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**ON STARLIKENESS AND CONVEXITY OF CERTAIN
ANALYTIC FUNCTIONS**

V. V. ANH AND P. D. TUAN

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Let N be the class of normalised regular functions

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \quad |z| < 1.$$

For $0 \leq \lambda < 1, \gamma \geq 1$, let $f(z), g(z) \in N$ be such that

$$|f(z)/[\lambda f(z) + (1 - \lambda)g(z)] - \gamma| < \gamma, \quad |z| < 1.$$

We establish the radius of starlikeness of $f(z)$ under the assumption $\operatorname{Re}\{g(z)/z\} > 0$, or $\operatorname{Re}\{g(z)/z\} > 1/2$, or $\operatorname{Re}\{zg'(z)/g(z)\} > \alpha, 0 \leq \alpha < 1$, or $\operatorname{Re}\{1 + zg''(z)/g'(z)\} > 0$ for $|z| < 1$. The analysis may be extended to the problem of finding the radius of convexity for certain subclasses of N .

1. Introduction and notation. Let S, S^*, S^c denote the subclasses of N which are univalent, univalent starlike, univalent convex in $|z| < 1$ respectively.

A necessary and sufficient condition for $f(z) \in N$ to be univalent starlike in $|z| < r$ is

$$\operatorname{Re} \left\{ \frac{zf'(z)}{f(z)} \right\} > 0, \quad |z| < r.$$

A necessary and sufficient condition for $f(z) \in N$ to be univalent convex in $|z| < r$ is

$$\operatorname{Re} \left\{ 1 + \frac{zf''(z)}{f'(z)} \right\} > 0, \quad |z| < r.$$

A function $f(z)$ belongs to $S^*(\beta)$, i.e., is starlike of order β , $0 \leq \beta < 1$, if it satisfies the condition

$$\operatorname{Re} \left\{ \frac{zf'(z)}{f(z)} \right\} > \beta, \quad |z| < 1.$$

A function $f(z)$ belongs to $S^c(\beta)$, i.e., is convex of order β , $0 \leq \beta < 1$, if it satisfies the condition

$$\operatorname{Re} \left\{ 1 + \frac{zf''(z)}{f'(z)} \right\} > \beta, \quad |z| < 1.$$

Let \mathcal{S}_α denote the class of regular functions of the form

$$p(z) = 1 + \sum_{k=1}^{\infty} c_k z^k, \quad |z| < 1,$$

satisfying the inequality $\operatorname{Re}\{p(z)\} > \alpha$ for $|z| < 1$, $0 \leq \alpha < 1$ and \mathcal{Q}_γ the class of functions $q(z)$ with expansion of the above form but satisfying the inequality $|q(z) - \gamma| < \gamma$ for $|z| < 1$, $\gamma \geq 1$. We note that both \mathcal{P}_0 and \mathcal{Q}_∞ reduce to the class \mathcal{P} of functions with positive real part.

Let N_n , $n \geq 1$, denote the subclass of N consisting of functions of the form $f(z) = z + \sum_{k=n+1}^{\infty} a_k z^k$. Then $N_1 = N$.

Shah [8] considered the problem of determining the radius of starlikeness of $f(z) \in N_n$ for the following cases:

(a) $f(z)/[\lambda f(z) + (1 - \lambda)g(z)] \in \mathcal{P}$ with $g(z) \in N_n$ and $g(z)/z \in \mathcal{P}$, or $g(z)/z \in \mathcal{P}_{1/2}$ (with $n = 1$), or $g(z) \in S^*(\alpha)$;

(b) $f(z)/[\lambda f(z) + (1 - \lambda)g(z)] \in \mathcal{Q}_1$ with $g(z) \in N_n$ and $g(z)/z \in \mathcal{P}$, or $g(z) \in S^*(\alpha)$.

