

# Pacific Journal of Mathematics

**PRODUCTS OF COMPACT SPACES WITH BI- $k$  AND RELATED  
SPACES**

ANDREW J. BERNER

## PRODUCTS OF COMPACT SPACES WITH $bi-k$ AND RELATED SPACES

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**The main theorem of this paper characterizes  $bi-k$  spaces as those spaces whose product with every compact space is sequentially  $k$ .**

1. **Introduction.** The classes of  $bi-k$  spaces, countably  $bi-k$  spaces and singly  $bi-k$  spaces were studied in [5], and the class of sequentially  $k$  spaces was introduced in [3]. The following implications hold among these spaces, without the assumption of any separation axioms:  $bi-k \rightarrow$  countably  $bi-k \rightarrow$  singly  $bi-k \rightarrow$  sequentially  $k$ . Also, all  $k$  spaces are sequentially  $k$ , and all Hausdorff sequentially  $k$  spaces are  $k$  spaces. (These classes will be defined at the end of this introduction.)

**THEOREM 1.1.** *The following are equivalent:*

- (a)  $X$  is a  $bi-k$  space.
- (b)  $X \times Y$  is a singly  $bi-k$  space for every compact Hausdorff space  $Y$ .
- (c)  $X \times Y$  is sequentially  $k$  for every compact space  $Y$ .

This theorem is proved in §2.

**REMARK 1.2.** Cohen [4] proved that the product of a  $k$  space with a (locally) compact Hausdorff space is a  $k$  space. Noble [6] showed this is false without the Hausdorff assumption, but in Noble's example, the product was a  $bi-k$  space. Theorem 1.1 ( $c \leftrightarrow a$ ) shows that the product of a  $k$  space with a compact space need not even be sequentially  $k$ .

**REMARK 1.3.** Michael [5] has asked whether the product of two countably  $bi-k$  spaces must be countably  $bi-k$ . Examples have been given showing it is consistent with Zermelo-Fraenkel set theory that this is false. Theorem 1.1 can be used to give an absolute counterexample. All we need is a countably  $bi-k$  space that is not  $bi-k$ . Let  $X$  be the subspace of the product of uncountably many copies of  $\{0, 1\}$  consisting of points that are 1 on only countably many coordinates (i.e., a  $\Sigma$ -product centered at the point all of whose coordinates are 0). Arhangel'skii proved that this space is countably  $bi-k$  (in fact, countably  $bi$ -sequential) but not  $bi-k$  [2]. Thus, by Theorem 1.1, there is a compact Hausdorff space  $Y_1$  and a

compact  $T_1$  space  $Y_2$  such that  $X \times Y_1$  is not singly  $bi - k$ , thus not countably  $bi - k$ , and  $X \times Y_2$  is not even sequentially  $k$ .

**DEFINITION 1.4.** [3] If  $S$  is a subset of a topological space  $X$  and  $(S_i)$  is a nested sequence of subsets of  $X$ , then  $(S_i)$  is an  $S$ -sequence if whenever  $(s_i)$  is a sequence of points with  $s_i \in S_i$  for each  $i$ , then  $(s_i)$  has an accumulation point in  $S$ .

**DEFINITION 1.5.** A space  $X$  is a  $bi - k$  space if whenever  $\mathcal{S}$  is a filter base containing the open sets around a point  $p \in X$ , there is a compact set  $S \subset X$  and a nested sequence of sets  $(S_i)$  such that  $(S_i)$  meshes with  $\mathcal{S}$  and  $(S_i)$  is an  $S$ -sequence.

**DEFINITION 1.6.** A space  $X$  is a countably  $bi - k$  space if whenever  $(F_i)$  is a nested sequence of sets accumulating at a point  $p$  (i.e.,  $p \in \text{cl}(F_i)$  for each  $i$ ) there is a nested sequence of sets  $(S_i)$  accumulating at  $p$  and a compact set  $S$  such that  $S_i \subset F_i$  for each  $i$  and  $(S_i)$  is an  $S$ -sequence.

