

# Pacific Journal of Mathematics

**ON THE DISTRIBUTION OF  $a$ -POINTS OF A STRONGLY  
ANNULAR FUNCTION**

AKIO OSADA

## ON THE DISTRIBUTION OF $a$ -POINTS OF A STRONGLY ANNULAR FUNCTION

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**This paper gives an example of a strongly annular function which omits 0 near an arc  $I$  on the unit circle  $C$  and which omits 1 near the complementary arc  $C-I$ . This example affirmatively answers the following question of Bonar: Does there exist any annular function for which we can find two or more complex numbers  $w$  such that the limiting set of its  $w$ -points does not cover  $C$ ?**

**1. Introduction.** The purpose of this paper is to study the distribution of  $a$ -points of annular functions. We recall that a holomorphic function in the open unit disk  $D : |z| < 1$  is said to be annular [1] if there is a sequence  $\{J_n\}$  of closed Jordan curves about the origin in  $D$ , converging out to the unit circle  $C : |z| = 1$ , such that the minimum modulus of  $f(z)$  on  $J_n$  increases to infinity as  $n$  increases. When the  $J_n$  can be taken as circles concentric with  $C$ ,  $f(z)$  will be called strongly annular. Given a finite complex number  $a$ , the minimum modulus principle guarantees that every annular function  $f$  has infinitely many  $a$ -points in  $D$  and hence their limit points form a nonempty closed subset, say  $Z'(f, a)$ , of  $C$ . On the other hand, by virtue of the Koebe-Gross theorem concerning meromorphic functions omitting three points, it follows from the annularity of  $f$  that open sets  $C - Z'(f, a)$  and  $C - Z'(f, b)$  on the circle can not overlap if  $a \neq b$  and consequently that the set of all values  $a$  for which  $Z'(f, a) \neq C$  must be at most countable. Therefore we may well say such  $a$  to be singular for  $f$ .

For this reason we will be concerned with the set  $S(f) = \{a : Z'(f, a) \neq C\}$  in this paper. We denote by  $|S(f)|$  the cardinality of  $S(f)$  and then, from the simple fact observed above, we have that  $0 \leq |S(f)| \leq \aleph_0$ , which in turn conversely tempt us to raise the following question: Given a cardinality  $N (0 \leq N \leq \aleph_0)$ , can we find any annular function  $f$  for which  $|S(f)| = N$ ? ([1], [2]).

We know many examples of strongly annular functions such that  $|S(f)| = 0$  [4]. In particular if an annular function  $f$  belongs to the MacLane class, i.e., the family of all nonconstant holomorphic functions in  $D$  which have asymptotic values at each point of everywhere dense subsets of  $C$ , the set  $S(f)$  becomes necessarily empty. As for  $N = 1$ , Barth and Schneider [3] constructed an example of an annular function  $f$  for which  $|S(f)| = 1$ . The example involved in their construction,

however, did not appear to be strongly annular. An example of a strongly annular  $f$  with  $|S(f)| = 1$  was constructed independently by Barth, Bonar and Carroll [2] and the author [5]. The aim of this paper is to give an example of a strongly annular function  $f$  for which  $|S(f)| = 2$ .

2. For this purpose we consider a class of functions holomorphic in  $D$ . Let  $I_0$  and  $I_1$  be a pair of complementary open arcs on the unit circle  $C$  and choose a Jordan arc  $J_j$  connecting the end points of  $I_j$ , which is contained, except for its end points, in the open sector

$$\{z : 0 < |z| < 1, z/|z| \in I_j\} \quad (j = 0, 1).$$

Further denote by  $G_j$  the Jordan domain surrounded by  $I_j$  and  $J_j$  and consider

$$S(G_0, G_1) = \{g \in H(D) : g \text{ is bounded away from } 0 \text{ (or } 1) \text{ in } G_0 \text{ (or } G_1)\}$$

where  $H(D)$  denotes the set of all functions holomorphic in  $D$ . In terms of this notation our purpose is in amount to find a strongly annular function which is locally a uniform limit of a sequence in  $S(G_0, G_1)$ . To construct such a function, we make essential use of the approximation theorem of Runge, which asserts that if  $K$  is a compact set with connected complement relative to the plane and a function  $g$  is holomorphic in an open set containing  $K$ , for any  $\rho > 0$ , there is a polynomial  $P$  such that

$$|P(z) - g(z)| < \rho \quad (z \in K).$$

We call such  $P$  an approximating polynomial with respect to the triple  $(K, g, \rho)$ . In our arguments to follow we may restrict ourselves to the special pair of  $G_0$  and  $G_1$  such that

$$G_0 = \{z = x + iy : |z| < 1, 2x + |y| > 1\} \quad \text{and} \quad G_1 = \{z : -z \in G_0\}$$

with no loss of generality, which serves to simplify the geometric formulation. Then the Runge theorem, in cooperation with our previous lemma, yields the following:

