Pacific Journal of Mathematics

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Vol. 69, No. 2 June 1977

PACIFIC JOURNAL OF MATHEMATICS
Vol. 69, No. 2, 1977

A NOTE ON EDELSTEIN'S ITERATIVE TEST AND SPACES OF CONTINUOUS FUNCTIONS

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In this note a question posed by Nadler is answered. It is shown that if X is a compact Hausdorff space that contains a sequence of distinct points that converge then there exists a linear contractive selfmap f of $C(X)$ such that, for some x, the sequence of iterates $\{f^n(x)\}$ does not converge. In particular, the iterative test is not conclusive for c.

Our setting is a metric space (X, d) and a contractive selfmap $f: X \to X$. In [1], Nadler introduces and motivates the following terminology: the *iterative test* (of Edelstein) *is conclusive* (for contractive maps) provided that if f is a contractive selfmap of X with a fixed point then, for all $x \in X$, $\{f^n(x)\}$ converges. Nadler shows that the iterative test is conclusive (ITC) for finite dimensional Banach spaces, but that the iterative test is not conclusive (ITNC) for the spaces $l_p (1 \leqq p < \infty)$ and c_0 (the space of sequences convergent to zero). The technique used there does not seem to apply directly to the space c of convergent sequences, and part of Nadler's Problem 1 is exactly the question of whether c has ITC.

LEMMA 1. *The iterative test is not conclusive for c.*

Proof. Let $\{\alpha_n\}$ be an increasing positive sequence with (infinite) product 1/2. Define $f: c \to c$ by $f(\{x_n\}) = \{y_n\}$ where

$$y_1 = 0, \quad y_2 = -y^2 = \alpha_1 x_1 ,$$
$$y_{2n} = -y_{2n+1} = \frac{\alpha_n}{2}(x_{2n-2} - x_{2n-1}) , \qquad n = 2, 3, \cdots .$$

Since f is linear, f has fixed point 0, and it suffices to show f is contractive at 0; if

$$\{z_n\} \in c,\ \{x_n\} \neq 0,\ d(f(\{x_n\}), 0) = \sup\{|y_n|\} = |y_{n_0}| ,$$

since $y_n \to 0$. If $n_0 = 1$ or 2 then it is easy to see that

$$d(f(\{x_n\}), 0) < d(\{y_n\}, 0) .$$

If $n_0 = 2k (k > 1)$, we have

$$|y_{n_0}| = \frac{\alpha_k}{2} |x_{n_0-2} - x_{n_0-1}| \leqq \frac{\alpha_k}{2}\{|x_{n_0-2}| + |x_{n_0-1}|\}$$
$$\leqq \alpha_k d(\{x_n\}, 0) < d(\{x_n\}), 0) .$$

Let e_k be the sequence $\{\delta_{kn}\} = \{0, 0, 0, \cdots, 1, 0, \cdots\}$ (1 in the kth coordinate). We have

$$f^j(e_1) = \left(\prod_{i=1}^{j} \alpha_i\right)(e_{2j} - e_{2j+1}) .$$

In particular, $d(f^j(e_1), 0) = \prod_{i=1}^{j} \alpha_i \to 1/2$, and so $\{f^j(e_1)\}$ does not converge. (If $\{f^j(e_1)\}$ converges, then, since f is contractive, $f^j(e_1)$ must converge to the fixed point 0 of f.)

It is of definite interest that the map $f: c \to c$ constructed above is linear. It would seem to be easier to solve Nadler's Problem 1 (if a Banach space has ITC then it is finite dimensional) when restricted to linear maps.

LEMMA 2. *Let Y be a normed space and let X be a subspace of Y. Let P be a projection of norm 1 from Y onto X. Then if the iterative test is not conclusive for X, it is not conclusive for Y.*

Proof. Let $f: X \to X$ be a contractive map with fixed point such that, for some x_0, $\{f^n(x_0)\}$ does not converge. Define $g: Y \to Y$ by $g = f \circ P$. Since f is contractive and $||P|| = 1$, then g is contractive. Also, $g^n(x_0) = f^n(x_0)$ (since $P(x_0) = x_0$), and so $\{g^n(x_0)\}$ does not converge.

If X is a compact Hausdorff space with a convergent sequence of distinct points, a projection P of norm 1 can be constructed from $C(X)$ onto a subspace that is linearly isometric to c.

Let $\{x_n\}$ be any sequence of distinct points of X that converges and furthermore $x_n \to \bar{x}$. Let $P_1: C(X) \to c$ be defined as follows: if

$$f \in C(X),\ P_1(f) = \{y_n\} \quad \text{where} \quad y_n = f(x_n) .$$

Since f is continuous $y_n \to f(x)$ and $P_1(f) \in c$. P_1 is nonexpansive for

$$||P_1(f)|| = \sup_n |f(x_n)| \leqq \sup_{x \in X} |f(x)| = ||f|| .$$

An isometric linear map Q is now constructed from c into $C(X)$ such that $P_1 \circ Q(x) = x$. Let $\{U_i\}$ be a sequence of open sets such that $x_i \in U_i$, $U_i \cap U_j = \emptyset$ if $i \neq j$, and $\bar{x} \notin U_i$ for all i. For each i define f_i to be a function such that $f_i(x_i) = 1$, $f_i(X - U_i) = 0$ and $0 \leqq f_i(x) \leqq 1$ for all x. If $\{y_n\} \in c$ and $y_n \to y$ then define $Q(\{y_n\}) = f$ where

$$f(x) = \sum_{n=1}^{\infty} f_n(x)(y_n - y) + y .$$

It is easily verified that f is continuous, $f(x_i) = y_i$ and $||f|| = ||\{y_n\}||$. Hence $Q: c \to C(X)$ is a linear isometry and

$$(P_1 \circ Q)(\{y_n\}) = P_1(\{y_n\}) = \{y_n\} .$$

Define $P: C(X) \to C(X)$ as $P = Q \circ P_1$. Since P_1 is onto and Q is an isometry then $||P|| = 1$ and P is a projection, for

$$P^2 = Q \circ P_1 \circ Q \circ P_1 = Q \circ P_1 = P .$$

Thus P is a projection of norm 1 from $C(X)$ onto $Q(c)$.

Combining this construction with Lemmas 1 and 2 we have:

THEOREM. *Let X be a compact Hausdorff space that contains an infinite sequence of distinct points that converge. Then the iterative test is not conclusive for $C(X)$.*

In each of the above, there is a linear selfmap for which the iterative test fails.

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1. S. B. Nadler, Jr., *A note on an iterative test of Edelstein*, Canad. Math. Bull., to appear.

Received November 11, 1971.

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The *Pacific Journal of Mathematics* is issued monthly as of January 1966. Regular subscription rate: $72 00 a year (6 Vols., 12 issues). Special rate: $36.00 a year to individual members of supporting institutions.

Subscriptions, orders for back numbers, and changes of address should be sent to Pacific Journal of Mathematics, 103 Highland Boulevard, Berkeley, California, 94708.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION

Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.).
8-8, 3-chome, Takadanobaba, Shinjuku-ku, Tokyo 160, Japan.

Pacific Journal of Mathematics

Vol. 69, No. 2 June, 1977