**DEFINITION 1.7.** A space  $X$  is a singly  $bi - k$  space if whenever  $p \in \text{cl}(F)$ , there is a compact set  $S$  and a nested sequence of sets  $(S_i)$  accumulating at  $p$  such that  $S_i \subset F$  for each  $i$  and  $(S_i)$  is an  $S$ -sequence.

**DEFINITION 1.8.** A space  $X$  is sequentially  $k$  if whenever a set  $F$  is not closed there is a point  $p \in \text{cl}(F) - F$ , a compact set  $S$  and a nested sequence of sets  $(S_i)$  accumulating at  $p$  such that  $S_i \subset F$  for each  $i$  and  $(S_i)$  is an  $S$ -sequence.

**2. Proof of Theorem 1.1.** In [2], Michael proved that a space  $X$  is countably  $bi - k$  if and only if  $X \times I$  is singly  $bi - k$  (where  $I$  is the unit interval). The heart of the proof (in one direction) involves coding a bad nested sequence  $(S_i)$  of subsets of  $X$  (i.e., a witnessing sequence to the statement ' $X$  is not countably  $bi - k$ ') as a single bad subset of  $X \times I$ . This idea of coding is hinted at in the following proof of Theorem 1.1.

$$a \longrightarrow b \quad \text{and} \quad a \longrightarrow c:$$

The product of two (or even countably many)  $bi - k$  spaces is  $bi - k$  [5]. Since compact spaces are  $bi - k$ , the product of a  $bi - k$  space and a compact space is  $bi - k$ , and thus singly  $bi - k$  and sequentially  $k$ .

$$\text{not } a \longrightarrow \text{not } b \quad \text{and} \quad \text{not } a \longrightarrow \text{not } c:$$

Both implications make use of the following construction.

Suppose  $X$  is not  $bi - k$ . Then there is a point  $p \in X$  and a filter base  $\mathcal{F}$  of subsets of  $X$  such that  $\mathcal{F}$  contains the open sets around  $p$ , but there is no compact  $S \subset X$  and nested sequence of sets  $(S_i)$  such that  $(S_i)$  meshes with  $\mathcal{F}$  and is an  $S$ -sequence. Thus, in particular, there is an  $F \in \mathcal{F}$  such that  $p \notin F$  and therefore if  $F_1 \in \mathcal{F}$  and  $F_2 \in \mathcal{F}$ , then  $F_1 \cap F_2 - \{p\} \neq \emptyset$ .

Define a base for a new topology on  $X$  as follows. If  $x \in X - \{p\}$ , then  $\{x\}$  is open, and if  $F \in \mathcal{F}$ , then  $F \cup \{p\}$  is a neighborhood of  $p$ . This refines the original topology on  $X$ . Let  $X'$  be  $X$  with this new topology. Note that  $X'$  is completely regular.

Let  $Y_1 = \beta(X')$ , the Stone-Ćech compactification of  $X'$  (actually, any Hausdorff compactification will do), and let  $Y_2$  be the one point compactification of  $X'$ . Note that  $Y_2$  is a  $T_1$  space, but is definitely not Hausdorff.

*Claim 1.*  $X \times Y_1$  is not singly  $bi - k$ .

