UNIMODALITY OF THE LÉVY SPECTRAL FUNCTION

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A. Ya. Khinchin proved that if $\Phi$ and $\Psi$ are characteristic functions and $\Phi(t) = t^{-1} \int_{0}^{t} \Psi(u)du$, then the distribution function of $\Phi$ is convex on $(-\infty, 0)$ and concave on $(0, +\infty)$. A similar theorem is proved here for logarithms of infinitely divisible characteristic functions and their Lévy spectral functions.

Suppose $\Phi(t)$ is a characteristic function (ch. f) of a distribution function (df), $F$, so that $\Phi(t) = \int_{\mathbb{R}} e^{ixt}dF(x)$. An application of Bochner's theorem (see [2]) shows that $\tilde{\Phi}(t) = t^{-1} \int_{0}^{t} \Phi(u)du$ is also a ch. f. Khinchin proved that $\tilde{\Phi}$ is a ch. f by constructing its df. In fact, he showed that a ch. f is of the form $\tilde{\Phi}$ if and only if its df is unimodal at 0; that is, the df is convex on $(-\infty, 0)$ and concave on $(0, +\infty)$. We shall prove a "unimodal theorem" for the function $\tilde{\phi}(t) = t^{-1} \int_{0}^{t} \phi(u)du$ under the assumptions that $\Phi(t)$ is infinitely divisible and $\phi(t) = \ln \Phi(t)$. Johansen’s characterization of infinitely divisible ch. fs. ([1], Theorem 2) insures that $\phi$, defined above, may also be written $\phi(t) = \ln \Psi(t)$, for some infinitely divisible ch. f $\Psi$, and hence provided the motivation for our work. To begin with, we state Lévy’s form of infinitely divisible ch. fs. (See [2].)

**Theorem 1.** A ch. f $\Phi$ is infinitely divisible if and only if $\phi(t) = \ln \Phi(t)$ may be uniquely represented as

$$
\phi(t) = i\mu t - \sigma^2 t^2 + f_{\mathbb{R}} \left( e^{i\pi t} - 1 - \frac{i\pi t}{1 + \pi^2} \right) dM(x)
$$

where $\mu \in \mathbb{R}$, $\sigma^2 \geq 0$, and the function $M$ has the following properties:

(i) $M$ is defined on $\mathbb{R}\{0\}$

(ii) $M$ is nondecreasing on $(-\infty, 0)$ and on $(0, +\infty)$ and is right continuous

(iii) $M(-\infty) = 0 = M(+\infty)$

(iv) $\int_{(-\varepsilon, \varepsilon)} x^2 dM(x)$ is finite for all $\varepsilon > 0$.

When (1) is in force, $M$ and $(\mu, \sigma^2, M)$ are respectively called the Lévy spectral function and the Lévy triple of $\Phi$. Moreover, every function which satisfies (i)–(iv) is a Lévy spectral function of
The main result of this article is Theorem 2 below; two preliminary lemmas are proven first.

**Lemma 1.** For every Lévy spectral function, $M$, the following relations hold:

(i) \[ \lim_{x \to -\infty} x \int_{-\infty}^{x} \frac{dM(z)}{z} = 0 = \lim_{x \to \infty} x \int_{-\infty}^{x} \frac{dM(z)}{z} \]

(ii) \[ \lim_{x \to 0^+} x^3 \int_{x}^{+\infty} \frac{dM(z)}{z} = 0 = \lim_{x \to 0^-} x^3 \int_{-\infty}^{x} \frac{dM(z)}{z} . \]

**Proof.** It is known that to each Lévy spectral function, $M$, there exists a df, $G$, and nonnegative number $c$ such that

\[
M(x) = \begin{cases} 
  c \int_{-\infty}^{x} u^{-2}(1 + u^2)dG(u) & \text{if } x < 0 \\
  -c \int_{x}^{+\infty} u^{-2}(1 + u^2)dG(u) & \text{if } x > 0 .
\end{cases}
\]

Then, according as $x > 1$ or $0 < x < 1$, we have $x \int_{-\infty}^{x} u^{-1}dM(u) \leq 2cx \int_{-\infty}^{x} u^{-1}dG(u)$ or $x^3 \int_{-\infty}^{x} u^{-1}dM(u) \leq 2cx \int_{-\infty}^{x} u^{-1}dG(u)$. Similar statements hold for negative $x$. Now, if we apply Lemma 4.5.1 of [2] to the integrals involving $G$, the assertions of Lemma 1 follow at once.

