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**ON STARLIKENESS AND CONVEXITY OF CERTAIN  
ANALYTIC FUNCTIONS**

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## ON STARLIKENESS AND CONVEXITY OF CERTAIN ANALYTIC FUNCTIONS

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**Let  $N$  be the class of normalised regular functions**

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \quad |z| < 1.$$

**For  $0 \leq \lambda < 1, \gamma \geq 1$ , let  $f(z), g(z) \in N$  be such that**

$$|f(z)/[\lambda f(z) + (1 - \lambda)g(z)] - \gamma| < \gamma, \quad |z| < 1.$$

**We establish the radius of starlikeness of  $f(z)$  under the assumption  $\operatorname{Re}\{g(z)/z\} > 0$ , or  $\operatorname{Re}\{g(z)/z\} > 1/2$ , or  $\operatorname{Re}\{zg'(z)/g(z)\} > \alpha, 0 \leq \alpha < 1$ , or  $\operatorname{Re}\{1 + zg''(z)/g'(z)\} > 0$  for  $|z| < 1$ . The analysis may be extended to the problem of finding the radius of convexity for certain subclasses of  $N$ .**

1. Introduction and notation. Let  $S, S^*, S^c$  denote the subclasses of  $N$  which are univalent, univalent starlike, univalent convex in  $|z| < 1$  respectively.

A necessary and sufficient condition for  $f(z) \in N$  to be univalent starlike in  $|z| < r$  is

$$\operatorname{Re} \left\{ \frac{zf'(z)}{f(z)} \right\} > 0, \quad |z| < r.$$

A necessary and sufficient condition for  $f(z) \in N$  to be univalent convex in  $|z| < r$  is

$$\operatorname{Re} \left\{ 1 + \frac{zf''(z)}{f'(z)} \right\} > 0, \quad |z| < r.$$

A function  $f(z)$  belongs to  $S^*(\beta)$ , i.e., is starlike of order  $\beta, 0 \leq \beta < 1$ , if it satisfies the condition

$$\operatorname{Re} \left\{ \frac{zf'(z)}{f(z)} \right\} > \beta, \quad |z| < 1.$$

A function  $f(z)$  belongs to  $S^c(\beta)$ , i.e., is convex of order  $\beta, 0 \leq \beta < 1$ , if it satisfies the condition

$$\operatorname{Re} \left\{ 1 + \frac{zf''(z)}{f'(z)} \right\} > \beta, \quad |z| < 1.$$

Let  $\mathcal{P}_\alpha$  denote the class of regular functions of the form

$$p(z) = 1 + \sum_{k=1}^{\infty} c_k z^k, \quad |z| < 1,$$

satisfying the inequality  $\operatorname{Re}\{p(z)\} > \alpha$  for  $|z| < 1$ ,  $0 \leq \alpha < 1$  and  $\mathcal{Q}_\gamma$  the class of functions  $g(z)$  with expansion of the above form but satisfying the inequality  $|g(z) - \gamma| < \gamma$  for  $|z| < 1$ ,  $\gamma \geq 1$ . We note that both  $\mathcal{S}_0$  and  $\mathcal{Q}_\infty$  reduce to the class  $\mathcal{S}$  of functions with positive real part.

Let  $N_n$ ,  $n \geq 1$ , denote the subclass of  $N$  consisting of functions of the form  $f(z) = z + \sum_{k=n+1}^{\infty} a_k z^k$ . Then  $N_1 = N$ .

Shah [8] considered the problem of determining the radius of starlikeness of  $f(z) \in N_n$  for the following cases:

(a)  $f(z)/[\lambda f(z) + (1 - \lambda)g(z)] \in \mathcal{S}$  with  $g(z) \in N_n$  and  $g(z)/z \in \mathcal{S}$ , or  $g(z)/z \in \mathcal{S}_{1/2}$  (with  $n = 1$ ), or  $g(z) \in S^*(\alpha)$ ;

(b)  $f(z)/[\lambda f(z) + (1 - \lambda)g(z)] \in \mathcal{Q}_1$  with  $g(z) \in N_n$  and  $g(z)/z \in \mathcal{S}$ , or  $g(z) \in S^*(\alpha)$ .

