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A NOTE ON SPECTRAL CONTINUITY AND ON SPECTRAL PROPERTIES OF ESSENTIALLY G_1 OPERATORS

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A bounded operator on a separable Hilbert space is essentially G_1 if the image of T in the Calkin algebra satisfies condition G_1. This paper contains results describing (1) isolated points of the essential spectrum of essentially G_1 operators, and (2) essentially G_1 operators whose essential spectrum lies on a smooth Jordan curve. Finally, the continuity of the essential spectrum, Weyl spectrum, and spectrum is discussed.

Notation and definitions. Throughout this paper H denotes a separable Hilbert space, $\mathscr{B}(H)$ denotes all bounded operators on H, $\mathscr{K}$ denotes all compact operators in $\mathscr{B}(H)$, $\mathscr{B}(H)/\mathscr{K}$ denotes the Calkin algebra, and $\pi: \mathscr{B}(H) \rightarrow \mathscr{B}(H)/\mathscr{K}$ denotes the quotient map. Since $\mathscr{B}(H)/\mathscr{K}$ is a C^*-algebra, there exists a Hilbert space H_0 and an isometric *-isomorphism ν of $\mathscr{B}(H)/\mathscr{K}$ into $\mathscr{B}(H_0)$ [see 2]. The *essential spectrum* of $T \in \mathscr{B}(H)$, denoted by $\sigma_e(T)$, is the spectrum of $\pi(T)$ in the Calkin algebra. T is *essentially* G_1 if $\|(\pi(T) - z)^{-1}\| = 1/d(z, \sigma_e(T))$ for all $z \notin \sigma_e(T)$. T is *essentially hyponormal, essentially normal, or essentially self-adjoint* if $\pi(T^*T - TT^*) \geqq 0$, $\pi(T^*T - TT^*) = 0$, or $\pi(T^* - T) = 0$, respectively. If λ is an eigenvalue of T then λ is a *normal eigenvalue* if $\{x \in H: Tx = \lambda x\} = \{x \in H: T^*x = \lambda^*x\}$. If λ is an approximate eigenvalue of T then λ is a *normal approximate eigenvalue* if $\|(T - \lambda I)x_n\| \rightarrow 0$ if and only if $\|(T - \lambda I)^*x_n\| \rightarrow 0$, where $\|x_n\|=1$ for all n.

1. Spectral properties of essentially G_1 operators. Recall that an isolated point λ of the spectrum of an operator $T \in \mathscr{B}(\mathscr{K})$ that satisfies condition G_1 (i.e., $\|(T - zI)^{-1}\| = 1/d(z, \sigma(T))$ for all $z \notin \sigma(T)$) must be a normal eigenvalue. What happens when λ is an isolated point of $\sigma_e(T)$ and T is essentially G_1? We first look at the special case when T is a compact operator ($\sigma_e(T) = \{0\}$). The following is a "folk" theorem whose proof is included for completeness.

THEOREM 1. *If T is a compact operator with* $\ker T = \ker T^* = \{0\}$, *then* 0 *is a normal approximate eigenvalue of* T.

Proof. Since T is compact, 0 is in the approximate point spectrum of T. Suppose $\|Tx_n\| \to 0$ and $\|x_n\| = 1$ for all n. If $x_n \to 0$ weakly; then, since T^* is compact, $\|T^*x_n\| \to 0$. If $\{x_n\}$ does not converge to zero weakly; then since the closed unit ball is weakly compact, there exists $0 < \|x\| \leqq 1$ and a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \to x$ weakly. Since T is compact, $Tx_{n_k} \to Tx$ (in norm). But $Tx_{n_k} \to 0$ so $Tx = 0$ which implies $x \in \ker T$. But $\ker T = \{0\}$, so $x = 0$. Contradiction. Therefore $\{x_n\}$ must converge weakly to zero; hence $\|Tx_n\| \to 0$ implies $\|T^*x_n\| \to 0$. By replacing T by T^* in the above argument one obtains that $\|T^*x_n\| \to 0$, $\|x_n\| = 1$ implies $\|Tx_n\| \to 0$.

For an arbitrary essentially G_1 operator with an isolated point of the essential spectrum, we have the following theorem.

