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**ON CHARACTERIZATIONS AND INTEGRALS OF
GENERALIZED NUMERICAL RANGES**

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Let $c=(\gamma_1, \dots, \gamma_n)$ be given. The generalized numerical range of an $n \times n$ matrix A , associated with c , is the set $W_c(A)=\{\sum \gamma_j(Ax_j, x_j)\}$ where (x_1, \dots, x_n) varies over orthonormal systems in C^n . Characterizations of this range, for real c , are given. Next, we study integrals of the form $\int W_c(A)d\mu(c)$ where $\mu(c)$ is a measure defined on a domain in R^n . The above characterizations are used to study the inclusion $\int W_c(A)d\mu(c) \subset \lambda W_{c'}(A)$. We determine those λ , for which this inclusion holds for all $n \times n$ matrices A . Such relations lead to more elementary ones, when the integral reduces to a finite linear combination of ranges. In particular, we obtain the inclusion relations of the form $W_c(A) \subset \lambda W_{c'}(A)$ which hold for all A .

1. Introduction. The generalized numerical range of an $n \times n$ complex matrix A , associated with a fixed vector $c = (\gamma_1, \dots, \gamma_n) \in C^n$, is the set of complex numbers

$$(1.1) \quad W_c = W_{(\gamma_1, \dots, \gamma_n)}(A) = \left\{ \sum_{j=1}^n \gamma_j(Ax_j, x_j) : (x_1, \dots, x_n) \in A_n \right\},$$

where A_n is the set of all orthonormal n -tuples of vectors in C^n . We call W_c a generalized range since for $c = (1, 0, \dots, 0)$ it reduces to the classical range

$$W(A) = \{(Ax, x) : \|x\| = 1\}.$$

It is clear from (1.1) that W_c remains invariant under permutations of the components of c ; that is, W_c depends on the unordered set $\{\gamma_1, \dots, \gamma_n\}$ rather than on c .

Westwick, [5], has shown that if c is a real vector then W_c is convex, but if $c \in C^n$ with $n \geq 3$, then $W_c(x)$ may fail to be convex even for normal A . For this reason we restrict our attention, in this paper, to generalized numerical ranges with *real* coefficients.

Our first purpose is to characterize the sets W_c . In §2 we show that

$$W_c(A) = \{\text{tr}(HA) : H \in \mathcal{H}_c\},$$

where \mathcal{H}_c is a class of Hermitian matrices depending on c .

In §3 we define integrals of the form $\int_{\mathcal{D}} W_c(A)d\mu(c)$ where \mathcal{D}

is a domain in \mathbf{R}^n and $\mu(c)$ is a nonnegative measure on \mathcal{D} . Since the sets W_c are convex, such integrals are convex as well, and we may define them in terms of their support functions.

Finally, using the above characterization of W_c , we investigate inclusion relations of the form

$$(1.2) \quad \int_{\mathcal{D}} W_c(A) d\mu(c) \subset \lambda W_{c'}(A), \quad \lambda = \text{constant},$$

which hold, uniformly, for all $A \in \mathbf{C}_{n \times n}$, i.e., for all n -square matrices. If the measure $\mu(c)$ is concentrated on a finite number of vectors c , then (1.2) is reduced to inclusion relations involving finite linear combinations of generalized numerical ranges. Such relations were considered in earlier works [2, 3].

In particular, for given vectors c, c' we obtain necessary and sufficient conditions under which

$$W_c(A) \subset \lambda W_{c'}, \quad \forall A \in \mathbf{C}_{n \times n}.$$

2. Characterization of generalized ranges. For any vector $c = (\gamma_1, \dots, \gamma_n)$ consider the diagonal matrix

$$C = \text{diag}(c) = \text{diag}(\gamma_1, \dots, \gamma_n),$$

and construct the class of matrices

$$\mathcal{U}_c = \text{conv} \{UCU^*: U \text{ unitary}\},$$

where conv denotes the convex hull.

Since we restrict attention to $c \in \mathbf{R}^n$ it is evident that the elements of \mathcal{U}_c are Hermitian.

Using \mathcal{U}_c we have the following characterization of ranges with real coefficients.