The conditions were shown to be sharp only when  $\lambda = 0$ . In this paper, we solve the problem for the subclasses of  $N$  mentioned at the beginning, subject to certain restrictions on the values of  $\lambda$ . Letting  $\gamma \rightarrow \infty$  we obtain the radii of starlikeness of  $f(z)$  satisfying  $f(z)/[\lambda f(z) + (1 - \lambda)g(z)] \in \mathcal{S}$ . All the bounds obtained are best possible. Furthermore, the same technique may be used to establish the radius of convexity of  $f(z) \in N$  satisfying  $f'(z)/[\lambda f'(z) + (1 - \lambda)g'(z)] \in \mathcal{Q}_\gamma$ , where  $g(z)$  belongs to various subclasses of  $N$ . The results proved here generalize those of MacGregor [3, 4, 5] and Ratti [6, 7].

It should be remarked that parallel results for subclasses of  $N_n$ ,  $n > 1$ , may be derived in an analogous manner. The manipulations involved are, however, more complicated.

The lemmas required for the proofs of our theorems are given in §2. Section 3 contains theorems giving the conditions for starlikeness. We outline the conditions for convexity in §4.

2. Some lemmas. Let  $\mathcal{B}$  denote the class of functions  $w(z)$  regular in  $|z| < 1$  and satisfying  $w(0) = 0$ ,  $|w(z)| < 1$  for  $|z| < 1$ .

LEMMA 2.1 [9]. *If  $w(z) \in \mathcal{B}$ , then for  $|z| < 1$ ,*

$$|zw'(z) - w(z)| \leq \frac{|z|^2 - |w(z)|^2}{1 - |z|^2}.$$

*Proof.* Write  $w(z) = z\phi(z)$ , where  $\phi(z)$  is regular in  $|z| < 1$  and  $|\phi(z)| \leq 1$ . The assertion now follows from the well-known result due to Caratheodory

$$|\phi'(z)| \leq \frac{1 - |\phi(z)|^2}{1 - |z|^2}.$$

LEMMA 2.2. *Let  $w_1(z) = [1 - w(z)]/[1 + \beta w(z)]$ , where  $w(z) \in \mathcal{B}$ ,*

$\beta \geq 0$ . Then, for  $|z| = r < \min(1, 1/\beta)$ ,

$$\begin{aligned} \operatorname{Re} \left\{ -\beta w_1(z) + \frac{1}{w_1(z)} \right\} + \frac{r^2 |1 + \beta w_1(z)|^2 - |1 - w_1(z)|^2}{(1 - r^2) |w_1(z)|} \\ \leq \frac{1 - \beta + (3\beta + 1)r + \beta(\beta + 3)r^2 + \beta(\beta - 1)r^3}{(1 - r^2)(1 + \beta r)}. \end{aligned}$$

*Proof.* By Schwarz's lemma,  $|w(z)| \leq r$  on  $|z| = r < 1$ . The transformation  $w_1(z) = [1 - w(z)]/[1 + \beta w(z)]$  maps the disc  $|w(z)| \leq r$ ,  $r < \min(1, 1/\beta)$ , onto the disc  $|w_1(z) - a| \leq d$ , where

$$a = \frac{1 - \beta r^2}{1 - \beta^2 r^2}, \quad d = \frac{(1 + \beta)r}{1 - \beta^2 r^2}.$$

Clearly,

$$0 < a - d = \frac{1 + r}{1 + \beta r} < a + d = \frac{1 + r}{1 - \beta r}.$$

Put  $w_1(z) = a + u + iv$ ,  $R = |a + u + iv|$ ; then

$$\begin{aligned} (2.1) \quad S(u, v) &= \operatorname{Re} \left\{ -\beta w_1(z) + \frac{1}{w_1(z)} \right\} + \frac{r^2 |1 + \beta w_1(z)|^2 - |1 - w_1(z)|^2}{(1 - r^2) |w_1(z)|} \\ &= -\beta(a + u) + \frac{a + u}{R^2} + \frac{1 - \beta^2 r^2}{1 - r^2} \cdot \frac{d^2 - u^2 - v^2}{R}. \end{aligned}$$

Now,

$$\frac{\partial S}{\partial v} = -\frac{v}{R^4} \left\{ 2(a + u) + \frac{1 - \beta^2 r^2}{1 - r^2} [(d^2 - u^2 - v^2)R + 2R^3] \right\}.$$

The terms inside the curly brackets are always positive for  $r < \min(1, 1/\beta)$ . Hence the maximum of  $S(u, v)$  in the disc  $|w_1(z) - a| \leq d$  is attained when  $v = 0$  and  $u \in [-d, d]$ . Setting  $v = 0$  in (2.1) we obtain

$$(2.2) \quad S(u, 0) = \frac{2(1 - \beta^2 r^2)a}{1 - r^2} - \frac{(1 + \beta)(1 - \beta r^2)}{1 - r^2} (a + u).$$

Since  $dS(u, 0)/du < 0$  for  $r < \min(1, 1/\beta)$ , the maximum of  $S(u, 0)$  occurs at the end point  $u = -d$  and the result follows.