The conditions were shown to be sharp only when $\lambda = 0$. In this paper, we solve the problem for the subclasses of N mentioned at the beginning, subject to certain restrictions on the values of λ . Letting $\gamma \rightarrow \infty$ we obtain the radii of starlikeness of $f(z)$ satisfying $f(z)/[\lambda f(z) + (1 - \lambda)g(z)] \in \mathcal{P}$. All the bounds obtained are best possible. Furthermore, the same technique may be used to establish the radius of convexity of $f(z) \in N$ satisfying $f'(z)/[\lambda f'(z) + (1 - \lambda)g'(z)] \in \mathcal{Q}_\gamma$, where $g(z)$ belongs to various subclasses of N . The results proved here generalize those of MacGregor [3, 4, 5] and Ratti [6, 7].

It should be remarked that parallel results for subclasses of N_n , $n > 1$, may be derived in an analogous manner. The manipulations involved are, however, more complicated.

The lemmas required for the proofs of our theorems are given in §2. Section 3 contains theorems giving the conditions for starlikeness. We outline the conditions for convexity in §4.

2. Some lemmas. Let \mathcal{B} denote the class of functions $w(z)$ regular in $|z| < 1$ and satisfying $w(0) = 0$, $|w(z)| < 1$ for $|z| < 1$.

LEMMA 2.1 [9]. *If $w(z) \in \mathcal{B}$, then for $|z| < 1$,*

$$|zw'(z) - w(z)| \leq \frac{|z|^2 - |w(z)|^2}{1 - |z|^2}.$$

Proof. Write $w(z) = z\phi(z)$, where $\phi(z)$ is regular in $|z| < 1$ and $|\phi(z)| \leq 1$. The assertion now follows from the well-known result due to Caratheodory

$$|\phi'(z)| \leq \frac{1 - |\phi(z)|^2}{1 - |z|^2}.$$

LEMMA 2.2. *Let $w_1(z) = [1 - w(z)]/[1 + \beta w(z)]$, where $w(z) \in \mathcal{B}$,*

$\beta \geq 0$. Then, for $|z| = r < \min(1, 1/\beta)$,

$$\begin{aligned} \operatorname{Re} \left\{ -\beta w_1(z) + \frac{1}{w_1(z)} \right\} + \frac{r^2 |1 + \beta w_1(z)|^2 - |1 - w_1(z)|^2}{(1 - r^2) |w_1(z)|} \\ \leq \frac{1 - \beta + (3\beta + 1)r + \beta(\beta + 3)r^2 + \beta(\beta - 1)r^3}{(1 - r^2)(1 + \beta r)}. \end{aligned}$$

Proof. By Schwarz's lemma, $|w(z)| \leq r$ on $|z| = r < 1$. The transformation $w_1(z) = [1 - w(z)]/[1 + \beta w(z)]$ maps the disc $|w(z)| \leq r$, $r < \min(1, 1/\beta)$, onto the disc $|w_1(z) - a| \leq d$, where

$$a = \frac{1 - \beta r^2}{1 - \beta^2 r^2}, \quad d = \frac{(1 + \beta)r}{1 - \beta^2 r^2}.$$

Clearly,

$$0 < a - d = \frac{1 + r}{1 + \beta r} < a + d = \frac{1 + r}{1 - \beta r}.$$

Put $w_1(z) = a + u + iv$, $R = |a + u + iv|$; then

$$\begin{aligned} (2.1) \quad S(u, v) &= \operatorname{Re} \left\{ -\beta w_1(z) + \frac{1}{w_1(z)} \right\} + \frac{r^2 |1 + \beta w_1(z)|^2 - |1 - w_1(z)|^2}{(1 - r^2) |w_1(z)|} \\ &= -\beta(a + u) + \frac{a + u}{R^2} + \frac{1 - \beta^2 r^2}{1 - r^2} \cdot \frac{d^2 - u^2 - v^2}{R}. \end{aligned}$$