**Lemma 2.** Let $M_1$ and $M_2$ be two Lévy spectral functions and assume they are related by

\[
M_2(x) = \begin{cases} 
  -\int_{-\infty}^{x} \int_{-\infty}^{y} \frac{dM_1(z)}{z} dy & \text{if } x < 0 \\
  -\int_{x}^{+\infty} \int_{y}^{+\infty} \frac{dM_1(z)}{z} dy & \text{if } x > 0 .
\end{cases}
\]

Suppose $\phi(t) = i\mu t - \sigma^2 t^2 + \int_{R} (e^{itz} - 1 - ixt/(1 + x^2))dM_1(x)$ where $\mu \in R$, $\sigma^2 \geq 0$. Then

\[
t^{-1} \int_{0}^{t} \phi(u)du = \int_{R} \left( e^{itz} - 1 - \frac{iuxt}{1 + x^2} \right) dM_1(x) .
\]

**Proof.** Let $T > 0$ be fixed and define $K(u, x) = e^{itu} - 1 - iux/(1 + x^2)$. Then $K(u, x) = O(x^2)$ as $x \to 0$ uniformly for $|u| \leq T$. Let $\eta > 0$. Then
\[
\begin{align*}
& t^{-1} \int_0^t du \lim_{\tau \to 0^+} \int_{-\infty}^{\infty} K(u, x) dM_1(x) = t^{-1} \int_0^t du O\left( \int_{0^+}^{\infty} x^2 dM_1(x) \right) \\
& \quad + t^{-1} \int_{\tau}^{\infty} \int_{0^+}^{t} K(u, x) dudM_1(x) = O\left( \int_{0^+}^{\infty} x^2 dM_1(x) \right) + \int_{\tau}^{\infty} L(t, x) \frac{dM_1(x)}{x}
\end{align*}
\]

where

\[
L(t, x) = \frac{e^{itx} - 1}{it} - x - \frac{itx^2}{2(1 + x^2)}.
\]

Letting $\eta \to 0^+$, we have that

\[
t^{-1} \int_0^t \int_{0^+}^{\infty} K(u, x) dM_1(x) du = \int_{0^+}^{\infty} L(t, x) \frac{dM_1(x)}{x}.
\]

A similar statement for the negative axis shows that

\[
t^{-1} \int_0^t \phi(u) du = (i\mu t/2) - (\sigma t^2/3)
\]

(4)

\[
+ \int_{R} \left( \frac{e^{itx} - 1}{it} - x - \frac{itx^2}{2(1 + x^2)} \right) \frac{dM_1(x)}{x}.
\]

Now apply integration by parts to the integral in (4), to conclude that

\[
t^{-1} \int_0^t \phi(u) du = (i\mu t/2) - (\sigma t^2/3) + \lim_{\tau \to 0^+} \left[ -L(t, x) \int_{x}^{\infty} z^{-\tau} dM_1(z) \right]_{z=-\tau}^{z=\infty}
\]

\[
+ \int_{x}^{\infty} \frac{\partial L(t, x)}{\partial x} \int_{x}^{\infty} z^{-\tau} dM_1(z) dz + L(t, x) \int_{-\infty}^{x} z^{-\tau} dM_1(z) \bigg|_{z=-\infty}^{z=x}
\]

\[
+ \int_{-\infty}^{x} \frac{\partial L(t, x)}{\partial x} \int_{-\infty}^{x} z^{-\tau} dM_1(z) dz
\]

\[
= (i\mu t/2) - (\sigma t^2/3) + \int_{R} K(t, x) dM_1(x)
\]

\[
+ it \int_{R} \frac{x^3}{(1 + x^2)^2} dM_1(x).
\]

The last equality follows by observing that $L(t, x)/x^3$ is bounded for $|t| \leq T$ as $x \to 0$ and using Lemma 1. This completes the proof of Lemma 2.

**Theorem 2.** A necessary and sufficient condition for $\phi(t)$ to be the logarithm of an infinitely divisible ch.f whose Lévy spectral function is convex on $(-\infty, 0)$ and concave on $(0, +\infty)$ is that $\phi(t)$ may be written $\phi(t) = t^{-1} \int_0^t \psi(u) du$, where $\psi$ is the logarithm of a certain infinitely divisible ch.f.

**Proof.** Suppose $\phi(t) = t^{-1} \int_0^t \psi(u) du$ where $\psi$ and $\phi$ are as in the
statement of the theorem and let \( M_1 \) and \( M_2 \) be the Lévy spectral functions of \( \psi \) and \( \phi \) respectively. Since the Lévy representation is unique, Lemma 2 shows that \( M_1 \) and \( M_2 \) are related by (3). Clearly \( M_2 \) is convex on \((-\infty, 0)\) and concave on \((0, +\infty)\) and so the sufficiency of the condition holds.

