DECOMPOSITIONS FOR NONCLOSED PLANAR $m$-CONVEX SETS

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Let $S$ be an $m$-convex set in the plane having the property that $(\text{int } \text{cl } S) - S$ contains no isolated points. If $T$ is an $m$-convex subset of $S$ having convex closure, then $T$ is a union of $\sigma(m)$ or fewer convex sets, where

$$\sigma(m) = (m - 1)(1 + (2^{m-2} - 1)2^{m-3}).$$

Hence for $m \geq 3$, $S$ is expressible as a union of $(m-1)2^{m-3}\sigma(m)$ or fewer convex sets.

In case $S$ is $m$-convex and $(\text{int } \text{cl } S) - S$ contains isolated points, an example shows that no such decomposition theorem is possible.

1. Introduction. For $S$ a subset of Euclidean space, $S$ is said to be $m$-convex, $m \geq 2$, if and only if for every $m$ distinct points of $S$, at least one of the line segments determined by these points lies in $S$. Several decomposition theorems have been proved for $m$-convex sets in the plane. A closed planar $3$-convex set is expressible as a union of $3$ or fewer convex sets (Valentine [4]), and an arbitrary planar $3$-convex set is a union of $6$ or fewer convex sets (Breen [1]). Concerning the general case, a recent study shows that for $m \geq 3$, a closed planar $m$-convex set may be decomposed into $(m - 1)2^{m-3}$ or fewer convex sets (Kay and Breen [2]). This leads naturally to the problem considered here, that of determining whether such a bound exists for an arbitrary $m$-convex set $S \subseteq \mathbb{R}^2$: With the restriction that $(\text{int } \text{cl } S) - S$ contain no isolated points, a bound in terms of $m$ is obtained; without this restriction, an example reveals that no bound is possible.

The following terminology will be used: For points $x, y$ in $S$, we say $x$ sees $y$ via $S$ if and only if the corresponding segment $[x, y]$ lies in $S$. Points $x_1, \ldots, x_n$ in $S$ are visually independent via $S$ if and only if for $1 \leq i < j \leq n$, $x_i$ does not see $x_j$ via $S$. Throughout the paper, $\text{conv } S$, $\text{bdry } S$, $\text{int } S$, and $\text{cl } S$ will be used to denote the convex hull of $S$, the boundary of $S$, the interior of $S$ and the closure of $S$, respectively.

2. The decomposition theorem. We shall be concerned with the proof of the following result, which yields the decomposition theorem as a corollary.
Theorem. Let $T$ be an $m$-convex set in the plane having the property that $(\text{int} \, \text{cl} \, T) \sim T$ contains no isolated points. If $\text{cl} \, T$ is convex, then $T$ is a union of $\sigma(m)$ or fewer convex sets, where

$$\sigma(m) = (m - 1)[1 + (2^{m-2} - 1)2^{m-3}] .$$

The main steps in the proof will be accomplished by a sequence of lemmas. The first lemma, which generalizes [1, Theorem 5], will require the following result by Lawrence, Hare and Kenelly [3, Theorem 2].

Lawrence, Hare, Kenelly theorem. Let $\Gamma$ be a subset of a linear space such that each finite subset $F \subseteq T$ has a $k$-partition $\{F_1, \ldots, F_k\}$, where $\text{conv} \, F_i \subseteq T$, $1 \leq i \leq k$. Then $T$ is a union of $k$ or fewer convex sets.

Lemma 1. Let $T$ be an $m$-convex set in the plane, $m \geq 3$, such that $\text{cl} \, T$ is convex. If all points of $(\text{cl} \, T) \sim T$ are in bdry$(\text{cl} \, T)$, then $T$ is a union of $\max(m - 1, 3)$ or fewer convex sets. The result is best possible.