LEMMA 2.3. If  $w(z) \in \mathcal{S}$ ,  $\beta \geq 0$ , then for  $|z| = r < \min(1, 1/\beta)$ ,

$$(2.3) \quad \operatorname{Re} \left\{ \frac{zw'(z)}{[1 - w(z)][1 + \beta w(z)]} \right\} \leq \frac{r}{(1 - r)(1 + \beta r)}.$$

*Proof.* From Lemma 2.1, we have

$$\operatorname{Re} \left\{ \frac{zw'(z)}{(1-w(z))(1+\beta w(z))} \right\} \leq \operatorname{Re} \left\{ \frac{w(z)}{(1-w(z))(1+\beta w(z))} \right\} + \frac{r^2 - |w(z)|^2}{(1-r^2)|1-w(z)||1+\beta w(z)|}.$$

Put  $w_1(z) = [1-w(z)]/[1+\beta w(z)]$ , then the above inequality becomes

$$\operatorname{Re} \left\{ \frac{zw'(z)}{(1-w(z))(1+\beta w(z))} \right\} \leq \frac{1}{(1+\beta)^2} \left[ \beta - 1 + \operatorname{Re} \left\{ -\beta w_1(z) + \frac{1}{w_1(z)} \right\} + \frac{r^2|1+\beta w_1(z)|^2 - |1-w_1(z)|^2}{(1-r^2)|w_1(z)|} \right].$$

An application of Lemma 2.2 to the right hand side will give the result which is easily seen to be sharp for  $w(z) = z$  at  $z = r$ .

The following lemma is a consequence of [2, Theorem 3].

LEMMA 2.4. *If  $p(z) \in \mathcal{P}$ , then on  $|z| = r$ ,*

$$(2.4) \quad \operatorname{Re} \left\{ \frac{zp'(z)}{1+p(z)} \right\} \geq \begin{cases} -\frac{r}{1+r}, & \text{for } r < \frac{1}{3} \\ \frac{r^2 + 2^{3/2}(1-r^2)^{1/2} - 3}{1-r^2}, & \text{for } \frac{1}{3} \leq r < 1. \end{cases}$$

$$(2.5) \quad \operatorname{Re} \left\{ \frac{zp'(z)}{p(z)} \right\} \geq -\frac{2r}{1-r^2}.$$

### 3. Radii of starlikeness.

THEOREM 3.1. *Let  $f(z) \in N$  be such that  $f(z)/[\lambda f(z) + (1-\lambda)g(z)] \in \mathcal{C}_r$ , where  $g(z) \in N$  and  $g(z)/z \in \mathcal{P}$ ,  $0 \leq \lambda < (1 + \sqrt{3} + 1/2\gamma)/(2 + \sqrt{3})$ . Then the radius of starlikeness  $\sigma_1$  of  $f(z)$  is given by the only positive root in  $(0, 1)$  of the equation*

$$\beta r^3 + (2 + 3\beta)r^2 + 3r - 1 = 0,$$

where  $\beta = [(1 + \lambda)\gamma - 1]/(1 - \lambda)\gamma$ .

*Proof.* Put  $\psi(z) = 1 - f(z)/\gamma[\lambda f(z) + (1-\lambda)g(z)]$ . Then  $|\psi(z)| < 1$  for  $|z| < 1$  and  $\psi(0) = 1 - 1/\gamma = A$ . Let  $w(z) = [\psi(z) - A]/[1 - A\psi(z)]$ . It is clear that  $w(z) \in \mathcal{B}$  and  $\psi(z) = [w(z) + A]/[1 + Aw(z)]$  from which we deduce

$$(3.1) \quad \frac{zf'(z)}{f(z)} = \frac{zg'(z)}{g(z)} - \frac{1+A}{1-\lambda} \cdot \frac{zw'(z)}{(1-w(z))(1+\beta w(z))},$$

$\beta = (A + \lambda)/(1 - \lambda)$ , provided  $1 - \lambda(1 - w(z))/(1 + Aw(z)) \neq 0$ . Since  $|w(z)| \leq r$  for  $|z| = r$  by Schwarz's lemma, it follows that

$$1 - \lambda(1 - w(z))/(1 + Aw(z)) \neq 0$$

if, in particular,  $|z| < 1/\beta$ .

