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# ON THE DISTRIBUTION OF $a$-POINTS OF A STRONGLY ANNULAR FUNCTION

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**This paper gives an example of a strongly annular function which omits 0 near an arc $I$ on the unit circle $C$ and which omits 1 near the complementary arc $C-I$. This example affirmatively answers the following question of Bonar: Does there exist any annular function for which we can find two or more complex numbers $w$ such that the limiting set of its $w$-points does not cover $C$?**

**1. Introduction.** The purpose of this paper is to study the distribution of $a$-points of annular functions. We recall that a holomorphic function in the open unit disk $D : |z| < 1$ is said to be annular [**1**] if there is a sequence $\{J_n\}$ of closed Jordan curves about the origin in $D$, converging out to the unit circle $C : |z| = 1$, such that the minimum modulus of $f(z)$ on $J_n$ increases to infinity as $n$ increases. When the $J_n$ can be taken as circles concentric with $C$, $f(z)$ will be called strongly annular. Given a finite complex number $a$, the minimum modulus principle guarantees that every annular function $f$ has infinitely many $a$-points in $D$ and hence their limit points form a nonempty closed subset, say $Z'(f, a)$, of $C$. On the other hand, by virtue of the Koebe–Gross theorem concerning meromorphic functions omitting three points, it follows from the annularity of $f$ that open sets $C - Z'(f, a)$ and $C - Z'(f, b)$ on the circle can not overlap if $a \neq b$ and consequently that the set of all values $a$ for which $Z'(f, a) \neq C$ must be at most countable. Therefore we may well say such $a$ to be singular for $f$.

For this reason we will be concerned with the set $S(f) = \{a : Z'(f, a) \neq C\}$ in this paper. We denote by $|S(f)|$ the cardinality of $S(f)$ and then, from the simple fact observed above, we have that $0 \leqq |S(f)| \leqq \aleph_0$, which in turn conversely tempt us to raise the following question: Given a cardinality $N (0 \leqq N \leqq \aleph_0)$, can we find any annular function $f$ for which $|S(f)| = N$? ([**1**], [**2**]).

We know many examples of strongly annular functions such that $|S(f)| = 0$ [**4**]. In particular if an annular function $f$ belongs to the MacLane class, i.e., the family of all nonconstant holomorphic functions in $D$ which have asymptotic values at each point of everywhere dense subsets of $C$, the set $S(f)$ becomes necessarily empty. As for $N = 1$, Barth and Schneider [**3**] constructed an example of an annular function $f$ for which $|S(f)| = 1$. The example involved in their construction, however, did not appear to be strongly annular. An example of a strongly annular $f$ with $|S(f)| = 1$ was constructed independently by Barth, Bonar and Carroll **[2]** and the author **[5]**. The aim of this paper is to give an example of a strongly annular function $f$ for which $|S(f)| = 2$.

**2.** For this purpose we consider a class of functions holomorphic in $D$. Let $I_0$ and $I_1$ be a pair of complementary open arcs on the unit circle $C$ and choose a Jordan arc $J_j$ connecting the end points of $I_j$, which is contained, except for its end points, in the open sector

$$\{z : 0 < |z| < 1,\ z/|z| \in I_j\} \qquad (j = 0, 1).$$

Further denote by $G_j$ the Jordan domain surrounded by $I_j$ and $J_j$ and consider

$$S(G_0, G_1) = \{g \in H(D) : g \text{ is bounded away from 0 (or 1) in } G_0 \text{ (or } G_1)\}$$

where $H(D)$ denotes the set of all functions holomorphic in $D$. In terms of this notation our purpose is in amount to find a strongly annular function which is locally a uniform limit of a sequence in $S(G_0, G_1)$. To construct such a function, we make essential use of the approximation theorem of Runge, which asserts that if $K$ is a compact set with connected complement relative to the plane and a function $g$ is holomorphic in an open set containing $K$, for any $\rho > 0$, there is a polynomial $P$ such that

$$|P(z) - g(z)| < \rho \qquad (z \in K).$$

We call such $P$ an approximating polynomial with respect to the triple $(K, g, \rho)$. In our arguments to follow we may restrict ourselves to the special pair of $G_0$ and $G_1$ such that

$$G_0 = \{z = x + iy : |z| < 1,\ 2x + |y| > 1\} \quad \text{and} \quad G_1 = \{z : -z \in G_0\}$$

with no loss of generality, which serves to simplify the geometric formulation. Then the Runge theorem, in cooperation with our previous lemma, yields the following:

LEMMA. *Let there be given positive numbers $\epsilon$ and $k$, numbers $a$ and $b$ with $0 < a < b < 1$, and a function $f$ in $S(G_0, G_1)$ (simply $S$), which is bounded in $G_1$. Then there exists a function $g$ in $S$, which is also bounded in $G_1$, such that*

$$|g(z)| > k \qquad (|z| = b) \tag{1}$$

*and*

$$(2) \qquad |g(z)-f(z)|<\epsilon \qquad (|z|\leqq a).$$

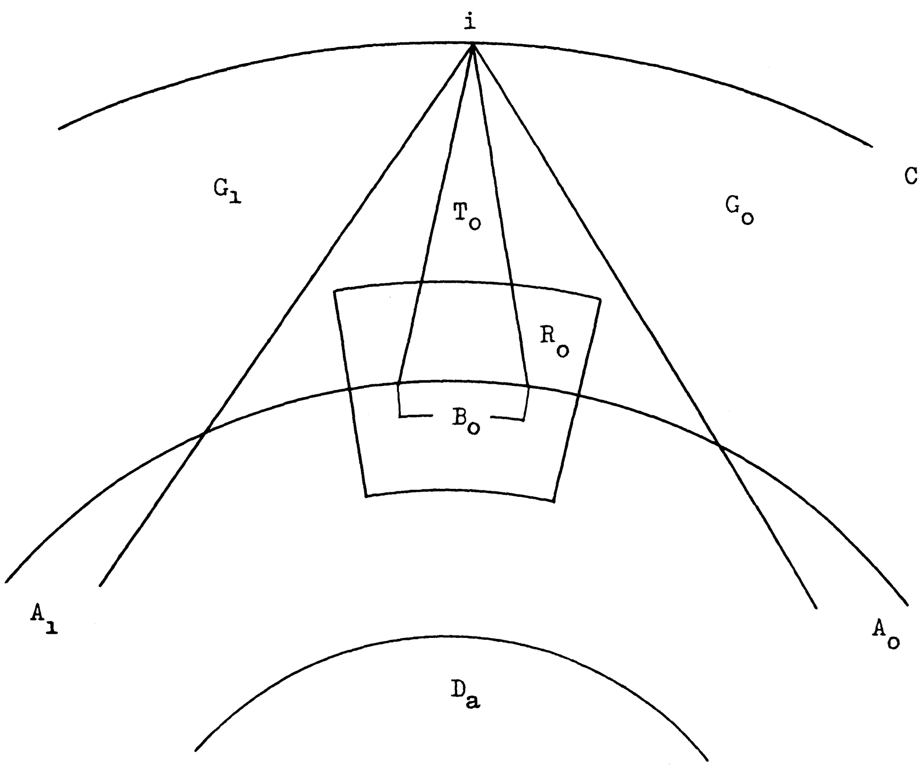
*Proof.* We first divide the circle $|z|=b$ into 4 closed arcs as follows:

$$A_0=[-bie^{it}, bie^{-it}], \qquad A_1=\{z:-z\in A_0\}$$
$$B_0=[bie^{-it}, bie^{it}], \qquad B_1=\{z:\bar{z}\in B_0\}.$$

Here $t(>0)$ should be chosen so small that we may apply our lemma **[5]** to an appropriately small open annular sector $R_0$, which is contained in

$$\{z=x+iy:y>0, |z|>a, 2|x|+|y|<1\}$$

and contains the arc $B_0$. Set $R_1=\{z:\bar{z}\in R_0\}$.




Next, to make use of the Runge theorem, we prepare two triples, which are defined, except for $c_j$ and $\rho_j$, by the following:

$$(3)\quad \begin{cases} K_j = \bar{G}_j \cup A_j \cup A_{1-j} \cup \bar{D}_a, \ \bar{D}_a = \{z : |z| \leqq a\} \\ g_j(z) = 0 \qquad (z \in \bar{G}_j \cup A_j \cup \bar{D}_a) \\ g_j(z) = c_j (>0) \qquad (z \in A_{1-j}) \end{cases} \qquad (j = 0, 1).$$