THEOREM 1. *If $c \in \mathbf{R}^n$ then*

$$W_c(A) = \{\text{tr}(HA): H \in \mathcal{U}_c\}.$$

Proof. It follows from the definition of $W_c(A)$ in (1.1) that

$$W_c(A) = \{\text{tr}(CU^*AU): U \text{ unitary}\}.$$

Thus

$$(2.1) \quad W_c(A) = \{\text{tr}((UCU^*)A): U \text{ unitary}\},$$

which implies that

$$W_c(A) \subset \{\text{tr}(HA): H \in \mathcal{U}_c\}.$$

For the converse inclusion let

$$H = \sum_i \lambda_i (U_i C U_i^*); \lambda_i \geq 0, \quad \sum_i \lambda_i = 1,$$

be an arbitrary element of \mathcal{W}_c . By the convexity of W_c and by (2.1) we have

$$\text{tr}(HA) = \sum \lambda_i \text{tr}((U_i C U_i^*)A) \in W_c(A).$$

So,

$$\{\text{tr}(HA): H \in \mathcal{W}_c\} \subset W_c(A),$$

and the theorem follows.

We introduce two definitions which lead to another characterization of $W_c(A)$.

DEFINITION 1. (i) A real vector $c = (\gamma_1, \dots, \gamma_n)$ is called *ordered* if

$$\gamma_1 \geq \gamma_2 \geq \dots \geq \gamma_n.$$

(ii) We say that c, c' satisfy $c' < c$ if there exists a doubly stochastic matrix S (i.e., a matrix with nonnegative entries whose row sums and columns sums equal 1), such that $c' = Sc$.

In Theorem 5 of [3] we proved the following.

LEMMA 1. For ordered c, c' we have $c' < c$ if and only if

$$\sum_{j=1}^l \gamma'_j \leq \sum_{j=1}^l \gamma_j, \quad l = 1, \dots, n,$$

with equality for $l = n$.

DEFINITION 2. Let $c \in \mathbf{R}^n$, and let $A_l (1 \leq l \leq n)$ be the set of all orthonormal l -tuples of vectors in \mathbf{C}^n . We define \mathcal{H}_c to be the class of all Hermitian matrices H for which

$$(2.2) \quad \sum_{j=1}^l (Hx_j, x_j) \leq \sum_{j=1}^l \gamma_j, \quad \forall (x_1, \dots, x_l) \in A_l, \quad l = 1, \dots, n,$$

with equality for $l = n$.

Let e_1, \dots, e_n be the standard basis of \mathbf{C}^n . Note that if $\sum \gamma_j = 0$ (which is the case assumed in §3), then the equality for $l = n$ in (2.2) implies that

$$\sum_{j=1}^n (He_j, e_j) = \sum \gamma_j = 0;$$

i.e., all members of \mathcal{H}_c have trace 0.

LEMMA 2. *If c is ordered then $\mathcal{H}_c = \mathcal{U}_c$.*

Proof. Take a unitary matrix U and orthonormal vectors x_1, \dots, x_l , ($1 \leq l \leq n$). Since the vectors $y_j = U^*x_j$, $j = 1, \dots, l$, are orthonormal as well, it is not hard to verify that

$$(2.3) \quad \sum_{j=1}^l (UCU^*x_j, x_j) = \sum_{j=1}^l (Cy_j, y_j) \leq \gamma_1 + \dots + \gamma_l, \\ C = \text{diag}(c),$$

with equality for $l = n$. Therefore, if

$$H = \sum_i \lambda_i U_i C U_i^*, \quad \left(\lambda_i \geq 0, \quad \sum_i \lambda_i = 1 \right),$$

is any (Hermitian) matrix in \mathcal{U}_c , we find by (2.3) that

$$\sum_{j=1}^l (Hx_j, x_j) = \sum_{j=1}^l \sum_i \lambda_i (U_i C U_i^* x_j, x_j) \leq \sum_i \lambda_i \sum_{j=1}^l \gamma_j = \sum_{j=1}^l \gamma_j,$$

with equality for $l = n$. So, by Definition 2, $H \in \mathcal{H}_c$, and consequently $\mathcal{U}_c \subset \mathcal{H}_c$.