Proof. By the Lawrence, Hare, Kenelly theorem, it suffices to consider finite subsets of $T$, so without loss of generality we may assume that $\text{cl} \, T$ is a convex polygon. Consider the collection of all intervals in $\text{cl} \, T$ having endpoints in $T$ and some relatively interior point not in $T$, and let $\mathcal{L}$ denote the collection of corresponding lines. Since $(\text{cl} \, T \sim T) \subseteq \text{bdry}(\text{cl} \, T)$, each line $L$ in $\mathcal{L}$ supports $\text{cl} \, T$ along an edge, and by the $m$-convexity of $T$, $L \cap T$, has at most $m - 1$ components. We will examine the components of $B = \bigcup \{L \cap T : L \in \mathcal{L}\}$.

Order the vertices of $\text{cl} \, T$ in a clockwise direction along $\text{bdry}(\text{cl} \, T)$, letting $p_i$ denote the $i$th vertex in our ordering, $1 \leq i \leq k$. If $p_i$ lies in some component of $B$, let $c_i$ denote this component. Otherwise, let $c_i = \emptyset$. Define sets $A_i'$, $1 \leq i \leq \max(3, m - 1)$, each an appropriate collection of components of $B$: For $i$ odd, $i < k$, assign $c_i$ to $A_i'$; for $i$ even, $i < k$, assign $c_i$ to $A_i'$; assign $c_k$ to $A_i'$.

Now consider the remaining components of $B$. If the line $L(p_i, p_{i+1})$ determined by $p_i$ and $p_{i+1}$ is in $\mathcal{L}$, $1 \leq i \leq k$ (where $p_{k+1} = p_0$), assign each remaining component on this line to some $A'$ set not containing $c_i \neq \emptyset$ or $c_{i+1} \neq \emptyset$, and assign at most one component to each $A'$ set. Since there are at most $m - 1$ components on each line, at most $m - 1$ $A'$ sets are required at each stage of the argument. Furthermore, no two components on any line will be assigned to the same $A'$ set.

Finally, let $A_i = T \sim \bigcup \{A_{i}' : j \neq i\}$, $1 \leq i \leq \max(m - 1, 3)$. It
is easy to show that the $A_i$ sets are convex and that their union is $T$, completing the proof.

To see that the result in Lemma 1 is best possible, consider the following example.

**Example 1.** Let $T$ be a pentagonal region having exactly $m-2$ points deleted from the relative interior of each edge, $m \geq 3$. Then $T$ is $m$-convex and is not expressible as a union of fewer than max $(m-1, 3)$ convex sets.

Lemmas 2, 3 and 4 concern points in $(\text{int cl } S) \sim S$.

**Lemma 2.** Let $S$ be an arbitrary set in the plane. If $(\text{int cl } S) \sim S$ contains at least $r$ noncollinear segments, where $r = 2^n$, $n \geq 0$, then $S$ contains $n + 2$ visually independent points.

**Proof.** The proof is by induction. If $n = 0$, then $r = 1$ and certainly $S$ contains 2 visually independent points. Assume the theorem true for numbers less than $n$, $n \geq 1$, to prove for $n$. Let $L$ be the line determined by one of the $2^n$ (or more) noncollinear segments $C$ in $(\text{int cl } S) \sim S$. Then at least half of the $2^n - 1$ remaining segments contain points in one of the open halfspaces $H_i$ determined by $L$. Hence $S' = S \cap H_i$ has the property that $(\text{int cl } S') \sim S'$ contains at least $r'$ noncollinear segments, where $r' \geq (2^n - 1)/2 = 2^{n-1} - 1/2$. Since $r'$ is an integer, $r' \geq 2^{n-1}$, so by our induction hypothesis, $S'$ contains $n + 1$ visually independent points $y_1, \ldots, y_{n+1}$. Letting $H_2$ denote the opposite open halfspace determined by $L$, select $y_i$ in $H_2 \cap S$ so that $(y_o, y_i)$ cuts $C$ for $1 \leq i \leq n + 1$. Then $\{y_o, \ldots, y_{n+1}\}$ is a set of $n + 2$ visually independent points of $S$.

**Corollary.** If $S$ is planar and $m$-convex, then $(\text{int cl } S) \sim S$ contains at most $2^{m-2} - 1$ noncollinear segments.