THEOREM 2. *Suppose T is essentially G_1 and λ is an isolated point of $\sigma_e(T)$.*

(1) *If $\sigma_e(T) = \{\lambda\}$, then $T - \lambda$ is compact.*

(2) *If $\sigma_e(T)$ contains more than one point, then there exists an operator S such that $T - \lambda \oplus S$ is compact and $\sigma_e(S) = \sigma_e(T) \sim \{\lambda\}$.*

The following two corollaries follow immediately from the proof of Theorem 2. Notice that Corollary 1 does not say that λ is a normal approximate eigenvalue of T but something very close to it.

COROLLARY 1. *If T is essentially G_1 and λ is an isolated point of $\sigma_e(T)$, then there exists an orthonormal sequence $\{x_n\}$ such that $\|(T - \lambda)x_n\| + \|(T - \lambda)^*x_n\| \to 0$.*

COROLLARY 2. *If T is essentially G_1 with $\sigma_e(T) = \{\lambda_1, \lambda_2, \cdots, \lambda_n\}$, then there exists orthogonal projections $P_1, P_2, \cdots, P_n$ of infinite rank such that*

(1) $1 = P_1 + P_2 + \cdots + P_n$,

(2) $P_iP_j = 0$ *for all* $i \neq j$, *and*

(3) $T - (\lambda_1 P_1 + \cdots + \lambda_n P_n)$ *is compact.*

Proof of Theorem 2. Recall that ν is a isometric *-isomorphism of $\mathscr{B}(H)/\mathscr{K}$ into $\mathscr{B}(H_0)$. By Theorem 4.28 of [2], the spectrum of $\pi(T) \in \mathscr{B}(H)/\mathscr{K}$ is equal to the spectrum of $\nu\circ\pi(T) \in \mathscr{B}(H_0)$, i.e., $\sigma_e(T) = \sigma(\nu\circ\pi(T))$. Therefore $\pi(T)$ satisfies conditions G_1 in $\mathscr{B}(H)/\mathscr{K}$ if and only if $\nu\circ\pi(T)$ satisfies condition G_1 in $\mathscr{B}(H_0)$.

If $\sigma_e(T) = \{\lambda\}$, then $\nu\circ\pi(T)$ is a G_1 operator in $\mathscr{B}(H_0)$ with spectrum $\{\lambda\}$. It is well known that this implies $\nu\circ\pi(T)=\lambda$ so that $\pi(T-\lambda)=0$. Hence $T-\lambda$ is compact. This completes part (1) of Theorem 2.

Now assume $\sigma_e(T)$ contains at least two points. Choose $\varepsilon>0$ so that $d(z,\sigma_e(T))=|z-\lambda|$ for all $|z-\lambda|=\varepsilon$. Since $\nu\circ\pi(T)$ is G_1 in $\mathscr{B}(H_0)$, it is well known that λ must be a normal eigenvalue whose eigenspace is the kernel of the orthogonal projection P_0, where

$$P_0=-\frac{1}{2\pi i}\int_{|z-\lambda|=\varepsilon}(\nu\circ\pi(T)-z)^{-1}dz\in\mathscr{B}(H_0)\,.$$

Furthermore $(\nu\circ\pi(T)-\lambda)P_0=P_0(\nu\circ\pi(T)-\lambda)=0$ and $\nu\circ\pi(T)=\lambda\oplus T_0$, where $\sigma(T_0)=\sigma(\nu\circ\pi(T))\sim\{\lambda\}=\sigma_e(T)\sim\{\lambda\}$. Define $P_k=-(1/2\pi i)\int_{|z-\lambda|=\varepsilon}(\pi(T)-z)^{-1}dz\in\mathscr{B}(H)/\mathscr{K}$. Then $\nu(P_k)=P_0$ so that P_k in an orthogonal projection $(P_k^2=P_k=P_k^*)$ in the Calkin algebra and $(\pi(T)-\lambda)P_k=P_k(\pi(T)-\lambda)=0$. Therefore there exists an orthogonal projection $P\in\mathscr{B}(H)$ such that $\pi(P)=P_k$. Thus $(T-\lambda)P$, $P(T-\lambda)$, and $(T-\lambda)^*P$ are compact. Let $M=P(H)$. Then relative to $H=M\oplus M^\perp$, write $P=\begin{pmatrix}1&0\\0&0\end{pmatrix}$ and $T=\begin{pmatrix}A&B\\C&S\end{pmatrix}$. Since

$$(T-\lambda)P=\begin{pmatrix}A-\lambda&B\\C&S-\lambda\end{pmatrix}\begin{pmatrix}1&0\\0&0\end{pmatrix}=\begin{pmatrix}A-\lambda&0\\C&0\end{pmatrix}$$