Now, as  $g(z)/z \in \mathcal{S}$ , write  $g(z)/z = p(z)$ , some  $p(z) \in \mathcal{S}$ . Then  $zg'(z)/g(z) = 1 + zp'(z)/p(z)$ . An application of (2.5) gives

$$(3.2) \quad \operatorname{Re} \left\{ \frac{zg'(z)}{g(z)} \right\} \geq \frac{1 - 2r - r^2}{1 - r^2}, \quad |z| = r < 1.$$

This result together with (3.1) and (2.3) yield

$$\operatorname{Re} \left\{ \frac{zf'(z)}{f(z)} \right\} \geq \frac{1 - 3r - (2 + 3\beta)r^2 - \beta r^3}{(1 - r)(1 + \beta r)}.$$

For the cubic polynomial

$$F(r) = \beta r^3 + (2 + 3\beta)r^2 + 3r - 1,$$

$F(0) < 0$ ,  $F(1) = 4 + 4\beta > 0$ ,  $F(1/\beta) = (3 + 6\beta - \beta^2)/\beta^2$ . Thus the equation  $F(r) = 0$  has exactly one root in  $(0, 1)$  which is in the range  $(0, 1/\beta)$  if  $\beta < 3 + 2\sqrt{3}$ , i.e., if  $\lambda < (1 + \sqrt{3} + 1/2\gamma)/(2 + \sqrt{3})$ .

REMARK 3.1. The theorem is sharp for

$$f(z) = \frac{1 - z}{1 + \beta z} \cdot \frac{z(1 - z)}{(1 + z)}.$$

When  $\lambda = 0$ ,  $f(z)$  is starlike in  $|z| < \sqrt{5} - 2$  if  $\gamma \rightarrow \infty$  and in  $|z| < (\sqrt{17} - 3)/4$  if  $\gamma = 1$  as previously shown by Ratti [6, Theorems 1 and 4].

THEOREM 3.2. Let  $f(z) \in N$  be such that  $f(z)/[\lambda f(z) + (1 - \lambda)g(z)] \in \mathcal{O}_\gamma$ , where  $g(z) \in N$  and  $g(z)/z \in \mathcal{S}_{1/2}$ . Then the radius of starlikeness of  $f(z)$  is

$$\sigma_2 = \begin{cases} r_1, & \text{for } 0 \leq \lambda \leq 1/2\gamma, \\ r_2 = [2^{1/2}(1 + \beta)^{1/2} - 1]/(1 + 2\beta), & \text{for } 1/2\gamma < \lambda < (\sqrt{5} + 1 + 1/\gamma)/(\sqrt{5} + 3), \end{cases}$$

where  $\beta = [(1 + \lambda)\gamma - 1]/(1 - \lambda)\gamma$  and  $r_1$  is the smallest positive root in  $(0, 1)$  of the equation

$$(1 + 2\beta + 9\beta^2)r^4 + 2(1 + 12\beta + 3\beta^2)r^3 + (13 + 10\beta + \beta^2)r^2 + 4(1 - \beta)r - 4 = 0.$$

Proof. Since  $g(z)/z \in \mathcal{S}_{1/2}$ , there exists  $p(z) \in \mathcal{S}$  so that  $g(z)/z = 1/2 + p(z)/2$ . Hence

$$(3.3) \quad \frac{zg'(z)}{g(z)} = 1 + \frac{zp'(z)}{1+p(z)}.$$

Applying (2.4) to this equation gives, on  $|z| = r$ ,

$$(3.4) \quad \operatorname{Re} \left\{ \frac{zg'(z)}{g(z)} \right\} \geq \begin{cases} 1/(1+r), & \text{for } 0 < r < 1/3 \\ 2[2^{1/2}(1-r^2)^{1/2} - 1]/(1-r^2), & \text{for } 1/3 \leq r < 1. \end{cases}$$