Conversely, take any $H \in \mathcal{H}_c$. Since H is Hermitian, it is unitarily similar to a real diagonal matrix, i.e., there exists a unitary V such that

$$(2.4) \quad C' \equiv V^* H V = \text{diag}(\gamma'_1, \dots, \gamma'_n),$$

where we may assume that $c' = (\gamma'_1, \dots, \gamma'_n)$ is ordered. Using (2.2) and the orthonormal vectors $x_j = V e_j$, $j = 1, \dots, l$, we find that

$$\sum_{j=1}^l \gamma'_j = \sum_{j=1}^l (C' e_j, e_j) = \sum_{j=1}^l (V^* H V e_j, e_j) = \sum_{j=1}^l (H x_j, x_j) \leq \sum_{j=1}^l \gamma_j,$$

with equality for $l = n$. That is, by Lemma 1, $c' < c$. Hence, there exists a doubly stochastic matrix S such that $c' = S c$. Now recall that doubly stochastic matrices are convex combinations of permutation matrices P_σ . In particular $S = \sum_\sigma \lambda_\sigma P_\sigma$. Thus

$$(2.5) \quad c' = \sum_{\sigma \in S_n} \lambda_\sigma P_\sigma c; \quad \lambda_\sigma \geq 0, \quad \sum \lambda_\sigma = 1,$$

where S_n is the symmetric group. Since for every B , $P_\sigma B P_\sigma^*$ has both the rows and columns of B permuted according to σ , we have

$$(2.6) \quad \text{diag}(P_\sigma c) = P_\sigma \text{diag}(c) P_\sigma^* = P_\sigma C P_\sigma^*.$$

So, by (2.5), (2.6),

$$(2.7) \quad C' = \text{diag}(c') = \sum_\sigma \lambda_\sigma \text{diag}(P_\sigma c) = \sum_\sigma \lambda_\sigma P_\sigma C P_\sigma^*.$$

From (2.4) and (2.7) we obtain

$$(2.8) \quad H = VC'V^* = \sum_{\sigma} \lambda_{\sigma} [(VP_{\sigma})C(VP_{\sigma})^*] = \sum_{\sigma} \lambda_{\sigma} (U_{\sigma}CU_{\sigma}^*),$$

$$\lambda_{\sigma} \geq 0, \quad \sum \lambda_{\sigma} = 1,$$

where $U_{\sigma} \equiv VP_{\sigma}$ are, of course, unitary. Hence, $H \in \mathcal{U}_c$, i.e., $\mathcal{H}_c \subset \mathcal{U}_c$ and the proof is complete.

Theorem 1 together with Lemma 2 imply a second characterization of generalized numerical ranges with real coefficients.

THEOREM 2. *If c is ordered then*

$$W_c(A) = \{ \text{tr}(HA) : H \in \mathcal{H}_c \}.$$

Another simple consequence of the last lemma and the convexity of \mathcal{U}_c is that for ordered c , \mathcal{H}_c is convex.

At this point we recall the definition of the k -numerical range, ($1 \leq k \leq n$), given by Halmos [1, § 167], which after a convenient normalization becomes

$$W_k(A) = \left\{ \frac{1}{k} \text{tr}(PAP) : P = \text{orthogonal projection of rank } k \right\}.$$

It can be verified that $W_k(A)$ may be written as

$$W_k(A) = \left\{ \frac{1}{k} \sum_{j=1}^k (Ax_j, x_j) : (x_1, \dots, x_k) \in A_k \right\}.$$

Hence we see that

$$W_k(A) = W_{c_k}(A), \quad \text{with } c_k = \frac{1}{k}(e_1 + \dots + e_k).$$

That is, the k -numerical range is a special case of the generalized numerical range.

The matrices \mathcal{H}_{c_k} are those Hermitian matrices which satisfy Definition 2 with $c = c_k$. Using this definition one can show that

$$\mathcal{H}_{c_k} = \left\{ \text{Hermitian } H : 0 \leq H \leq \frac{1}{k}I, \text{tr}(H) = 1 \right\}.$$

Thus Theorem 2 generalizes the result

$$W_k(A) = \left\{ \text{tr}(HA) : 0 \leq H \leq \frac{1}{k}I, \text{tr}(H) = 1 \right\}, \quad k = 1, \dots, n$$

of Fillmore and Williams [1, Theorem 1.2].