Now,

$$\frac{\partial S}{\partial v} = -\frac{v}{R^4} \left\{ 2(a + u) + \frac{1 - \beta^2 r^2}{1 - r^2} [(d^2 - u^2 - v^2)R + 2R^3] \right\}.$$

The terms inside the curly brackets are always positive for $r < \min(1, 1/\beta)$. Hence the maximum of $S(u, v)$ in the disc $|w_1(z) - a| \leq d$ is attained when $v = 0$ and $u \in [-d, d]$. Setting $v = 0$ in (2.1) we obtain

$$(2.2) \quad S(u, 0) = \frac{2(1 - \beta^2 r^2)a}{1 - r^2} - \frac{(1 + \beta)(1 - \beta r^2)}{1 - r^2} (a + u).$$

Since $dS(u, 0)/du < 0$ for $r < \min(1, 1/\beta)$, the maximum of $S(u, 0)$ occurs at the end point $u = -d$ and the result follows.

LEMMA 2.3. If $w(z) \in \mathcal{B}$, $\beta \geq 0$, then for $|z| = r < \min(1, 1/\beta)$,

$$(2.3) \quad \operatorname{Re} \left\{ \frac{zw'(z)}{[1 - w(z)][1 + \beta w(z)]} \right\} \leq \frac{r}{(1 - r)(1 + \beta r)}.$$

Proof. From Lemma 2.1, we have

$$\operatorname{Re} \left\{ \frac{zw'(z)}{(1-w(z))(1+\beta w(z))} \right\} \leq \operatorname{Re} \left\{ \frac{w(z)}{(1-w(z))(1+\beta w(z))} \right\} + \frac{r^2 - |w(z)|^2}{(1-r^2)|1-w(z)||1+\beta w(z)|}.$$

Put $w_1(z) = [1-w(z)]/[1+\beta w(z)]$, then the above inequality becomes

$$\operatorname{Re} \left\{ \frac{zw'(z)}{(1-w(z))(1+\beta w(z))} \right\} \leq \frac{1}{(1+\beta)^2} \left[\beta - 1 + \operatorname{Re} \left\{ -\beta w_1(z) + \frac{1}{w_1(z)} \right\} + \frac{r^2|1+\beta w_1(z)|^2 - |1-w_1(z)|^2}{(1-r^2)|w_1(z)|} \right].$$

An application of Lemma 2.2 to the right hand side will give the result which is easily seen to be sharp for $w(z) = z$ at $z = r$.

The following lemma is a consequence of [2, Theorem 3].

LEMMA 2.4. *If $p(z) \in \mathcal{P}$, then on $|z| = r$,*

$$(2.4) \quad \operatorname{Re} \left\{ \frac{zp'(z)}{1+p(z)} \right\} \geq \begin{cases} -\frac{r}{1+r}, & \text{for } r < \frac{1}{3} \\ \frac{r^2 + 2^{3/2}(1-r^2)^{1/2} - 3}{1-r^2}, & \text{for } \frac{1}{3} \leq r < 1. \end{cases}$$

$$(2.5) \quad \operatorname{Re} \left\{ \frac{zp'(z)}{p(z)} \right\} \geq -\frac{2r}{1-r^2}.$$

3. Radii of starlikeness.

THEOREM 3.1. *Let $f(z) \in N$ be such that $f(z)/[\lambda f(z) + (1-\lambda)g(z)] \in \mathcal{Q}_r$, where $g(z) \in N$ and $g(z)/z \in \mathcal{P}$, $0 \leq \lambda < (1 + \sqrt{3} + 1/2\gamma)/(2 + \sqrt{3})$. Then the radius of starlikeness σ_1 of $f(z)$ is given by the only positive root in $(0, 1)$ of the equation*

$$\beta r^3 + (2 + 3\beta)r^2 + 3r - 1 = 0,$$

where $\beta = [(1 + \lambda)\gamma - 1]/(1 - \lambda)\gamma$.