LEMMA. *Let there be given positive numbers  $\epsilon$  and  $k$ , numbers  $a$  and  $b$  with  $0 < a < b < 1$ , and a function  $f$  in  $S(G_0, G_1)$  (simply  $S$ ), which is bounded in  $G_1$ . Then there exists a function  $g$  in  $S$ , which is also bounded in  $G_1$ , such that*

$$(1) \quad |g(z)| > k \quad (|z| = b)$$

and

$$(2) \quad |g(z) - f(z)| < \epsilon \quad (|z| \leq a).$$

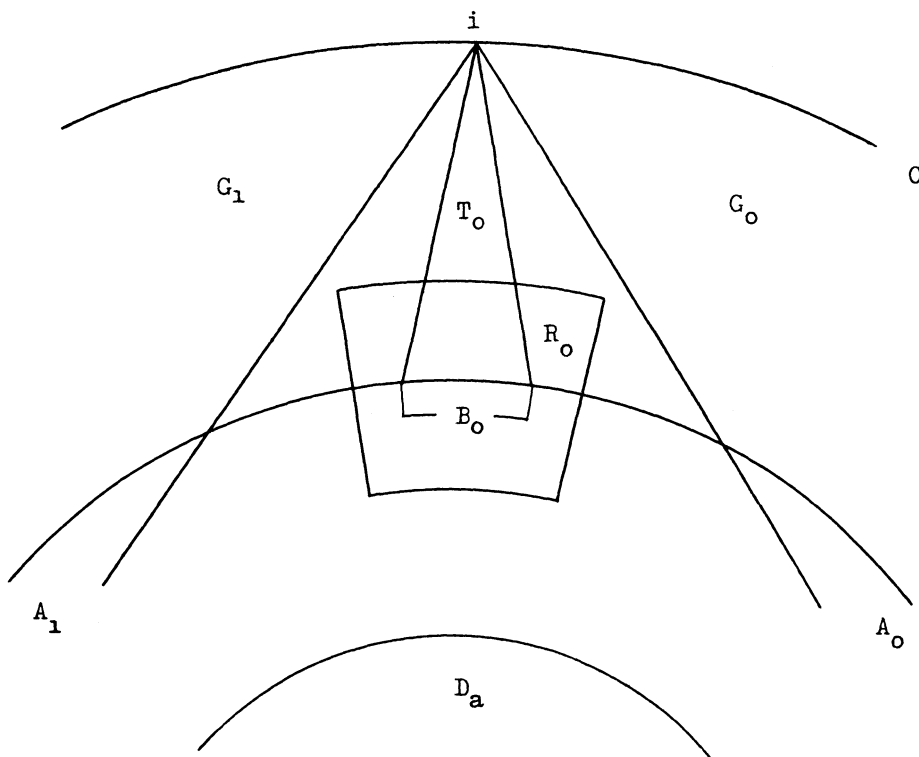
*Proof.* We first divide the circle  $|z| = b$  into 4 closed arcs as follows:

$$\begin{aligned} A_0 &= [-bie^u, bie^{-u}], & A_1 &= \{z : -z \in A_0\} \\ B_0 &= [bie^{-u}, bie^u], & B_1 &= \{z : \bar{z} \in B_0\}. \end{aligned}$$

Here  $t (> 0)$  should be chosen so small that we may apply our lemma [5] to an appropriately small open annular sector  $R_0$ , which is contained in

$$\{z = x + iy : y > 0, |z| > a, 2|x| + |y| < 1\}$$

and contains the arc  $B_0$ . Set  $R_1 = \{z : \bar{z} \in R_0\}$ .



Next, to make use of the Runge theorem, we prepare two triples, which are defined, except for  $c_j$  and  $\rho_j$ , by the following:

$$(3) \quad \begin{cases} K_j = \bar{G}_j \cup A_j \cup A_{1-j} \cup \bar{D}_a, \bar{D}_a = \{z : |z| \leq a\} \\ g_j(z) = 0 & (z \in \bar{G}_j \cup A_j \cup \bar{D}_a) \\ g_j(z) = c_j (> 0) & (z \in A_{1-j}) \end{cases} \quad (j = 0, 1).$$