3. Integrals of generalized ranges. In this section we are

interested in linear combinations, or more generally, in integrals of the sets $W_c(A)$, where A is arbitrary but fixed, and c varies in some domain of \mathbf{R}^n .

Let $c = (\gamma_1, \dots, \gamma_n)$ be a real vector with $\gamma \equiv \Sigma \gamma_j \neq 0$, and consider the vector $b = (\beta_1, \dots, \beta_n)$ defined by

$$b = c - \left(\frac{\gamma}{n}, \dots, \frac{\gamma}{n} \right).$$

We have $\Sigma \beta_j = 0$ and

$$B \equiv \text{diag}(b) = \text{diag}(c) - \frac{\gamma}{n} I = C - \frac{\gamma}{n} \dot{I}.$$

So, by Theorem 1,

$$\begin{aligned} W_b(A) &= \{ \text{tr}(UBU^*A) : U \text{ unitary} \} \\ &= \left\{ \text{tr} \left[U \left(C - \frac{\gamma}{n} I \right) U^* A \right] : U \text{ unitary} \right\} = W_c(A) - \left\{ \frac{\gamma}{n} \text{tr}(A) \right\}. \end{aligned}$$

This argument suggests that it is convenient to restrict attention to those vectors c for which $\Sigma \gamma_j = 0$. The limitation merely involves a translation of the ranges by multiples of the trace, or, equivalently, the restriction to matrices of trace 0.

Since W_c is invariant under permutations of the γ_j , we may assume that each vector c in our domain is ordered. Hence, we consider the set of ordered vectors c with $\Sigma \gamma_j = 0$, which form a conical subset \mathcal{C} of an $(n-1)$ -dimensional subspace of \mathbf{R}^n .

We are ready now to study integrals of $W_c(A)$ relative to an arbitrary measure μ on \mathcal{C} , that is integrals of the form

$$(3.1) \quad J_\mu = J_\mu(A) = \int_{\mathcal{C}} W_c(A) d\mu(c).$$

One way of defining the integral in (3.1) is by carrying linear sums, over partitions of \mathcal{C} , to the limit. Alternatively, one realizes that J_μ , being an integral of the convex sets W_c , is a convex set as well. Hence J_μ may be characterized by its support function (e.g., [4] part V),

$$u(J_\mu, \theta) = \sup_{z \in J_\mu} \text{Re}(ze^{-i\theta}), \quad 0 \leq \theta < \pi$$

In order to evaluate $u(J_\mu, \theta)$, we consider the support functions of our closed convex integrands W_c . We have

$$u(W_c, \theta) = u(c, \theta) = \max_{z \in W_c} \text{Re}(ze^{-i\theta}), \quad 0 \leq \theta < \pi.$$

Since $u(c, \theta)$ is a linear function of c in the sense that

$$u(\lambda W_c + \lambda' W_{c'}, \theta) = \lambda u(c, \theta) + \lambda' u(c', \theta), \quad \forall \lambda, \lambda' \geq 0,$$

we have

$$u(J_\mu, \theta) = u\left(\int W_c d\mu(c), \theta\right) = \int u(W_c, \theta) d\mu(c) = \int u(c, \theta) d\mu(c).$$

Of course, the measure μ may be concentrated at a finite number of points $c_1, \dots, c_m \in \mathcal{C}$. In this case the integral J_μ reduces to the finite linear combination

$$\mu(c_1)W_{c_1}(A) + \dots + \mu(c_m)W_{c_m}(A).$$

Since $W_{\lambda c} = \lambda W_c$ for scalar λ , we shall avoid integration over proportional vectors of \mathcal{C} . This can be achieved by restricting integration to the domain

$$\mathcal{D} = \{c: c = (\gamma_1, \dots, \gamma_n), \Sigma \gamma_j = 0, \gamma_1 = 1\},$$

which is the bounded set of all vectors in \mathcal{C} with $\gamma_1 = 1$.