This result together with (3.1) and (2.3) yield, for  $|z| = r < 1/3$ ,

$$\operatorname{Re} \left\{ \frac{zf'(z)}{f(z)} \right\} \geq \frac{1-2r-(1+2\beta)r^2}{(1-r)(1+\beta r)} = G(r)$$

and for  $1/3 \leq r < 1$ ,

$$\operatorname{Re} \left\{ \frac{zf'(z)}{f(z)} \right\} \geq -\frac{(1+\beta)r}{(1-r)(1+\beta r)} + \frac{2[2^{1/2}(1-r^2)^{1/2} - 1]}{1-r^2},$$

which yields the equation giving the condition of starlikeness of  $f(z)$  to be

$$(1+2\beta+9\beta^2)r^4 + 2(1+12\beta+3\beta^2)r^3 + (13+10\beta+\beta^2)r^2 + 4(1-\beta)r - 4 = 0.$$

The only root in  $(0, 1)$  of the numerator of  $G(r)$  is  $r_2$  which is less than  $1/3$  if  $\beta > 1$ , i.e., if  $\lambda > 1/2\gamma$ , and is the range  $(0, 1/\beta)$  if  $\beta < \sqrt{5} + 2$ , i.e., if  $\lambda < (\sqrt{5} + 1 + 1/\gamma)/(\sqrt{5} + 3)$ . Thus  $f(z)$  is starlike in  $|z| < r_2$  if  $1/2\gamma < \lambda < (\sqrt{5} + 1 + 1/\gamma)/(\sqrt{5} + 3)$ . Now, for  $0 \leq \lambda \leq 1/2\gamma$ ,  $\beta < 1$ , and  $r_1$  is in the interval  $(0, 1/\beta)$  and the theorem is proved.

**REMARK 3.2.** The results are sharp. The extremal functions are

$$f(z) = \begin{cases} \frac{1-z}{1+\beta z} \cdot \frac{z}{2} \left[ 1 + \frac{1}{2} \left( \frac{1+ze^{-i\theta}}{1-ze^{-i\theta}} + \frac{1+ze^{i\theta}}{1-ze^{i\theta}} \right) \right], & \text{for } 0 \leq \lambda \leq 1/2\gamma \\ \frac{1-z}{1+\beta z} \cdot \frac{z}{1+z}, & \text{for } 1/2\gamma < \lambda < (\sqrt{5} + 1 + 1/\gamma)/(\sqrt{5} + 3), \end{cases}$$

where  $\theta$  satisfies the equation

$$H(r_1)(1+r_1^2) + r_1^2 - [3H(r_1) + 1/2 + r_1^2(H(r_1) + 1/2)]r_1 \cos \theta + 2H(r_1)r_1^2 \cos^2 \theta = 0$$

with

$$H(r_1) = [r_1^2 + 2^{3/2}(1-r_1^2)^{1/2} - 3]/2(1-r_1^2).$$

When  $\lambda = 0$ , the cases  $\gamma \rightarrow \infty$  and  $\gamma = 1$  give Theorems 2 and 5 of [6].

REMARK 3.3. For  $g(z) \in S^c$ , the result [10]

$$\operatorname{Re} \left\{ \frac{zg'(z)}{g(z)} \right\} \geq \frac{1}{1+r}, \quad |z| = r < 1$$

together with (3.1) and (2.3) give the radius of starlikeness of  $f(z) \in N$  with  $f(z)/[\lambda f(z) + (1-\lambda)g(z)] \in \mathcal{O}_r$  to be  $[2^{1/2}(1+\beta)^{1/2}-1]/(1+2\beta)$  for  $0 \leq \lambda < (\sqrt{5}+1+1/\gamma)/(\sqrt{5}+3)$ ,  $\beta = [(1+\lambda)\gamma-1]/(1-\lambda)\gamma$ . The bound is attained for the function

$$f(z) = \frac{1-z}{1+\beta z} \cdot \frac{z}{1+z}.$$

When  $\lambda = 0$ , the cases  $\gamma \rightarrow \infty$  and  $\gamma = 1$  become Theorem 4 of [4] and Theorem 4 of [5] respectively.

THEOREM 3.3. Let  $f(z) \in N$  be such that  $f(z)/[\lambda f(z) + (1-\lambda)g(z)] \in \mathcal{O}_r$ , where  $g(z) \in S^*(\alpha)$ ,  $0 \leq \lambda < \lambda_0$ , some  $\lambda_0 < 1$ . Then the radius of starlikeness  $\sigma_3$  of  $f(z)$  is given by the smallest positive root in  $(0, 1)$  of the equation

$$\beta(2\alpha-1)r^3 + (3\beta+2\alpha-2\alpha\beta)r^2 + (3-2\alpha)r - 1 = 0,$$

where  $\beta = [(1+\lambda)\gamma-1]/(1-\lambda)\gamma$ .