Proof. Put $\psi(z) = 1 - f(z)/\gamma[\lambda f(z) + (1-\lambda)g(z)]$. Then $|\psi(z)| < 1$ for $|z| < 1$ and $\psi(0) = 1 - 1/\gamma = A$. Let $w(z) = [\psi(z) - A]/[1 - A\psi(z)]$. It is clear that $w(z) \in \mathcal{B}$ and $\psi(z) = [w(z) + A]/[1 + Aw(z)]$ from which we deduce

$$(3.1) \quad \frac{zf'(z)}{f(z)} = \frac{zg'(z)}{g(z)} - \frac{1+A}{1-\lambda} \cdot \frac{zw'(z)}{(1-w(z))(1+\beta w(z))},$$

$\beta = (A + \lambda)/(1 - \lambda)$, provided $1 - \lambda(1 - w(z))/(1 + Aw(z)) \neq 0$. Since $|w(z)| \leq r$ for $|z| = r$ by Schwarz's lemma, it follows that

$$1 - \lambda(1 - w(z))/(1 + Aw(z)) \neq 0$$

if, in particular, $|z| < 1/\beta$.

Now, as $g(z)/z \in \mathcal{S}$, write $g(z)/z = p(z)$, some $p(z) \in \mathcal{S}$. Then $zg'(z)/g(z) = 1 + zp'(z)/p(z)$. An application of (2.5) gives

$$(3.2) \quad \operatorname{Re} \left\{ \frac{zg'(z)}{g(z)} \right\} \geq \frac{1 - 2r - r^2}{1 - r^2}, \quad |z| = r < 1.$$

This result together with (3.1) and (2.3) yield

$$\operatorname{Re} \left\{ \frac{zf'(z)}{f(z)} \right\} \geq \frac{1 - 3r - (2 + 3\beta)r^2 - \beta r^3}{(1 - r)(1 + \beta r)}.$$

For the cubic polynomial

$$F(r) = \beta r^3 + (2 + 3\beta)r^2 + 3r - 1,$$

$F(0) < 0$, $F(1) = 4 + 4\beta > 0$, $F(1/\beta) = (3 + 6\beta - \beta^2)/\beta^2$. Thus the equation $F(r) = 0$ has exactly one root in $(0, 1)$ which is in the range $(0, 1/\beta)$ if $\beta < 3 + 2\sqrt{3}$, i.e., if $\lambda < (1 + \sqrt{3} + 1/2\gamma)/(2 + \sqrt{3})$.

REMARK 3.1. The theorem is sharp for

$$f(z) = \frac{1 - z}{1 + \beta z} \cdot \frac{z(1 - z)}{(1 + z)}.$$

When $\lambda = 0$, $f(z)$ is starlike in $|z| < \sqrt{5} - 2$ if $\gamma \rightarrow \infty$ and in $|z| < (\sqrt{17} - 3)/4$ if $\gamma = 1$ as previously shown by Ratti [6, Theorems 1 and 4].

THEOREM 3.2. Let $f(z) \in N$ be such that $f(z)/[\lambda f(z) + (1 - \lambda)g(z)] \in \mathcal{O}_\gamma$, where $g(z) \in N$ and $g(z)/z \in \mathcal{S}_{1/2}$. Then the radius of starlikeness of $f(z)$ is

$$\sigma_2 = \begin{cases} r_1, & \text{for } 0 \leq \lambda \leq 1/2\gamma, \\ r_2 = [2^{1/2}(1 + \beta)^{1/2} - 1]/(1 + 2\beta), & \text{for } 1/2\gamma < \lambda < (\sqrt{5} + 1 + 1/\gamma)/(\sqrt{5} + 3), \end{cases}$$

where $\beta = [(1 + \lambda)\gamma - 1]/(1 - \lambda)\gamma$ and r_1 is the smallest positive root in $(0, 1)$ of the equation

$$(1 + 2\beta + 9\beta^2)r^4 + 2(1 + 12\beta + 3\beta^2)r^3 + (13 + 10\beta + \beta^2)r^2 + 4(1 - \beta)r - 4 = 0.$$