The above concept of integration can be extended in order to consider the integral

$$(3.2) \quad \mathcal{H}_\mu \equiv \int_{\mathcal{D}} \mathcal{H}_c d\mu(c).$$

We recall that the integrands \mathcal{H}_c are convex sets in the $(n^2 - 1)$ real dimensional) space \mathbf{H} of Hermitian matrices of trace 0. It follows that \mathcal{H}_μ is also a convex set in \mathbf{H} . Again, the convexity of \mathcal{H}_c and \mathcal{H}_μ implies that the integral may be defined in terms of the support functions of \mathcal{H}_c . Here, in analogy to the previous case, the support function of \mathcal{H}_c assigns to each point H_1 on the unit sphere of \mathbf{H} , the distance from the origin O of \mathbf{H} to the plane of support of \mathcal{H}_c perpendicular to the direction \vec{OH}_1 .

Having the integrals J_μ and \mathcal{H}_μ defined we state our main result.

THEOREM 3. *Let μ be a nonnegative measure on \mathcal{D} , and let $c' \neq 0$ be an ordered vector with $\Sigma \gamma'_j = 0$. Then*

$$(3.3) \quad \int_{\mathcal{D}} W_c(A) d\mu(c) \subset \lambda W_{c'}(A), \quad \forall A \in \mathbf{C}_{n \times n},$$

if and only if $\lambda \geq \eta(c')$ or $\lambda \leq \zeta(c')$ where

$$(3.4a) \quad \eta(c') = \max_{1 \leq l < n} \int_{\mathcal{D}} \frac{\gamma_1 + \dots + \gamma_l}{\gamma'_1 + \dots + \gamma'_l} d\mu(c),$$

$$(3.4b) \quad \zeta(c') = \min_{1 \leq l < n} \int_{\mathcal{D}} \frac{\gamma_1 + \dots + \gamma_l}{\gamma'_n + \dots + \gamma'_{n-l+1}} d\mu(c).$$

Proof. In the proof of Lemma 8 of [3] we have shown that if $c' \neq 0$ with $\Sigma\gamma'_j = 0$, then

$$(3.5) \quad \gamma'_1 + \cdots + \gamma'_l > 0, \quad \gamma'_n + \cdots + \gamma'_{n-l+1} < 0; \quad l = 1, \dots, n-1.$$

This establishes that η, ζ of (3.4) are well defined and since μ is a nonnegative measure we see that $\eta \geq 0, \zeta \leq 0$.

Next we show that $\lambda \geq \eta(c')$ or $\lambda \leq \zeta(c')$ imply (3.3). For this purpose we use the definition of \mathcal{H}_μ , Theorem 2, and the linearity of the trace to evaluate the set on the left of (3.3):

$$(3.6) \quad \int_{\mathcal{D}} W_c(A) d\mu(c) = \int_{\mathcal{D}} \{\text{tr}(HA): H \in \mathcal{H}_c\} d\mu(c) \\ = \left\{ \text{tr}(HA): H \in \int_{\mathcal{D}} \mathcal{H}_c d\mu(c) \right\} = \{\text{tr}(HA): H \in \mathcal{H}_\mu\}.$$

Now choose λ with $\lambda \geq \eta(c')$. Since $\lambda \geq 0$, the vector $\lambda c'$ remains ordered. Hence, by Theorem 2,

$$(3.7) \quad \lambda W_{c'}(A) = W_{\lambda c'}(A) = \{\text{tr}(HA): H \in \mathcal{H}_{\lambda c'}\}.$$