is compact, $A-\lambda$ and C are compact. Since

$$P(T-\lambda)=\begin{pmatrix}1&0\\0&0\end{pmatrix}\begin{pmatrix}A-\lambda&B\\C&S-\lambda\end{pmatrix}=\begin{pmatrix}A-\lambda&B\\0&0\end{pmatrix}$$

is compact, B is compact. Therefore there exists a compact operator K such that $T=(\lambda\oplus S)+K$. Since $\nu\circ\pi(T)=\lambda\oplus T_0$ where $\sigma(T_0)=\sigma_e(T)\sim\{\lambda\}$, it is easily seen that $\sigma_e(S)=\sigma_e(T)\sim\{\lambda\}$ and the proof of Theorem 2 is complete.

From Corollary 2 of Theorem 2 we see that an essentially G_1 operator T with finite essential spectrum can be written as the sum of a normal operator N and a compact operator such that $\sigma(N)=\sigma_e(N)=\sigma_e(T)$. The following theorem shows that this is not true then we replace the "finite essential spectrum" hypothesis with "$\sigma_e(T)$ is countable".

THEOREM 3. *There exists an essentially G_1 operator T with countable spectrum that is not essentially hyponormal and hence cannot be written as a normal plus compact operator.*

Proof. Let $H = M_1 \oplus M_2 \oplus M_3$ where each M_i has infinite dimension. Let $A \in \mathscr{B}(M_1 \oplus M_2)$ be $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, and let $N \in \mathscr{B}(M_3)$ be a normal operator such that $\sigma_e(N) = \sigma(N)$ is countable and such that $T = A \oplus N$ is G_1 [6, Theorem 5]. Since $\sigma_e(T) = \sigma(T)$ and since T is G_1, we have for all $z \notin \sigma_e(T)$,

$$||(\pi(T) - z^{-1}|| \leqq ||(T - z)^{-1}|| = \frac{1}{d(z, \sigma(T))} = \frac{1}{d(z, \sigma_e(T))} .$$

Since we always have $||(\pi(T) - z)^{-1}|| \geqq 1/d(z, \sigma_e(T))$, we see that T is essentially G_1. However, T is not essentially hyponormal because A is not essentially hyponormal [8].

Let Γ be a C^2-smooth Jordan curve. Stampfli has shown that if T is a G_1 operator with $\sigma(T) \subseteqq \Gamma$, then T is a normal operator [10, Theorem 2]. Suppose T is essentially G_1 with $\sigma_e(T) \subseteqq \Gamma$. Let ν be the isometric embedding of $\mathscr{B}(H)/\mathscr{K}$ into $\mathscr{B}(H)$. Since $\pi(T)$ is G_1 in $\mathscr{B}(H)/\mathscr{K}$, $\nu \circ \pi(T)$ is G_1 in $\mathscr{B}(H_0)$ and $\sigma_e(T) = \sigma(\nu \circ \pi(T))$. Since $\sigma(\nu \circ \pi(T)) = \sigma_e(T) \subseteqq \Gamma$ and $\nu \circ \pi(T)$ is G_1, $\nu \circ \pi(T)$ is normal [11, theorem 2] and hence T is essentially normal. If $\sigma_e(T) = \Gamma$, then T is not necessarily the sum of a normal operator and a compact operator (for example, let T be a unilateral shift of finite multiplicity). However, if $\sigma_e(T) \neq \Gamma$ then, since T is essentially normal, we may apply a result of Brown-Douglas-Fillmore [1, p. 62] to obtain the following.

REMARK 1. If T is essentially G_1 with $\sigma_e(T) \subset \Gamma$ and $\sigma_e(T) \neq \Gamma$, then $T = N + K$ where K is compact and N is normal with $\sigma(N) = \sigma_e(N) = \sigma_e(T)$.

The previous remark is actually true with a weaker hypothesis on T, since to apply Stampfli's result [11, Theorem 2] it is only necessary to have the operator satisfy growth condition G_1 in a neighborhood of its spectrum.