Proof. Since $g(z)/z \in \mathcal{S}_{1/2}$, there exists $p(z) \in \mathcal{S}$ so that $g(z)/z = 1/2 + p(z)/2$. Hence

$$(3.3) \quad \frac{zg'(z)}{g(z)} = 1 + \frac{zp'(z)}{1+p(z)}.$$

Applying (2.4) to this equation gives, on $|z| = r$,

$$(3.4) \quad \operatorname{Re} \left\{ \frac{zg'(z)}{g(z)} \right\} \geq \begin{cases} 1/(1+r), & \text{for } 0 < r < 1/3 \\ 2[2^{1/2}(1-r^2)^{1/2} - 1]/(1-r^2), & \text{for } 1/3 \leq r < 1. \end{cases}$$

This result together with (3.1) and (2.3) yield, for $|z| = r < 1/3$,

$$\operatorname{Re} \left\{ \frac{zf'(z)}{f(z)} \right\} \geq \frac{1-2r-(1+2\beta)r^2}{(1-r)(1+\beta r)} = G(r)$$

and for $1/3 \leq r < 1$,

$$\operatorname{Re} \left\{ \frac{zf'(z)}{f(z)} \right\} \geq -\frac{(1+\beta)r}{(1-r)(1+\beta r)} + \frac{2[2^{1/2}(1-r^2)^{1/2} - 1]}{1-r^2},$$

which yields the equation giving the condition of starlikeness of $f(z)$ to be

$$(1+2\beta+9\beta^2)r^4 + 2(1+12\beta+3\beta^2)r^3 + (13+10\beta+\beta^2)r^2 + 4(1-\beta)r - 4 = 0.$$

The only root in $(0, 1)$ of the numerator of $G(r)$ is r_2 which is less than $1/3$ if $\beta > 1$, i.e., if $\lambda > 1/2\gamma$, and is the range $(0, 1/\beta)$ if $\beta < \sqrt{5} + 2$, i.e., if $\lambda < (\sqrt{5} + 1 + 1/\gamma)/(\sqrt{5} + 3)$. Thus $f(z)$ is starlike in $|z| < r_2$ if $1/2\gamma < \lambda < (\sqrt{5} + 1 + 1/\gamma)/(\sqrt{5} + 3)$. Now, for $0 \leq \lambda \leq 1/2\gamma$, $\beta < 1$, and r_1 is in the interval $(0, 1/\beta)$ and the theorem is proved.

REMARK 3.2. The results are sharp. The extremal functions are

$$f(z) = \begin{cases} \frac{1-z}{1+\beta z} \cdot \frac{z}{2} \left[1 + \frac{1}{2} \left(\frac{1+ze^{-i\theta}}{1-ze^{-i\theta}} + \frac{1+ze^{i\theta}}{1-ze^{i\theta}} \right) \right], & \text{for } 0 \leq \lambda \leq 1/2\gamma \\ \frac{1-z}{1+\beta z} \cdot \frac{z}{1+z}, & \text{for } 1/2\gamma < \lambda < (\sqrt{5} + 1 + 1/\gamma)(\sqrt{5} + 3), \end{cases}$$

where θ satisfies the equation

$$H(r_1)(1+r_1^2) + r_1^2 - [3H(r_1) + 1/2 + r_1^2(H(r_1) + 1/2)]r_1 \cos \theta + 2H(r_1)r_1^2 \cos^2 \theta = 0$$

with

$$H(r_1) = [r_1^2 + 2^{3/2}(1-r_1^2)^{1/2} - 3]/2(1-r_1^2).$$

When $\lambda = 0$, the cases $\gamma \rightarrow \infty$ and $\gamma = 1$ give Theorems 2 and 5 of [6].