Conversely suppose a Lévy spectral function \( M_2 \) is given and assume further that \( M_2 \) is unimodal at 0. Then we can write

\[
M_2(x) = \begin{cases} 
\int_{-\infty}^{x} p(u)du & \text{if } x < 0 \\
-\int_{x}^{\infty} p(u)du & \text{if } x > 0
\end{cases}
\]

where \( p \geq 0 \) and is nondecreasing on \((-\infty, 0)\) and nonincreasing on \((0, +\infty)\). Define \( M_i(x) = -\int_{-\infty}^{x} ud\rho(u) \) if \( x < 0 \) and \( M_i(x) = \int_{x}^{\infty} ud\rho(u) \) if \( x > 0 \). Then \( M_i \) is also a Lévy spectral function and

\[
M_i(x) = \int_{-\infty}^{x} \int_{-\infty}^{y} d\rho(z)dy = -\int_{-\infty}^{x} \int_{-\infty}^{y} z^{-1}dM_i(z)dy
\]

if \( x < 0 \), and similarly, \( M_i(x) = -\int_{x}^{\infty} \int_{x}^{y} z^{-1}dM_i(z)dy \) if \( x > 0 \). This shows that \( M_1 \) and \( M_2 \) are related by (3). So if \( \phi \) has the Lévy triple \((\mu, \sigma^2, M_2)\), define

\[
\psi(t) = it\left(2\mu - 2\int_{+\infty}^{0} \frac{x^3}{(1+x^2)^2}dM_i(x)\right) - 3\sigma^2t^2
\]

\[
+ \int_{+\infty}^{0} e^{itx} - 1 - \frac{itx}{1+x^2}dM_i(x)
\]

By Lemma 2, \( \phi(t) = t^{-1}\int_{0}^{t} \psi(u)du \), and hence, the proof of Theorem 2.

Some applications and consequences of Theorem 2 will be given.

(a) Suppose that a Lévy spectral function, \( M \), and a df, \( G \), are related by (2) for some \( c \geq 0 \). From (2), it is clear that the \((0)\)-unimodality of \( G \) entails that of \( M \). The converse is not true; a counterexample is provided by the function \( M(x) = c_1|x|^{-\alpha} \) or \( c_2x^{-\alpha} \) according as \( x < 0 \) or \( x > 0 \), where \( c_1, c_2 > 0 \) and \( 0 < \alpha < 1 \).

(b) Medgyessy ([3], Theorem 2.1) proved that if \( M \) is symmetric and convex on \((-\infty, 0)\), then the original df is unimodal at 0. Hence, combining our result with Khinchin’s theorem on unimodality, one obtains that if \( \Phi(t) \) is an infinitely divisible real ch.f and \( \ln \Phi(t) = t^{-1}\int_{0}^{t} \ln \Psi(u)du \) for some infinitely divisible ch.f \( \Psi \), then \( \Phi(t) = t^{-1}\int_{0}^{t} \chi(u)du \) for some ch.f \( \chi(u) \).

(c) Suppose \( \phi(t) = i\mu t - b|t|^{\alpha}(1 + (i\beta|t|)\omega(|t|, \alpha)) \) corresponds
to a stable law of index \( \alpha \). (See [2], p. 136.) In this case

\[
\phi(t) = i\gamma t + c \tilde{\phi}(t)
\]

where \( \gamma \in \mathbb{R}, \ c \geq 0, \) and \( \tilde{\phi}(t) = t^{-1} \int_0^t \phi(u)du \). Conversely suppose

\[
\phi(t) = \ln \Phi(t) \text{ for some infinitely divisible ch. f } \Phi \text{ and for some } \gamma \in \mathbb{R}, \ c \geq 0, \ (5) \text{ holds.}
\]

Let \((\mu, \sigma^2, M)\) be the Lévy triple of \( \Phi \). If \( M = 0 \), then \( \Phi \) is a normal ch. f and \( c = 3 \). Assume \( M \) is not identically zero. By Theorem 2, \( M \) is convex on \((-\infty, 0)\) and concave on \((0, +\infty)\), and so there exists a nonnegative function \( p(x) \) such that \( p \) is nondecreasing on \((-\infty, 0)\), nonincreasing on \((0, +\infty)\), and such that

\[
M(x) = \begin{cases} 
\int_{-\infty}^x p(u)du & \text{if } x < 0 \\
-\int_{-\infty}^x p(u)du & \text{if } x < 0 
\end{cases}
\]

Since the Lévy representation is unique, if (5) holds, the Lévy spectral functions of \( \phi \) and \( c \tilde{\phi} \) agree. Hence \( M \) satisfies the identity

\[
M(x) = \begin{cases} 
-c \int_{-\infty}^x z^{-1} dM(z)dy & \text{if } x < 0 \\
-c \int_{-\infty}^x z^{-1} dM(z)dy & \text{if } x > 0 
\end{cases}
\]

In terms of \( p \), (6) reduces to

\[
p(x) = \begin{cases} 
-c \int_{-\infty}^x u^{-1} p(u)du & \text{if } x < 0 \\
\int_{-\infty}^x u^{-1} p(u)du & \text{if } x > 0 
\end{cases}
\]

Employing the uniqueness theorem for first order differential equations, it follows that \( p(x) = p(-1)|x|^{-c} \) if \( x < 0 \) or \( p(1)x^{-c} \) if \( x > 0 \). But since \( \int_{x \in (-1, 1)} p(x) dx \) and \( \int_{x \in (-1, 1)} x^c p(x) dx \) are both finite, we must have that \( 1 < c < 3 \). This, in turn, forces \( \sigma^2 = 0 \). Combining this and the form of the Lévy spectral function for stable distributions, we see that (5) characterizes the stable laws.

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