From (3.6), (3.7) we see that in order to prove (3.3) it suffices to show that

$$(3.8) \quad \mathcal{H}_\mu \subset \mathcal{H}_{\lambda c'}.$$

Thus, let H_0 be a matrix in \mathcal{H}_μ . Then by (3.2), there exist elements $H_c \in \mathcal{H}_c$ for all $c \in \mathcal{D}$, such that

$$H_0 = \int_{\mathcal{D}} H_c d\mu(c).$$

The matrices H_c satisfy Definition 2, and since μ is a nonnegative measure on \mathcal{D} , it follows that for l -tuples x_1, \dots, x_l in A_k we have

$$(3.9) \quad \sum_{j=1}^l (H_0 x_j, x_j) = \int_{\mathcal{D}} \sum_{j=1}^l (H_c x_j, x_j) d\mu(c) \\ \leq \int_{\mathcal{D}} (\gamma_1 + \cdots + \gamma_l) d\mu(c); \quad l = 1, \dots, n,$$

with equality for $l = n$. Since $\Sigma\gamma_j = \Sigma\gamma'_j = 0$, the above equality for $l = n$ implies

$$(3.10a) \quad \sum_{j=1}^n (H_0 x_j, x_j) = 0 = \lambda \sum_{j=1}^n \gamma'_j.$$

For $1 \leq l < n$ we use the assumption $\lambda \geq \eta$ to obtain from (3.9) that

$$(3.10b) \quad \sum_{j=1}^l (H_0 x_j, x_j) \\ \cong (\gamma'_1 + \dots + \gamma'_l) \int_{\mathcal{D}} \frac{\gamma_1 + \dots + \gamma_l}{\gamma'_1 + \dots + \gamma'_l} d\mu(c) \leq \lambda(\gamma'_1 + \dots + \gamma'_l).$$

By Definition 2, the relations (3.10) mean that $H_0 \in \mathcal{H}_{\lambda c'}$. Hence, (3.8) holds, and consequently the inclusion in (3.3) follows.

For $\lambda \leq \zeta$ the situation is slightly different. Consider the vector $c'' \equiv (-\gamma'_n, \dots, -\gamma'_1)$. Since c' is ordered, c'' is too. Also, the condition $\lambda \leq \zeta(c')$ becomes

$$(3.11) \quad -\lambda \geq -\zeta(c'') = -\min_{1 \leq l < n} \int_{\mathcal{D}} \frac{\gamma_1 + \dots + \gamma_l}{\gamma'_n + \dots + \gamma'_{n-l+1}} d\mu(c) \\ = \max_{1 \leq l < n} \int_{\mathcal{D}} \frac{\gamma_1 + \dots + \gamma_l}{-\gamma' - \dots - \gamma'_{n-l+1}} d\mu(c) = \eta(c'').$$

Hence, by the previous part of the proof, we have that

$$(3.12) \quad \int_{\mathcal{D}} W_c(A) d\mu(c) \subset -\lambda W_{c''}(A), \quad \forall A \in \mathbf{C}_{n \times n}.$$

But $-\lambda c''$ is merely a reordering of $\lambda c'$. Thus, the set on the right of (3.12) satisfies

$$-\lambda W_{c''}(A) = W_{-\lambda c''}(A) = W_{\lambda c'}(A) = \lambda W_{c'}(A),$$

and we obtain (3.3).

To complete the proof we have to show that if $\zeta < \lambda < \eta$, then (3.3) does not hold for some $A \in \mathbf{C}_{n \times n}$. First assume $0 \leq \lambda < \eta$. That is, for some l , $1 \leq l < n$,

$$(3.13) \quad \lambda(\gamma'_1 + \dots + \gamma'_l) < \int_{\mathcal{D}} (\gamma_1 + \dots + \gamma_l) d\mu(c).$$

Consider the matrix $A_l = I_l \oplus O_{n-l}$. A simple computation shows that for an ordered vector c , the range $W_c(A_l)$ is a real interval with right end-point $\gamma_1 + \dots + \gamma_l$. Then, the left side of (3.3) represents a real interval with right end-point

$$\int_{\mathcal{D}} (\gamma_1 + \dots + \gamma_l) d\mu(c),$$

which, by (3.13), exceeds the right end-point $\lambda(\gamma'_1 + \dots + \gamma'_l)$ of $W_{\lambda c'}(A_l)$.

Finally, if $\zeta(c') < \lambda < 0$, then (3.11) implies that $0 < -\lambda < \eta(c'')$ where $c'' = (-\gamma'_n, \dots, -\gamma'_1)$. Thus by the above example the inclusion

$$\int_{\mathcal{D}} W_c(A_l) d\mu(c) \subset -\lambda W_{c''}(A_l) = \lambda W_{c'}(A_l)$$

fails to hold, and the theorem follows.