REMARK 3.3. For $g(z) \in S^c$, the result [10]

$$\operatorname{Re} \left\{ \frac{zg'(z)}{g(z)} \right\} \geq \frac{1}{1+r}, \quad |z| = r < 1$$

together with (3.1) and (2.3) give the radius of starlikeness of $f(z) \in N$ with $f(z)/[\lambda f(z) + (1-\lambda)g(z)] \in \mathcal{E}_r$ to be $[2^{1/2}(1+\beta)^{1/2}-1]/(1+2\beta)$ for $0 \leq \lambda < (\sqrt{5}+1+1/\gamma)/(\sqrt{5}+3)$, $\beta = [(1+\lambda)\gamma-1]/(1-\lambda)\gamma$. The bound is attained for the function

$$f(z) = \frac{1-z}{1+\beta z} \cdot \frac{z}{1+z}.$$

When $\lambda = 0$, the cases $\gamma \rightarrow \infty$ and $\gamma = 1$ become Theorem 4 of [4] and Theorem 4 of [5] respectively.

THEOREM 3.3. Let $f(z) \in N$ be such that $f(z)/[\lambda f(z) + (1-\lambda)g(z)] \in \mathcal{E}_r$, where $g(z) \in S^*(\alpha)$, $0 \leq \lambda < \lambda_0$, some $\lambda_0 < 1$. Then the radius of starlikeness σ_3 of $f(z)$ is given by the smallest positive root in $(0, 1)$ of the equation

$$\beta(2\alpha-1)r^3 + (3\beta+2\alpha-2\alpha\beta)r^2 + (3-2\alpha)r - 1 = 0,$$

where $\beta = [(1+\lambda)\gamma-1]/(1-\lambda)\gamma$.

Proof. Since $g(z) \in S^*(\alpha)$, we have

$$\operatorname{Re} \left\{ \frac{zg'(z)}{g(z)} \right\} \geq \frac{1+(2\alpha-1)r}{1+r}, \quad |z| = r < 1.$$

Applying this result and (2.3) to (3.1) gives the required equation from which σ_3 may be obtained. λ_0 is determined by the condition $\sigma_3 < 1/\beta$.

REMARK 3.4. The theorem is sharp for

$$f(z) = \frac{1-z}{1+\beta z} \cdot \frac{z}{(1+z)^{2-2\alpha}}.$$

When $\lambda = 0$, the cases $\gamma \rightarrow \infty$ and $\gamma = 1$ correspond to Theorems 3 and 6 of [6].

4. Radii of convexity. In this section, we briefly look at the problem of determining the radius of convexity of $f(z) \in N$ with $f'(z)/[\lambda f'(z) + (1-\lambda)g'(z)] \in \mathcal{E}_r$, where $g(z)$ belongs to various subclasses of N . For such $f(z)$, we can deduce in a similar manner as in Theorem 3.1 that

$$(4.1) \quad \operatorname{Re} \left\{ 1 + \frac{zf''(z)}{f'(z)} \right\} = \operatorname{Re} \left\{ 1 + \frac{zg''(z)}{g'(z)} \right\} - \frac{1+A}{1-\lambda} \cdot \frac{zw'(z)}{(1-w(z))(1+\beta w(z))},$$

provided $1 - \lambda(1 - w(z))/(1 + Aw(z)) \neq 0$, $w(z) \in \mathcal{S}$, $A = 1 - 1/\gamma$, $\beta = (A + \lambda)/(1 - \lambda)$. With some restriction on λ , we may apply (2.3) and the known bounds for $\operatorname{Re} \{1 + zg''(z)/g'(z)\}$ to (4.1) to get the equations from which the radii of convexity of $f(z)$ may be obtained. We consider the following six cases.

(i) $g'(z) \in \mathcal{S}$. The radius of convexity of $f(z)$ is equal to σ_1 as given by Theorem 3.1.

(ii) $g'(z) \in \mathcal{S}_{1/2}$. The radius of convexity of $f(z)$ is equal to σ_2 as given by Theorem 3.2.

