

# Pacific Journal of Mathematics

**A NOTE ON EDELSTEIN'S ITERATIVE TEST AND SPACES OF  
CONTINUOUS FUNCTIONS**

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## A NOTE ON EDELSTEIN'S ITERATIVE TEST AND SPACES OF CONTINUOUS FUNCTIONS

JACK BRYANT AND T. F. McCABE

**In this note a question posed by Nadler is answered. It is shown that if  $X$  is a compact Hausdorff space that contains a sequence of distinct points that converge then there exists a linear contractive selfmap  $f$  of  $C(X)$  such that, for some  $x$ , the sequence of iterates  $\{f^n(x)\}$  does not converge. In particular, the iterative test is not conclusive for  $c$ .**

Our setting is a metric space  $(X, d)$  and a contractive selfmap  $f: X \rightarrow X$ . In [1], Nadler introduces and motivates the following terminology: the *iterative test* (of Edelstein) *is conclusive* (for contractive maps) provided that if  $f$  is a contractive selfmap of  $X$  with a fixed point then, for all  $x \in X$ ,  $\{f^n(x)\}$  converges. Nadler shows that the iterative test is conclusive (ITC) for finite dimensional Banach spaces, but that the iterative test is not conclusive (ITNC) for the spaces  $l_p (1 \leq p < \infty)$  and  $c_0$  (the space of sequences convergent to zero). The technique used there does not seem to apply directly to the space  $c$  of convergent sequences, and part of Nadler's Problem 1 is exactly the question of whether  $c$  has ITC.

LEMMA 1. *The iterative test is not conclusive for  $c$ .*

*Proof.* Let  $\{\alpha_n\}$  be an increasing positive sequence with (infinite) product  $1/2$ . Define  $f: c \rightarrow c$  by  $f(\{x_n\}) = \{y_n\}$  where

$$y_1 = 0, \quad y_2 = -y^2 = \alpha_1 x_1,$$

$$y_{2n} = -y_{2n+1} = \frac{\alpha_n}{2}(x_{2n-2} - x_{2n-1}), \quad n = 2, 3, \dots$$

Since  $f$  is linear,  $f$  has fixed point 0, and it suffices to show  $f$  is contractive at 0; if

$$\{z_n\} \in c, \quad \{x_n\} \neq 0, \quad d(f(\{x_n\}), 0) = \sup \{|y_n|\} = |y_{n_0}|,$$

since  $y_n \rightarrow 0$ . If  $n_0 = 1$  or 2 then it is easy to see that

$$d(f(\{x_n\}), 0) < d(\{y_n\}, 0).$$

If  $n_0 = 2k (k > 1)$ , we have

$$|y_{n_0}| = \frac{\alpha_k}{2} |x_{n_0-2} - x_{n_0-1}| \leq \frac{\alpha_k}{2} (|x_{n_0-2}| + |x_{n_0-1}|)$$

$$\leq \alpha_k d(\{x_n\}, 0) < d(\{x_n\}, 0).$$

Let  $e_k$  be the sequence  $\{\delta_{kn}\} = \{0, 0, 0, \dots, 1, 0, \dots\}$  (1 in the  $k$ th coordinate). We have

$$f^j(e_1) = \left( \prod_{i=1}^j \alpha_i \right) (e_{2j} - e_{2j+1}).$$

In particular,  $d(f^j(e_1), 0) = \prod_{i=1}^j \alpha_i \rightarrow 1/2$ , and so  $\{f^j(e_1)\}$  does not converge. (If  $\{f^j(e_1)\}$  converges, then, since  $f$  is contractive,  $f^j(e_1)$  must converge to the fixed point  $0$  of  $f$ .)

It is of definite interest that the map  $f: c \rightarrow c$  constructed above is linear. It would seem to be easier to solve Nadler's Problem 1 (if a Banach space has ITC then it is finite dimensional) when restricted to linear maps.

**LEMMA 2.** *Let  $Y$  be a normed space and let  $X$  be a subspace of  $Y$ . Let  $P$  be a projection of norm 1 from  $Y$  onto  $X$ . Then if the iterative test is not conclusive for  $X$ , it is not conclusive for  $Y$ .*

*Proof.* Let  $f: X \rightarrow X$  be a contractive map with fixed point such that, for some  $x_0$ ,  $\{f^n(x_0)\}$  does not converge. Define  $g: Y \rightarrow Y$  by  $g = f \circ P$ . Since  $f$  is contractive and  $\|P\| = 1$ , then  $g$  is contractive. Also,  $g^n(x_0) = f^n(x_0)$  (since  $P(x_0) = x_0$ ), and so  $\{g^n(x_0)\}$  does not converge.

If  $X$  is a compact Hausdorff space with a convergent sequence of distinct points, a projection  $P$  of norm 1 can be constructed from  $C(X)$  onto a subspace that is linearly isometric to  $c$ .

Let  $\{x_n\}$  be any sequence of distinct points of  $X$  that converges and furthermore  $x_n \rightarrow \bar{x}$ . Let  $P_i: C(X) \rightarrow c$  be defined as follows: if

$$f \in C(X), P_i(f) = \{y_n\} \quad \text{where } y_n = f(x_n).$$

Since  $f$  is continuous  $y_n \rightarrow f(\bar{x})$  and  $P_i(f) \in c$ .  $P_i$  is nonexpansive for

$$\|P_i(f)\| = \sup_n |f(x_n)| \leq \sup_{x \in X} |f(x)| = \|f\|.$$

An isometric linear map  $Q$  is now constructed from  $c$  into  $C(X)$  such that  $P_i \circ Q(x) = x$ . Let  $\{U_i\}$  be a sequence of open sets such that  $x_i \in U_i$ ,  $U_i \cap U_j = \emptyset$  if  $i \neq j$ , and  $\bar{x} \notin U_i$  for all  $i$ . For each  $i$  define  $f_i$  to be a function such that  $f_i(x_i) = 1$ ,  $f_i(X - U_i) = 0$  and  $0 \leq f_i(x) \leq 1$  for all  $x$ . If  $\{y_n\} \in c$  and  $y_n \rightarrow y$  then define  $Q(\{y_n\}) = f$  where

$$f(x) = \sum_{n=1}^{\infty} f_n(x)(y_n - y) + y.$$

It is easily verified that  $f$  is continuous,  $f(x_i) = y_i$  and  $\|f\| = \|\{y_n\}\|$ . Hence  $Q: c \rightarrow C(X)$  is a linear isometry and

$$(P_1 \circ Q)(\{y_n\}) = P_1(\{y_n\}) = \{y_n\} .$$

Define  $P: C(X) \rightarrow C(X)$  as  $P = Q \circ P_1$ . Since  $P_1$  is onto and  $Q$  is an isometry then  $\|P\| = 1$  and  $P$  is a projection, for

$$P^2 = Q \circ P_1 \circ Q \circ P_1 = Q \circ P_1 = P .$$

Thus  $P$  is a projection of norm 1 from  $C(X)$  onto  $Q(c)$ .

Combining this construction with Lemmas 1 and 2 we have:

**THEOREM.** *Let  $X$  be a compact Hausdorff space that contains an infinite sequence of distinct points that converge. Then the iterative test is not conclusive for  $C(X)$ .*

In each of the above, there is a linear selfmap for which the iterative test fails.

#### REFERENCE

1. S. B. Nadler, Jr., *A note on an iterative test of Edelstein*, *Canad. Math. Bull.*, to appear.

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