2. Continuity of $\sigma_e(T)$, $\sigma_w(T)$, and $\sigma(T)$. Recall that $\sigma_w(T)$ is the Weyl spectrum and is defined to be $\bigcap_{K \in \mathscr{K}} \sigma(T + K) = \{\lambda : T - \lambda$ is not Fredholm of index zero$\}$. T is Fredholm if T has closed range, finite nullity, and finite co-rank. The index of T is equal to the dimension of the kernel of T minus the dimension ofthe kernel T^*. It is well-known that $\sigma(T)$ is not a continuous function of T [4, Problem 85]; it is also known [7] that $\sigma(T)$ is a continuous function of T if T is restricted to the class of all G_1 operators on H. What happens if T is restricted to the class of all essentially

G_1 operators, or to all essentially convexoid operators (T is essentially convexoid if the convex hull of the essential spectrum is equal to the essential numerical range of T [see 8])? In search for examples, one immediately thinks of obtaining an easy counter-example by taking $T_n \to T$, where each T_n is compact and $\sigma(T_n)$ does not approach $\sigma(T)$ (in the Hausdorff topology [4, p. 53]). However, this cannot be done because of the following theorem of J. D. Newburgh [9, Theorem 3]. Recall that the spectrum of a compact operatoris countable with the origin the only possible point of accumulation. The proof of Newburgh's theorem given here differs from Newburgh's proof in that it does not use spectral sets.

THEOREM 4 (Newburgh). *If $\sigma(T)$ is totally disconnected and if $T_n \to T$, then $\sigma(T_n) \to \sigma(T)$.*

Proof. From [4, Problem 86], $\sigma(T)$ is upper semicontinuous, i.e. for each $\varepsilon > 0$ there exists N such that for all $n \geqq N$, $\sigma(T_n) \subseteqq \sigma(T) + (\varepsilon) \equiv \{z + w: z \in \sigma(T),\ |w| < \varepsilon\}$. Therefore, we need only show that for each $\varepsilon > 0$ there exists N such that for all $n \geqq N$, $\sigma(T) \subseteqq \sigma(T_n) + (\varepsilon)$. If this were not the case, then we may assume that there exists $\varepsilon > 0$ and there exists a sequence $\{z_n\} \subseteqq \sigma(T)$ such that $d(z_n, \sigma(T_n)) \geqq \varepsilon$ for all n. Since $\sigma(T)$ is compact we may also assume $z_n \to z \in \sigma(T)$. If $|z_n - z| < \varepsilon/2$, then

$$d(z, \sigma(T_n)) \geqq d(z_n, \sigma(T_n)) - |z - z_n| \geqq \varepsilon - \varepsilon/2 = \varepsilon/2 .$$

Thus, for all n sufficiently large $\sigma(T_n)$ is no closer than $\varepsilon/2$ to $z \in \sigma(T)$. Since $\sigma(T)$ is compact and totally disconnected, we may apply a theorem of Zoretti [13, Theorem 3.11, p. 109] to conclude that there exists a simple closed (rectifiable) curve γ such that

(1) $\gamma \cap \sigma(T) = \varnothing$

(2) γ lies within a disc about z of radius $\varepsilon/4$.

(3) γ separates $\sigma(T)$ into two parts (inside γ and outside γ), each a positive distance from γ, and

(4) z lies inside γ.

Define $P = -(1/2\pi i)\int_\gamma (T - \lambda)^{-1}d\lambda$. Then $P \neq 0$ since the inside of γ contains part of $\sigma(T)$. Let $P_n = -(1/2\pi i)\int_\gamma (T_n - \lambda)^{-1}d\lambda$. Then, for all n sufficiently large, $P_n = 0$ because the inside of γ contains no points of $\sigma(T_n)$. By a standard argument, we obtain that $(T_n - \lambda)^{-1} \to (T - \lambda)^{-1}$ uniformly on $\lambda \in \gamma$. Hence

$$0 = \lim_{n\to\infty} P_n = \lim_{u\to\infty} -\frac{1}{2\pi i}\int (T_n - \lambda)^{-1}d\lambda .$$
$$= -\frac{1}{2\pi i}\int (T - \lambda)^{-1}d\lambda = P \neq 0 .$$

Contradiction. Therefore $\sigma(T_n) \to \sigma(T)$ and the proof is complete.

Even though the spectrum is a continuous function of T when T is compact and when T is G_1, we have the following

REMARK 2. $\sigma(T)$ is not a continuous function of T for T essentially (G_1) (or even for T essentially normal).