**Proof.** Assume that $S$ contains $r \geq 1$ noncollinear segments. Then $2^n \leq r < 2^{n+1}$ for an appropriate $n \geq 0$, and by the lemma, $S$ contains $n + 2$ visually independent points. Since $S$ is $m$-convex, we have $n + 2 \leq m - 1$, so $r < 2^{m-2}$.

The author wishes to thank the referee for his conjecture of the following result.

**Lemma 3.** Let $S$ be an $m$-convex set in the plane, $m \geq 3$. If $M$ is any line, then $M \cap [(\text{int cl } S) \sim S]$ has at most $m + [(m-3)/2]$ components. The result is best possible.
Proof. Assume that $M \cap [(\text{int } \text{cl } S) \sim S] \neq \emptyset$, for otherwise there is nothing to prove. Since $S$ is $m$-convex, it is easy to show that the set $\text{cl } S$ is $m$-convex, so $M \cap \text{cl } S$ has at most $m - 1$ components $M_i$, $1 \leq i \leq m - 1$. There exist disjoint convex neighborhoods $U_i$ of $M_i$, $1 \leq i \leq m - 1$, such that no point of $U_i \cap \text{cl } S$ sees any point of $U_j \cap \text{cl } S$ via $S$, $1 \leq i < j \leq m - 1$. Thus no point of $U_i \cap S$ sees any point of $U_j \cap S$ via $S$, $1 \leq i < j \leq m - 1$.

Note that if $M_i \cap [(\text{int } \text{cl } S) \sim S] \neq \emptyset$, there are at least two points in $U_i \cap S$ which are visually independent via $S$. Hence $M_i \cap [(\text{int } \text{cl } S) \sim S] \neq \emptyset$ for at most $[(m - 1)/2]$ of the $M_i$ sets.

We use an inductive argument to prove the lemma. If $S$ is 3-convex, then $M_i \cap [(\text{int } \text{cl } S) \sim S] \neq \emptyset$ for at most one component $M_i$ of $M \cap \text{cl } S$, and it is easy to see that $M_i \cap [(\text{int } \text{cl } S) \sim S]$ consists of at most three components. Assume that the result is true for $j$, $3 \leq j < m$, to prove for $m$. For some component $M_i$ of $M \cap \text{cl } S$, assume that $M_i \cap [(\text{int } \text{cl } S) \sim S]$ has $k$ components. Then clearly $1 \leq k \leq m$. For the neighborhood $U_i$ defined above, there correspond at least $\max(2, k - 1)$ visually independent points of $S$ in $U_i$. Examine the set $S' = \cup \{U_i \cap S : i \neq 1\}$. There are two cases to consider.

Case 1. If $k \geq 3$, the set $S'$ contains at most $m - k$ visually independent points, and $S'$ is $(m - k + 1)$-convex. By our inductive assumption applied to $S'$, $M \cap [(\text{int } \text{cl } S') \sim S']$ has at most $(m - k + 1) + [(m - k + 1 - 3)/2]$ components. Then $M \cap [(\text{int } \text{cl } S) \sim S]$ has at most $k + (m - k + 1) + [(m - k - 2)/2] = m + [(m - k)/2]$ components. This number is maximal when $k = 3$, giving the desired result.

Case 2. If $1 \leq k < 3$, then a similar argument shows that there are at most $2 + (m - 2) + [(m - 2 - 3)/2] = m + [(m - 5)/2] < m + [(m - 3)/2]$ components, finishing the proof of the lemma.

An inductive construction may be used to show that the result of Lemma 3 is best possible.

Example 2. For $3 \leq m \leq 4$, remove $m$ collinear segments appropriately from an open convex set to obtain an $m$-convex set having the required property. Inductively, for $m \geq 5$ let $S$ denote the union of an $(m - 2)$-convex set $S_1$ and a 3-convex set $S_2$, where $(\text{int } \text{cl } S_i) \sim S_i$ has the maximal number of collinear components, $(\text{int } \text{cl } S_i) \sim S_i$ and $(\text{int } \text{cl } S_i) \sim S_2$ are collinear, and $\text{cl } S_i \cap \text{cl } S_2 = \emptyset$. By our inductive construction, the set $(\text{int } \text{cl } S) \sim S$ will have exactly $m - 2 + [(m - 5)/2] + 3 = m + [(m - 3)/2]$ collinear components.