(iii) $g(z) \in S^\circ(\alpha)$. The radius of convexity of $f(z)$ is equal to σ_3 as given by Theorem 3.3.

(iv) $g(z) \in S$.

The result [1, p. 166]

$$\operatorname{Re} \left\{ 1 + \frac{zg''(z)}{g'(z)} \right\} \geq \frac{1 - 4r + r^2}{1 - r^2}, \quad |z| = r < 1,$$

together with (2.3) and (4.1) yield the radius of convexity of $f(z)$ to be the smallest positive root (less than 1) of the equation

$$\beta r^3 - 5\beta r^2 - 5r + 1 = 0,$$

with $0 \leq \lambda < (2 - \sqrt{6} + 1/2\gamma)/(3 - \sqrt{6})$.

(v) $g(z) \in S^*$. The radius of convexity of $f(z)$ is the same as that of part (iv).

(vi) $g(z) \in S^*(1/2)$. Theorem 4.1 of [9] with $\beta = 1/2$ gives

$$\operatorname{Re} \left\{ 1 + \frac{zg''(z)}{g'(z)} \right\} \geq \frac{1-r}{1+r}, \quad |z| = r < 1/2.$$

This result together with (2.3) and (4.1) yield the radius of convexity of $f(z)$ to be the smallest positive root ρ of the equation

$$\beta r^3 - 3\beta r^2 - 3r + 1 = 0,$$

with $0 \leq \lambda < (1 + \sqrt{2} + 1/2\gamma)/(2 + \sqrt{2})$.

All these results are best possible and generalise those obtained by Ratti [7, Theorems 1-6].

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Pacific Journal of Mathematics

Vol. 69, No. 1

May, 1977

V. V. Anh and P. D. Tuan, <i>On starlikeness and convexity of certain analytic functions</i>	1
Willard Ellis Baxter and L. A. Casciotti, <i>Rings with involution and the prime radical</i>	11
Manuel Phillip Berriozabal, Hon-Fei Lai and Dix Hayes Pettey, <i>Noncompact, minimal regular spaces</i>	19
Sun Man Chang, <i>Measures with continuous image law</i>	25
John Benjamin Friedlander, <i>Certain hypotheses concerning L-functions</i>	37
Moshe Goldberg and Ernst Gabor Straus, <i>On characterizations and integrals of generalized numerical ranges</i>	45
Pierre A. Grillet, <i>On subdirectly irreducible commutative semigroups</i>	55
Robert E. Hartwig and Jiang Luh, <i>On finite regular rings</i>	73
Roger Hugh Hunter, Fred Richman and Elbert A. Walker, <i>Finite direct sums of cyclic valuated p-groups</i>	97
Atsushi Inoue, <i>On a class of unbounded operator algebras. III</i>	105
Wells Johnson and Kevin J. Mitchell, <i>Symmetries for sums of the Legendre symbol</i>	117
Jimmie Don Lawson, John Robie Liukkonen and Michael William Mislove, <i>Measure algebras of semilattices with finite breadth</i>	125
Glenn Richard Luecke, <i>A note on spectral continuity and on spectral properties of essentially G_1 operators</i>	141
Takahiko Nakazi, <i>Invariant subspaces of weak-* Dirichlet algebras</i>	151
James William Pendergrass, <i>Calculations of the Schur group</i>	169
Carl Pomerance, <i>On composite n for which $\varphi(n) \mid n - 1$. II</i>	177
Marc Aristide Rieffel and Alfons Van Daele, <i>A bounded operator approach to Tomita-Takesaki theory</i>	187
Daniel Byron Shapiro, <i>Spaces of similarities. IV. (s, t)-families</i>	223
Leon M. Simon, <i>Equations of mean curvature type in 2 independent variables</i>	245
Joseph Nicholas Simone, <i>Metric components of continuous images of ordered compacta</i>	269
William Charles Waterhouse, <i>Pairs of symmetric bilinear forms in characteristic 2</i>	275