Proof. Let B be the bilateral shift (a normal operator) on the orthonormal basis e_n, $n = 0, \pm 1, \pm 2, \cdots$ and let P be the rank one operator that maps e_0 to e_1 and e_n to 0 for all $n \neq 0$. Then $B - ((n-1)/n)P \to B - P$ and by [4, Problem 85] $\sigma(B - ((n-1)/n)P)$ is the unit circle for all n and $\sigma(B - P)$ is the closed unit disc.

THEOREM 5. *The essential spectrum is not a continuous function of T. However $\sigma_e(T)$ is a continuous function of T for T essentially G_1.*

Proof. To see that $\sigma_e(T)$ is not a continuous function of T, we can generalize the example found in problem 87 of [4] as follows. Let $H = M_1 \oplus M_2 \oplus M_3 \oplus \cdots$ where each M_i has infinite dimension. Relative to this decomposition of H define a "generalized" unilateral weighted shift W with weights $a_1, a_2, a_3, \cdots$ as follows:

$$W = \begin{bmatrix} 0 & 0 & 0 & 0 & \\ a_1 & 0 & 0 & 0 & \\ 0 & a_2 & 0 & 0 & \\ 0 & 0 & a_3 & 0 & \\ & & & & \ddots \end{bmatrix}.$$

Now choose the a_n's exactly as in the solution to Problem 86 [4, p 248] to obtain nilpotent operators $T_n \to W$ where the essential spectral radius of W is strictly positive. Clearly $\sigma_e(T_n) = \sigma(T_n) = (0)$.

To see that $\sigma_e(T)$ is continuous when T is essentially G_1, let $T_n \to T$ where each T_n is essentially G_1. Since the essentially G_1 operators is a closed set in $\mathscr{B}(H)$ [8, Theorem 12], T is also essentially G_1. By previous remarks we have $\nu\circ\pi(T_n) \to \nu\circ\pi(T)$ in $\mathscr{B}(H_0)$ and $\nu\circ\pi(T_n)$ is G_1. Since the spectrum is a continuous function of T when T is G_1 [7], $\sigma(\nu\circ\pi(T_n)) \to \sigma(\nu\circ\pi(T))$. But $\sigma_e(T_n) = \sigma(\nu\circ\pi(T_n))$ and $\sigma_e(T) = \sigma(\nu\circ\pi(T))$. Thus $\sigma_e(T_n) \to \sigma_e(T)$ and the proof is complete.

Notice that in the above theorem we have actually shown that if T_n and T are essentially G_1 and if $\pi(T_n) \to \pi(T)$, then $\sigma_e(T_n) \to \sigma_e(T)$.

Recall that $T \in \mathscr{B}(H)$ is *essentially convexoid* if the convex hull of the essential spectrum equals the essential numerical range of T, $co\, \sigma_e(T) = W_e(T)$, [see 8, 10]. By [8] we know that the set of essentially convexoid operators is a larger class of operators than the essentially G_1 operators. One might guess that $\sigma_e(T)$ is continuous for T essentially convexoid; however, this is not true. To see this let $A_n \to A$ such that $\|A_n\| \leqq 1$ for all n, $\sigma_e(A_n) = (0)$, and $\sup_{z \in \sigma_e(A)} |z| > 0$ (see the proof of Theorem 5 for the construction of A_n and A). Let N be a normal operator such that $\sigma(N) = \sigma_e(N) = \{z: |z| = 2\}$. Then $co\, \sigma_e(N) \supsetneqq W_e(A_n)$ for all n, so that $T_n \equiv A_n \oplus N$ is essentially convexoid [see 8] and $\sigma_e(T_n) = \{0\} \cup \{z: |z| = 2\}$. Let $T = A \oplus N$. Then $\sigma_e(T) = \sigma_e(A) \cup \sigma_e(N)$ and hence $\sigma_e(T_n)$ cannot approach $\sigma_e(T)$, since $1 \geqq \sup_{z \in \sigma_e(A)} |z| > 0$.

We now look at the continuity of the Weyl spectrum. Before proceeding we need the following proposition.

PROPOSITION 1. *In general* $\sigma_w(A \oplus B) \subseteqq \sigma_w(A) \cup \sigma_w(B)$. *If* B *is normal, then* $\sigma_w(A \oplus B) = \sigma_w(A) \cup \sigma_w(B)$.

Proof. If A and B are Fredholm of index zero, then $A \oplus B$ is Fredholm of index zero. This implies $\sigma_w(A \oplus B) \subseteqq \sigma_w(A) \cup \sigma_w(B)$.