Lemma 4. Let $S$ be an $m$-convex set in the plane. If $x \in (\text{int }
cl(S) ~ S and x is not an isolated point, then x lies in a segment in (int cl S) ~ S.

Proof. Assume on the contrary that x is not in a segment in (int cl S) ~ S to obtain a contradiction. By the corollary to Lemma 2, (int cl S) ~ S contains at most $2^{m-1} - 1$ noncollinear segments. Also, by Lemma 3, for any line determined by such a segment, $M \cap [(int cl S) ~ S]$ has at most $m + [(m - 3)/2]$ components, so the segments in (int cl S) ~ S may be written as a finite union of segments. Hence we may select an open disk $N$ centered at $x$ which is disjoint from each of these segments, with $N \subseteq int cl S$. Let $N_0$ be an open disk centered at $x$ and properly contained in $N$. Let $L$ be any line through $x$, and let $C$ be any component of $(int cl S) ~ S$ containing $x$. Since $x$ is not an isolated point, there are points of $C \cap N_0$ in at least one of the open halfspaces $H_i$ determined by $L$, and we let $C_i$ be a component of $C \cap H_i \cap N_0$. Clearly $C_i$ is not a singleton set and cannot be collinear with $x$.

We assert that there is some point $z_i$ in $N \cap S$ and some neighborhood $N_i$ of $x$, $N_i \subseteq N$, such that $z_i$ sees no point of $N_i \cap S$ via $S$: Select points $s, t$ in $C_i$ such that $x, s, t$ are not collinear. Select $z_i \in S$ in the open convex region bounded by the rays $R(x, s), R(x, t)$ and in $N \sim N_0$ (where $R(x, s)$ denotes the ray emanating from $x$ through $s$). Since $[x, z_i] \subseteq N$, each component of $[x, z_i] \sim S$ is a singleton point. Also, there are at most $m - 2$ such components, so there is some point $q$ on $[x, z_i]$ such that $(x, q) \cap C_i = \emptyset$.

Let line $L_i$ be parallel to $L$ so that $s, t, z_i$ are on the same side of $L_i$, and so that $L_i$ contains some point $q_i \in (x, q)$. Repeating an argument from the preceding paragraph, components of $C_i \cap L_i$ are singleton sets. Hence there exist points $v, w$ in $L_i \cap N_i$, $v < q, w$, with $(v, w) \cap C_i = \emptyset$. Without loss of generality, assume that $v$ and $w$ are interior to the convex region determined by rays $R(z_i, s)$ and $R(z_i, t)$. Then for $v < y < w$, we see that $[z_i, y] \cap C_i = \emptyset$. Otherwise, the path $\lambda = [z_i, y] \cup [y, q_i] \cup [q_i, x]$ would be disjoint from $C_i$, with $s$ and $t$ on opposite sides of $\lambda$. Since $z_i, x \in H_1 \cap N_0$ and $C_i \subseteq H_i \cap N_0$, $\lambda$ would separate $C_i$, impossible.

Finally, let $N_i$ be any open disk about $x$ in the open convex region determined by $R(z_i, v)$ and $R(z_i, w)$ such that $N_i$ and $z_i$ are on opposite sides of $L_i$. Then for every $y$ in $N_i$, $[z_i, y]$ intersects $(v, w)$ and thus $[z_i, y]$ intersects $C_i$. Hence $z_i$ sees no point of $N_i \cap S$ via $S$, the desired result.