As for $c_j$ (or $\rho_j$) we shall later choose positive numbers large (or small) enough to satisfy our requirements. Obviously these definitions allow us to apply the Runge theorem to $(K_j, g_j, \rho_j)$ $(j = 0, 1)$ and hence we can find an approximating polynomial $P_j$. On the other hand, if necessary, adding a small vector we may assume that $f(z) \neq 0, 1$ on the circle $|z| = b$. Combining these functions, define a function $F$ holomorphic in $D$ by

$$F(z) = \{(f(z) - 1)\exp(P_0(z)) + 1\}\exp(P_1(z)).$$

Then carefully observing (3) and suitably choosing values of $c_j$ and $\rho_j$, we can conclude that the function $F$ is a member of $S$, bounded in $G_1$ and has the following properties:

$$(4)\qquad |F(z)| > 2k \qquad (z \in \{z : |z| = b\} - B_0 - B_1)$$

$$(5)\qquad |F(z) - f(z)| < \epsilon/2 \qquad (z \in \bar{D}_a).$$

In addition it may be supposed that $F$ does not vanish on $B_0 \cup B_1$.

Thus the last step in our construction of $g$ is to make $|F(z)|$ large on the remaining arcs $B_0$ and $B_1$ without losing the properties described above of $F$. Given $c_2 > 0$ and $\rho_2 > 0$, applying our lemma **[5]** to the annular sectors $R_0$ and $R_1$ previously chosen, and successively using the standard "pole sweeping" method for the resulting rational functions, we can find a holomorphic function $H_j$ in $D$ such that

$$(6)\qquad |H_j(z)| > c_2 \qquad (z \in B_j),$$

$$(7)\qquad \operatorname{Re} H_j(z) > -\rho_2 \qquad (z \in R_j \cap \{z : |z| = b\} - B_j)$$

and

$$(8)\qquad |H_j(z)| < 2\rho_2 \qquad (z \in D - T_j)$$

where $T_0$ (or $T_1$) denotes an appropriate "pole sweeping route" ending at $z = i$ (or $-i$) which is contained in

$$E_0 = \{z = x + iy : y > 0, \ |z| > b, \ 2|x| + |y| < 1\}$$

(or $E_1 = \{z : \bar{z} \in E_0\}$) (see Figure 1). Using these functions and $F$ defined above, set

$$F(z)\{1 + H_0(z)\}\{1 + H_1(z)\} = g(z).$$

Since $F$ does not vanish on $B_0 \cup B_1$, if we appropriately choose a large (or small) positive number as a value of $c_2$ (or $\rho_2$), by virtue of (4) and (5) together with (6), (7) and (8), we can show that the function $g$ belongs to the class $S$, is bounded in $G_1$ and further satisfies (1) and (2). This proves Lemma.

**3.** The following result is immediate from Lemma in 2.

THEOREM. *Let $\{r_n\}$ and $\{k_n\}$ be two sequences of positive numbers with $r_n \uparrow 1$ and $1 < k_n \uparrow +\infty$. Then there exists a function $f$, which is locally a uniform limit of a sequence in $S$ and which furthermore satisfies that $|f(z)| \geqq k_n$ on the circle $|z| = r_n$.*

*Proof.* It is sufficient to construct a sequence $\{f_n(z)\}$ in $S$ such that

(9) $\quad |f_n(z)| > k_j \quad$ if $\quad 1 \leqq j \leqq n \qquad (z \in C_j = \{z : |z| = r_j\})$,

(10) $\quad |f_n(z) - f_{n-1}(z)| < \epsilon_{n-1} \qquad (|z| \leqq r_{n-1},\ n \geqq 2)$

and

(11) $\quad f_n$ is bounded in $G_1$

where $\{\epsilon_n\}$ is a preassigned sequence of positive numbers with $\Sigma\epsilon_n < +\infty$. In order to construct $\{f_n\}$ inductively, let $f_1(z) = 2k_1$ and suppose that $f_1, \cdots, f_{n-1}$ have already been defined. In Lemma in 2, on setting $f = f_{n-1}$, $a = r_{n-1}$, $b = r_n$, $k = k_n$ and $\epsilon = \min\{\epsilon_{n-1}, m_1, \cdots, m_{n-1}\}$ where $m_j = \min\{|f_{n-1}(z)| - k_j : z \in C_j\}$, we can find a function $f_n$ in $S$ satisfying (9), (10) and (11). Thus we obtain a sequence $\{f_n\}$ in $S$, which, by virtue of (10), converges uniformly on any compact subset of $D$. Obviously its limit $f$ is a desired function in Theorem. Hence our proof is complete.

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