We remember of course, that we restricted integration to the domain \mathcal{D} for convenience only. Therefore, if so desired, $\mu(c)$ can be extended to the domain \mathcal{C} , and Theorem 3 remains valid.

If μ is concentrated at a finite number of vectors $c_1, \dots, c_m \in \mathcal{C}$, then Theorem 3 characterizes all λ for which

$$\sum_{i=1}^m \mu(c_i) W_{c_i}(A) \subset \lambda W_{c'}(A), \quad \forall A \in C_{n \times n}.$$

A result of this type is given in Theorem 1 of [2].

Of particular interest is the case where μ is concentrated at a single vector $c'' \in \mathcal{C}$. That is,

$$\int_{\mathcal{D}} W_c(A) d\mu(c) = W_{c''}(A),$$

and η, ζ of (3.13) are given now by

$$(3.14) \quad \eta(c') = \max_{1 \leq l < n} \frac{\gamma_1'' + \dots + \gamma_l''}{\gamma_1' + \dots + \gamma_l'}; \quad \zeta(c') = \min_{1 \leq l < n} \frac{\gamma_1'' + \dots + \gamma_l''}{\gamma_n' + \dots + \gamma_{n-l+1}'}.$$

Thus, from Theorem 3 we conclude,

COROLLARY. *Let $c' \neq 0$ and c'' be ordered vectors with $\Sigma \gamma_j' = \Sigma \gamma_j'' = 0$. Then*

$$W_{c''}(A) \subset \lambda W_{c'}(A), \quad \forall A \in C_{n \times n}$$

if and only if $\lambda \geq \eta(c')$ or $\lambda \leq \zeta(c')$ where η, ζ are given in (3.14).

This result was proved differently in Theorem 8 of [3].

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V. V. Anh and P. D. Tuan, <i>On starlikeness and convexity of certain analytic functions</i>	1
Willard Ellis Baxter and L. A. Casciotti, <i>Rings with involution and the prime radical</i>	11
Manuel Phillip Berriozabal, Hon-Fei Lai and Dix Hayes Pettey, <i>Noncompact, minimal regular spaces</i>	19
Sun Man Chang, <i>Measures with continuous image law</i>	25
John Benjamin Friedlander, <i>Certain hypotheses concerning L-functions</i>	37
Moshe Goldberg and Ernst Gabor Straus, <i>On characterizations and integrals of generalized numerical ranges</i>	45
Pierre A. Grillet, <i>On subdirectly irreducible commutative semigroups</i>	55
Robert E. Hartwig and Jiang Luh, <i>On finite regular rings</i>	73
Roger Hugh Hunter, Fred Richman and Elbert A. Walker, <i>Finite direct sums of cyclic valuated p-groups</i>	97
Atsushi Inoue, <i>On a class of unbounded operator algebras. III</i>	105
Wells Johnson and Kevin J. Mitchell, <i>Symmetries for sums of the Legendre symbol</i>	117
Jimmie Don Lawson, John Robie Liukkonen and Michael William Mislove, <i>Measure algebras of semilattices with finite breadth</i>	125
Glenn Richard Luecke, <i>A note on spectral continuity and on spectral properties of essentially G_1 operators</i>	141
Takahiko Nakazi, <i>Invariant subspaces of weak-* Dirichlet algebras</i>	151
James William Pendergrass, <i>Calculations of the Schur group</i>	169
Carl Pomerance, <i>On composite n for which $\varphi(n) \mid n - 1$. II</i>	177
Marc Aristide Rieffel and Alfons Van Daele, <i>A bounded operator approach to Tomita-Takesaki theory</i>	187
Daniel Byron Shapiro, <i>Spaces of similarities. IV. (s, t)-families</i>	223
Leon M. Simon, <i>Equations of mean curvature type in 2 independent variables</i>	245
Joseph Nicholas Simone, <i>Metric components of continuous images of ordered compacta</i>	269
William Charles Waterhouse, <i>Pairs of symmetric bilinear forms in characteristic 2</i>	275