Repeat the argument to obtain $z_2$ in $N_i \cap S$ and $N_2 \subseteq N_i$ with $z_2$ seeing no point of $N_2 \cap S$ via $S$. By an obvious induction, we obtain $\{z_1, \ldots, z_m\}$ a set of $m$ visually independent points in $S$. This contradicts the $m$-convexity of $S$, our original assumption is false,
and \( x \) must lie in a segment in \((\text{int} \text{ cl} \; S) \sim S\).

Finally, the following combinatorial result will be helpful.

**Lemma 5.** For each collection \( \mathcal{L} \) of \( r \geq 1 \) lines in the plane, \( \mathbb{R}^2 \sim (\cup \mathcal{L}) \) consists of at most \( f(r) = 1 + \sum_{i=1}^{r-1} k \) convex components.

**Proof.** We use an inductive argument. If \( r = 1 \), the result is clear. Assume the result true for \( r = n \geq 1 \) to prove for \( n + 1 \). For \( \mathcal{L} \) consisting of \( n + 1 \) lines, select any member \( L \) of \( \mathcal{L} \) and let \( \mathcal{L}' = \mathcal{L} \setminus \{L\} \). Then by our induction hypothesis, \( \mathbb{R}^2 \sim (\cup \mathcal{L}') \) consists of at most \( f(n) \) convex components. The line \( L \) cuts each member of \( \mathcal{L}' \) at most once, so there are at most \( n \) corresponding points of intersection. These \( n \) points in turn determine at most \( n + 1 \) intervals on \( L \) (two of which are unbounded), and each of these intervals cuts a component of \( \mathbb{R}^2 \sim (\cup \mathcal{L}') \), yielding two convex components where previously there was only one. Hence \( \mathbb{R}^2 \sim (\cup \mathcal{L}) \) consists of at most \( f(n) + n + 1 = f(n + 1) \) convex components.

**Theorem 1.** Let \( T \) be an \( m \)-convex set in the plane having the property that \((\text{int} \text{ cl} \; T) \sim T\) contains no isolated points. If \( \text{cl} \; T \) is convex, then \( T \) is a union of \( \sigma(m) \) or fewer convex sets, where

\[
\sigma(m) = (m - 1)[1 + (2^{m-2} - 1)2^{m-3}].
\]

**Proof.** If \( m = 2 \), the result is clear, so assume that \( m \geq 3 \). By Lemma 4, \((\text{int} \text{ cl} \; T) \sim T\) may be expressed as a union of segments, and by the corollary to Lemma 2, these segments determine a corresponding collection \( \mathcal{L} \) of at most \( r = 2^{m-2} - 1 \) lines. Using Lemma 4, \( \mathbb{R}^2 \sim (\cup \mathcal{L}) \) consists of at most \( f(r) \) convex components \( C_i, \; 1 \leq i \leq f(r) \), where \( f(r) = 1 + \sum_{i=1}^{r-1} k = 1 + (r(r + 1))/2 = 1 + (2^{m-2} - 1)(2^{m-3}) \).

Let \( T_i = (\text{cl} \; C_i) \cap T, \; 1 \leq i \leq f(r) \). Then clearly \( T_i \) is an \( m \)-convex set, \( m \geq 3 \), such that \( \text{cl} \; T_i \) is convex and \((\text{cl} \; T_i) \sim T_i \subseteq \text{bdry-} (\text{cl} \; T_i)\). Then by Lemma 1, \( T_i \) is a union of \( \max (m - 1, 3) \) or fewer convex sets. Hence if \( m \geq 4 \), \( T \) is a union of

\[
\sigma(m) = (m - 1)[1 + (2^{m-2} - 1)2^{m-3}]
\]
or fewer convex sets, the desired result.

In case \( m = 3 \), then by [1, Lemma 3], all points of \((\text{int} \text{ cl} \; T) \sim T\) are collinear. If \( L \) is the corresponding line, \( T \cap L \) contains at most two components \( L_1, L_2 \). Letting \( H_i, H_2 \) represent distinct open halfspaces determined by \( L \), define \( T_i = (H_i \cap T) \cup L_i, \; 1 \leq i \leq 2 \).
A proof similar to that of Lemma 1 shows that each $T_i$ is a union of two or fewer convex sets, so $T$ is a union of $\sigma(3) = 4$ or fewer convex sets, completing the proof of the theorem.

