neq 0.
\]

So

\[
\alpha_2\beta_1 - \alpha_1\beta_2 = p\delta, \ t \in N, \delta \in R, (\delta, p) = 1.
\]

If \(t \geq k\), then \((\alpha_2\beta_1 - \alpha_1\beta_2)yx = 0\), a contradiction. So \(t < k\). Hence

\[
p^{k-t}(\alpha_2\beta_1 - \alpha_1\beta_2)y = p^{k-t}p\deltayx = 0.
\]

Let \(o(y) = p^i, i \in N\). If \(i < k\), then \(y \in C\), a contradiction. So \(i \geq k\). Hence

\[
p^k \mid p^i \mid p^{k-t}(\alpha_2\beta_1 - \alpha_1\beta_2).
\]

So \(p^i \mid \alpha_2\beta_2 - \alpha_1\beta_1\) and \(\alpha_2\beta_2 - \alpha_1\beta_1 = p^i\gamma, \gamma \in R\). Then \(p^i\gammayx = p^i\gammay\). Hence \(p^i(\deltayx - \gammayx) = 0\). By induction hypothesis, \(\deltayx - \gammayx \in C\). So \([\deltayx - \gammayx, y] = 0\). Thus \(\delta[yx, y] = 0\). Since \((\delta, p) = 1, [yx, y] = 0\). This establishes (1).

Now let \(u \in A\) and suppose \([x, u] \neq 0\). Then also \([x, u + 1] \neq 0\). By (1), \([ux, u] = 0 = [(u + 1)x, u]\). So \([x, u] = 0\), a contradiction. So \(x \in C\) and the lemma is proved.

**Lemma 2.** Suppose \(A\) satisfies (*). Let \(a, b \in A, o(b) = 0\). If \(ba = 0\), then \(ab = 0\).
Proof. Suppose \( ab \neq 0 \). Then there exist \( \beta_1, \beta_2, \gamma_1, \gamma_2 \in R \) such that

\[
\begin{align*}
\beta_1(a + 1)b &= \beta_2b(a + 1), \quad (\beta_1, \beta_2) = 1, \\
\gamma_1a(b + 1) &= \gamma_2b(a + 1), \quad (\gamma_1, \gamma_2) = 1.
\end{align*}
\]

So

\[
\begin{align*}
(\gamma_2 - \gamma_1)\beta_1ab &= (\gamma_1\beta_1(ab)b \\
&= \gamma_1\beta_1(b(ab) \\
&= \beta_1(\gamma_2 - \gamma_1)ba \\
&= 0.
\end{align*}
\]

So \( o(ab) \neq 0 \), a contradiction. This proves the lemma.

Lemma 3. Suppose \( A \) satisfies (*)\#. Let \( b \in A, o(b) = 0. \) Then \( b \in C, \) the center of \( A. \)

Proof. Let \( a \in A. \) There exist \( \alpha_1, \alpha_2, \beta_1, \beta_2 \in R \) such that

\[
\begin{align*}
\alpha_1ab &= \alpha_2ba, \quad (\alpha_1, \alpha_2) = 1, \\
\beta_1(a + 1)b &= \beta_2b(a + 1), \quad (\beta_1, \beta_2) = 1.
\end{align*}
\]

Multiplying the second equation by \( \alpha_1 \) and substituting the first we obtain

\[
b[(\alpha_2\beta_1 - \alpha_1\beta_2)a - (\alpha_1\beta_2 - \alpha_1\beta_1) \cdot 1] = 0.
\]

By Lemma 2,

\[
[(\alpha_2\beta_1 - \alpha_1\beta_2)a - (\alpha_1\beta_2 - \alpha_1\beta_1) \cdot 1]b = 0.
\]

Let \( \mu = \alpha_2\beta_1 - \alpha_1\beta_2. \) Then \( \alpha_1(\beta_2 - \beta_1)b = \mu ab = \mu ba. \) By (6) \( \alpha_1\mu ab = \alpha_2\mu ba = \alpha_2\mu ab. \) So
\[(\alpha_2 - \alpha_1)\alpha_1(\beta_2 - \beta_1)b = 0.\]

Since \(o(b) = 0\), we obtain by (6) that either \(\alpha_1 = \alpha_2\) is a unit, \(\beta_1 = \beta_2\) is a unit or else \(\alpha_1 = 0\). The first two cases imply by (6) that \(ab = ba\). So let \(\alpha_1 = 0\). Then \(\alpha_2ba = 0\) and \(\alpha_2\) is a unit by (6). So \(ba = 0\). By Lemma 2, \(ab = 0\). Thus in any case \(ab = ba\) and we are done.

**Theorem 4.** Suppose \(A\) satisfies (*) Then \(A\) is commutative.

*Proof.* Suppose \(A\) is not commutative. We will obtain a contradiction. There exists \(x \in A\) such that \(x \notin C\), the center of \(A\). So \(x + 1 \notin C\). By Lemma 3 \(o(x) \neq 0\) and \(o(x + 1) \neq 0\). Hence \(o(1) \neq 0\). Let \(o(1) = d \neq 0\). Then \(d\) is not a unit and hence \(d = p_1^{\alpha_1} \cdots p_l^{\alpha_l}\) for some primes \(p_1, \ldots, p_l \in A\) and some positive integers \(\alpha_1, \ldots, \alpha_l\). Let \(A_i = \{a \mid a \in A, p_i^{\alpha_i}a = 0\}\). Then each \(A_i\) is a nonzero subalgebra of \(A\) and \(A = A_1 \oplus \cdots \oplus A_r\). Being subalgebras of \(A\), the \(A_i\)'s also satisfy (*). Being homomorphic images of \(A\), all the \(A_i\)'s have identity elements. By Lemma 1 each \(A_i\) and hence \(A\) is commutative, a contradiction. This proves the theorem.

**References**


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