As for  $c_j$  (or  $\rho_j$ ) we shall later choose positive numbers large (or small) enough to satisfy our requirements. Obviously these definitions allow us to apply the Runge theorem to  $(K_j, g_j, \rho_j)$  ( $j = 0, 1$ ) and hence we can find an approximating polynomial  $P_j$ . On the other hand, if necessary, adding a small vector we may assume that  $f(z) \neq 0, 1$  on the circle  $|z| = b$ . Combining these functions, define a function  $F$  holomorphic in  $D$  by

$$F(z) = \{(f(z) - 1) \exp(P_0(z)) + 1\} \exp(P_1(z)).$$

Then carefully observing (3) and suitably choosing values of  $c_j$  and  $\rho_j$ , we can conclude that the function  $F$  is a member of  $S$ , bounded in  $G_1$  and has the following properties:

$$(4) \quad |F(z)| > 2k \quad (z \in \{z : |z| = b\} - B_0 - B_1)$$

$$(5) \quad |F(z) - f(z)| < \epsilon/2 \quad (z \in \bar{D}_a).$$

In addition it may be supposed that  $F$  does not vanish on  $B_0 \cup B_1$ .

Thus the last step in our construction of  $g$  is to make  $|F(z)|$  large on the remaining arcs  $B_0$  and  $B_1$  without losing the properties described above of  $F$ . Given  $c_2 > 0$  and  $\rho_2 > 0$ , applying our lemma [5] to the annular sectors  $R_0$  and  $R_1$  previously chosen, and successively using the standard "pole sweeping" method for the resulting rational functions, we can find a holomorphic function  $H_j$  in  $D$  such that

$$(6) \quad |H_j(z)| > c_2 \quad (z \in B_j),$$

$$(7) \quad \operatorname{Re} H_j(z) > -\rho_2 \quad (z \in R_j \cap \{z : |z| = b\} - B_j)$$

and

$$(8) \quad |H_j(z)| < 2\rho_2 \quad (z \in D - T_j)$$

where  $T_0$  (or  $T_1$ ) denotes an appropriate "pole sweeping route" ending at  $z = i$  (or  $-i$ ) which is contained in

$$E_0 = \{z = x + iy : y > 0, |z| > b, 2|x| + |y| < 1\}$$

(or  $E_1 = \{z : \bar{z} \in E_0\}$ ) (see Figure 1). Using these functions and  $F$  defined above, set

$$F(z)\{1 + H_0(z)\}\{1 + H_1(z)\} = g(z).$$

Since  $F$  does not vanish on  $B_0 \cup B_1$ , if we appropriately choose a large (or small) positive number as a value of  $c_2$  (or  $\rho_2$ ), by virtue of (4) and (5) together with (6), (7) and (8), we can show that the function  $g$  belongs to the class  $S$ , is bounded in  $G_1$  and further satisfies (1) and (2). This proves Lemma.

3. The following result is immediate from Lemma in 2.

**THEOREM.** *Let  $\{r_n\}$  and  $\{k_n\}$  be two sequences of positive numbers with  $r_n \uparrow 1$  and  $1 < k_n \uparrow +\infty$ . Then there exists a function  $f$ , which is locally a uniform limit of a sequence in  $S$  and which furthermore satisfies that  $|f(z)| \geq k_n$  on the circle  $|z| = r_n$ .*

*Proof.* It is sufficient to construct a sequence  $\{f_n(z)\}$  in  $S$  such that

$$(9) \quad |f_n(z)| > k_j, \quad \text{if } 1 \leq j \leq n \quad (z \in C_j = \{z : |z| = r_j\}),$$

$$(10) \quad |f_n(z) - f_{n-1}(z)| < \epsilon_{n-1} \quad (|z| \leq r_{n-1}, n \geq 2)$$

and

$$(11) \quad f_n \text{ is bounded in } G_1$$

where  $\{\epsilon_n\}$  is a preassigned sequence of positive numbers with  $\sum \epsilon_n < +\infty$ . In order to construct  $\{f_n\}$  inductively, let  $f_1(z) = 2k_1$  and suppose that  $f_1, \dots, f_{n-1}$  have already been defined. In Lemma in 2, on setting  $f = f_{n-1}$ ,  $a = r_{n-1}$ ,  $b = r_n$ ,  $k = k_n$  and  $\epsilon = \min\{\epsilon_{n-1}, m_1, \dots, m_{n-1}\}$  where  $m_j = \min\{|f_{n-1}(z)| - k_j : z \in C_j\}$ , we can find a function  $f_n$  in  $S$  satisfying (9), (10) and (11). Thus we obtain a sequence  $\{f_n\}$  in  $S$ , which, by virtue of (10), converges uniformly on any compact subset of  $D$ . Obviously its limit  $f$  is a desired function in Theorem. Hence our proof is complete.

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