**Corollary.** If $S$ is an $m$-convex set in the plane, $m \geq 3$, having the property that $(\text{int cl } S) \sim S$ contains no isolated points, then $S$ is expressible as a union of $(m - 1)2^{m-3}\sigma(m)$ or fewer convex sets.

**Proof.** It is easy to show that the set $\text{cl } S$ is $m$-convex, and by [2, Theorem 6], $\text{cl } S$ may be decomposed into $(m - 1)2^{m-3}$ or fewer closed convex sets. If $C$ is one of these convex sets, let $T = C \cap S$. Clearly $T$ is $m$-convex. There are two cases to consider.

**Case 1.** If $C$ is contained in a line, then $T$ contains at most $m - 1 < \sigma(m)$ convex components.

**Case 2.** If $C$ is not contained in a line, then it is easy to show that $\text{cl } T = C$: First pick $p$ in $C$. Since $C \subseteq \text{cl } S$, every neighborhood of $p$ contains points of $S$. If $p$ is in $\text{int } C$, then points of $S$ contained in small discs centered at $p$ necessarily belong to $C \cap S = T$. Thus we conclude that $p \in \text{cl } T$. On the other hand, if $p \in \text{bdry } C$, then every neighborhood of $p$ contains points of $\text{int } C$. By our previous remarks, $\text{int } C \subseteq \text{cl } T$, so $p \in \text{cl } (\text{cl } T) = \text{cl } T$. Hence $C \subseteq \text{cl } T$. The reverse inclusion is obvious, so $C = \text{cl } T$ and $\text{cl } T$ is convex. Certainly $(\text{int cl } T) \sim T$ contains no isolated points, so by the theorem, $T$ is a union of $\sigma(m)$ or fewer convex sets. Thus $S$ is a union of $(m - 1)2^{m-3}\sigma(m)$ or fewer convex sets.

3. An example. The following example shows that no decomposition theorem is possible in case $S$ is an $m$-convex set having isolated points as components of $(\text{int cl } S) \sim S$.

**Example.** Let $k$ be an arbitrary integer and let $P$ be a regular polygon having $2k$ vertices $p_1, \ldots, p_{2k}$. Let $v_1, \ldots, v_{2k}$ be vertices of a regular polygon interior to $P$, where for $1 \leq i \leq 2k$, $v_i$ is sufficiently close to $p_i$ that the following holds: If $x$ and $y$ are visually independent points of $P' \equiv P \sim \{v_1, \ldots, v_{2k}\}$, then for every $i, j$, $1 \leq i, j \leq 2k$, either $(R(x, v_i) \sim [x, v_i]) \cap (R(y, v_j) \sim [y, v_j]) \cap P = \emptyset$ or $x, v_i, y, v_j$ are collinear. Hence three points $x, y, z$ are visually independent via $P'$ only if they are collinear with a pair of distinct points $v_i$ and $v_j$, and $P'$ is 4-convex.

However, $P'$ is not expressible as a union of fewer than $k + 2$
convex sets. (If the vertices $v_i$ are ordered in a clockwise direction, $1 \leq i \leq 2k$, consider the $k + 1$ subsets $P_i, \ldots, P_{k+1}$ of $P'$ bounded by and disjoint from the $k$ lines $L(v_i, v_{2i}), L(v_{2i}, v_{3i}), \ldots, L(v_{ki}, v_{k+1})$. Let $P_{k+2} = \text{conv}(\cup \{(v_i, v_{2i+1-i}) : 1 \leq i \leq k\})$. Assign each remaining segment of $P' \cap L(v_i, v_{2i+1-i})$ to one of the adjacent regions $P_i$ or $P_{i+1}$, $1 \leq i \leq k$, in the obvious manner. This yields a $(k + 2)$-member decomposition of $P'$. The number $k + 2$ is best possible.)

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Received May 1, 1975 and in revised form October 29, 1975.

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