Proof. Since  $g(z) \in S^*(\alpha)$ , we have

$$\operatorname{Re} \left\{ \frac{zg'(z)}{g(z)} \right\} \geq \frac{1+(2\alpha-1)r}{1+r}, \quad |z| = r < 1.$$

Applying this result and (2.3) to (3.1) gives the required equation from which  $\sigma_3$  may be obtained.  $\lambda_0$  is determined by the condition  $\sigma_3 < 1/\beta$ .

REMARK 3.4. The theorem is sharp for

$$f(z) = \frac{1-z}{1+\beta z} \cdot \frac{z}{(1+z)^{2-2\alpha}}.$$

When  $\lambda = 0$ , the cases  $\gamma \rightarrow \infty$  and  $\gamma = 1$  correspond to Theorems 3 and 6 of [6].

4. Radii of convexity. In this section, we briefly look at the problem of determining the radius of convexity of  $f(z) \in N$  with  $f'(z)/[\lambda f'(z) + (1-\lambda)g'(z)] \in \mathcal{O}_r$ , where  $g(z)$  belongs to various subclasses of  $N$ . For such  $f(z)$ , we can deduce in a similar manner as in Theorem 3.1 that

$$(4.1) \quad \operatorname{Re} \left\{ 1 + \frac{zf''(z)}{f'(z)} \right\} = \operatorname{Re} \left\{ 1 + \frac{zg''(z)}{g'(z)} \right\} - \frac{1+A}{1-\lambda} \cdot \frac{zw'(z)}{(1-w(z))(1+\beta w(z))},$$

provided  $1 - \lambda(1 - w(z))/(1 + Aw(z)) \neq 0$ ,  $w(z) \in \mathcal{B}$ ,  $A = 1 - 1/\gamma$ ,  $\beta = (A + \lambda)/(1 - \lambda)$ . With some restriction on  $\lambda$ , we may apply (2.3) and the known bounds for  $\operatorname{Re} \{1 + zg''(z)/g'(z)\}$  to (4.1) to get the equations from which the radii of convexity of  $f(z)$  may be obtained. We consider the following six cases.

(i)  $g'(z) \in \mathcal{P}$ . The radius of convexity of  $f(z)$  is equal to  $\sigma_1$  as given by Theorem 3.1.

(ii)  $g'(z) \in \mathcal{P}_{1/2}$ . The radius of convexity of  $f(z)$  is equal to  $\sigma_2$  as given by Theorem 3.2.

(iii)  $g(z) \in S^\circ(\alpha)$ . The radius of convexity of  $f(z)$  is equal to  $\sigma_3$  as given by Theorem 3.3.

(iv)  $g(z) \in S$ .

The result [1, p. 166]

$$\operatorname{Re} \left\{ 1 + \frac{zg''(z)}{g'(z)} \right\} \geq \frac{1 - 4r + r^2}{1 - r^2}, \quad |z| = r < 1,$$

together with (2.3) and (4.1) yield the radius of convexity of  $f(z)$  to be the smallest positive root (less than 1) of the equation

$$\beta r^3 - 5\beta r^2 - 5r + 1 = 0,$$

with  $0 \leq \lambda < (2 - \sqrt{6} + 1/2\gamma)/(3 - \sqrt{6})$ .

(v)  $g(z) \in S^*$ . The radius of convexity of  $f(z)$  is the same as that of part (iv).

(vi)  $g(z) \in S^*(1/2)$ . Theorem 4.1 of [9] with  $\beta = 1/2$  gives

$$\operatorname{Re} \left\{ 1 + \frac{zg''(z)}{g'(z)} \right\} \geq \frac{1-r}{1+r}, \quad |z| = r < 1/2.$$

This result together with (2.3) and (4.1) yield the radius of convexity of  $f(z)$  to be the smallest positive root  $\rho$  of the equation

$$\beta r^3 - 3\beta r^2 - 3r + 1 = 0,$$

with  $0 \leq \lambda < (1 + \sqrt{2} + 1/2\gamma)/(2 + \sqrt{2})$ .

All these results are best possible and generalise those obtained by Ratti [7, Theorems 1-6].

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