*Proof.* Let  $C = \{(x, x) : x \in X - \{p\}\}$ .  $C$  is a subset of  $X \times X' \subset X \times Y_1$ . If  $U \times V$  is a basic open set around  $(p, p)$ , then  $U$  and  $V \cap X'$  are both in  $\mathcal{F}$ , so  $U \cap V \cap X' - \{p\} \neq \emptyset$ . Thus  $(U \times V) \cap C \neq \emptyset$ , i.e.,  $(p, p) \in \text{cl}(C) - C$ . Suppose  $X \times Y_1$  is singly  $bi - k$ . Then there is a compact set  $K \subset X \times Y_1$ , and a nested sequence  $(K_i)$  of subsets of  $C$  such that  $(p, p) \in \text{cl}(K_i)$  for each  $i$ , and  $(K_i)$  is a  $K$ -sequence. Then, if  $\pi_x : X \times Y_1 \rightarrow X$  is the projection map, the sequence  $(\pi_x(K_i))$  is a  $\pi_x(K)$ -sequence and  $\pi_x(K)$  is compact in  $X$ . Suppose  $F \in \mathcal{F}$ . Then there is an open set  $V \subset Y_1$  such that  $V \cap X' = F \cup \{p\}$ . But then for each  $i$ , there is an  $x_i$  such that  $(x_i, x_i) \in K_i \cap (X \times V)$ . Then  $x_i \in F$ , so  $\pi_x(K_i) \cap F \neq \emptyset$ . Thus  $(\pi_x(K_i))$  is a  $\pi_x(K)$ -sequence meshing with  $\mathcal{F}$ . This violates the choice of  $\mathcal{F}$ , so  $X \times Y_1$  is not singly  $bi - k$ .

*Claim 2.*  $X \times Y_2$  is not sequentially  $k$ .

*Proof.* Let  $Y_2 = X' \cup \{\alpha\}$ .

Let  $C = (X \times \{\alpha\}) \cup \{(y, x) : y \in \text{cl}_x(\{x\}) \text{ and } x \in X' - \{p\}\}$ . (Nobody said  $X$  was a  $T_1$  space!) As in the proof of Claim 1,  $(p, p) \in \text{cl}(C) - C$ . Suppose  $(x, y) \in \text{cl}(C) - C$ . Then  $y \neq \alpha$ . Suppose  $y \neq p$ . Then since  $(x, y) \notin C$ , it follows that  $x \notin \text{cl}_x(\{y\})$  so there is an open (in  $X$ ) set  $U$  such that  $x \in U$  but  $y \notin U$ . But then  $U \times \{y\}$  is open (in  $X \times Y_2$ ) and  $(U \times \{y\}) \cap C = \emptyset$ . Thus  $y = p$ .

Could  $X \times Y_2$  be sequentially  $k$ ? If so, then there is a point  $(z, p) \in \text{cl}(C) - C$ , a compact set  $K \subset X \times Y_2$  and a sequence  $(K_i)$

such that  $(z, p) \in \text{cl}(K_i)$  and  $K_i \subset C$  for each  $i$ , and  $(K_i)$  is a  $K$ -sequence. Again, let  $\pi_x: X \times Y_2 \rightarrow X$  be the projection map. Let  $D_i = \{x: \text{there is a } y \in \pi_x(K_i) \text{ such that } y \in \text{cl}_x(\{x\})\}$ . Suppose  $x_i \in D_i$  for each  $i$ . There is, for each  $i$ , a point  $y_i \in \pi_x(K_i) \cap \text{cl}_x(\{x_i\})$ . Since  $(\pi_x(K_i))$  is a  $\pi_x(K)$ -sequence, the sequence  $(y_i)$  has an accumulation point  $k \in \pi_x(K)$ . But, since any open set containing  $y_i$  also contains  $x_i$ ,  $k$  is an accumulation point of  $(x_i)$ . Thus  $(D_i)$  is a  $\pi_x(K)$ -sequence.

Suppose  $F \in \mathcal{S}$ . Since  $F \cup \{p\}$  is open in  $Y_2$ , for each  $i$   $(X \times (F \cup \{p\})) \cap K_i \neq \emptyset$ . But  $\alpha \notin F \cup \{p\}$  so there is a point  $(y, x) \in K_i$  with  $x \in F$  and  $y \in \text{cl}_x(\{x\})$ . Thus  $x \in D_i \cap F$ . Therefore  $D_i$  is a  $\pi_x(K)$ -sequence meshing with  $\mathcal{S}$ , and  $\pi_x(K)$  is compact. As in the case of Claim 1, this contradicts the choice of  $\mathcal{S}$ , so  $X \times Y_2$  is not sequentially- $k$ .

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