Now suppose B is normal and $A \oplus B$ is Fredholm of index zero. Clearly A and B are Fredholm. Since B is normal, the Fredholm index of B, $i(B)$, is zero. Since $0 = i(A \oplus B) = i(A) + i(B) = i(A)$, A also has Fredholm index equal to zero. Therefore A and B both are Fredholm of index 0, so that $\sigma_w(A \oplus B) = \sigma_w(A) \cup \sigma_w(B)$. This completes the proof.

Let $T_n = A_n \oplus N \to T = A \oplus N$ be as in the example just before Proposition 1. Each T_n is essentially convexoid and, by Proposition 1, $\sigma_w(T_n) = \sigma_w(A_n) \cup \sigma_w(N)$. Since each A_n is nilpotent $\sigma(A_n) = (0)$ so that $\sigma_w(A_n) = 0$. Therefore $\sigma_w(T_n) = \{0\} \cup \{z: |z| = 2\}$. Since $\sup_{z \in \sigma_e(A)} |z| > 0$ and since $\sigma_e(A) \subseteqq \sigma_w(A)$, $\sup_{z \in \sigma_w(A)} |z| > 0$. Therefore $\sigma_w(T_n)$ does not converge to $\sigma_w(T)$. This completes the first part of the proof of the following theorem.

THEOREM 6. *The Weyl spectrum,* $\sigma_w(T)$, *is not a continuous function of* T. *However, if* $T_n \to T$ *and each* T_n *is essentially* G_1, *then* $\sigma_w(T_n) \to \sigma_w(T)$.

Proof. First we show that $\sigma_w(T)$ is an upper semicontinuous function of T. Let $T_n \to T$. By a result of Stampfli [12], there exists a compact operator K such that $\sigma_w(T) = \sigma(T + K)$. By the upper semicontinuity of the spectrum [4, Problem 86], there exists $\delta > 0$ such that whenever $\|A - (T + K)\| < \delta$, then $\sigma(A) \subseteqq \sigma(T+K) + (\varepsilon)$. If S_0 is a set in the complex plane, then $S_0 + (\varepsilon)$ denotes $\{z+w: z \in S_0 \text{ and } |w| < \varepsilon\}$. If $\|S - T\| < \delta$, then $\|(S + K) - (T+K)\| = \|S - T\| < \delta$, so that $\sigma_w(S) \subseteqq \sigma(S + K) \subseteqq \sigma(T + K) + (\varepsilon) = \sigma_w(T) + (\varepsilon)$. Therefore $\sigma_w(T)$ is an upper semicontinuous fnnction of T.

Suppose $T_n \to T$, T_n essentially G_1. Since $\sigma_w(T)$ is upper semicontinuous, we need only show lower semicontinuity. If it is not lower semicontinuous then there exists $\varepsilon > 0$ such that $\sigma_w(T) \nsubseteq \sigma_w(T_n) + (2\varepsilon)$ for an infinite number of n's. Without loss of generality assume that this holds for all n. Then there exists $z_n \in \sigma_w(T)$ such that $z_n \notin \sigma_w(T_n) + (2\varepsilon)$. Since $\sigma_w(T)$ is compact, we may assume, $z_n \to z \in \sigma_w(T)$. Then whenever $|z_n - z| < \varepsilon$, we have

$$\begin{aligned} d(z, \sigma_w(T_n)) &\geqq d(z_n, \sigma_w(T_n)) - |z - z_n| \\ &> d(z_n, \sigma_w(T_n)) - \varepsilon \geqq 2\varepsilon - \varepsilon = \varepsilon\,. \end{aligned}$$

Since the T_n's are essentially G_1, $\sigma_e(T_n) \to \sigma_e(T)$. Since $\sigma_e(T_n) \to \sigma_e(T)$ and $d(z, \sigma_e(T_n)) \geqq d(z, \sigma_w(T_n)) \geqq \varepsilon$ for n sufficiently large, $z \notin \sigma_e(T)$. Therefore $T - z$ is Fredholm and since $z \in \sigma_w(T)$, the index of $T - z$ is not zero. Since $z \notin \sigma_w(T_n)$ for n sufficiently large, $T_n - z$ is Fredholm of index zero. The Fredholm index is a continuous function on the Fredholm operators [2, Theorem 5.36] so $T_n - z \to T - z$ implies $T - z$ has Fredholm index equal to zero.

Contradiction. Therefore $\sigma_w(T)$ is continuous on the essentially G_1 operators.

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