A NOTE ON EDELSTEIN’S ITERATIVE TEST AND SPACES OF CONTINUOUS FUNCTIONS

JACK DOUGLAS BRYANT AND THOMAS FRANCIS MCCABE
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In this note a question posed by Nadler is answered. It is shown that if $X$ is a compact Hausdorff space that contains a sequence of distinct points that converge then there exists a linear contractive selfmap $f$ of $C(X)$ such that, for some $x$, the sequence of iterates $\{f^n(x)\}$ does not converge. In particular, the iterative test is not conclusive for $c$.

Our setting is a metric space $(X, d)$ and a contractive selfmap $f: X \to X$. In [1], Nadler introduces and motivates the following terminology: the iterative test (of Edelstein) is conclusive (for contractive maps) provided that if $f$ is a contractive selfmap of $X$ with a fixed point then, for all $x \in X$, $\{f^n(x)\}$ converges. Nadler shows that the iterative test is conclusive (ITC) for finite dimensional Banach spaces, but that the iterative test is not conclusive (ITNC) for the spaces $l_p(1 \leq p < \infty)$ and $c_0$ (the space of sequences convergent to zero). The technique used there does not seem to apply directly to the space $c$ of convergent sequences, and part of Nadler's Problem 1 is exactly the question of whether $c$ has ITC.

**Lemma 1.** The iterative test is not conclusive for $c$.

**Proof.** Let $\{\alpha_n\}$ be an increasing positive sequence with (infinite) product $1/2$. Define $f: c \to c$ by $f([x_n]) = [y_n]$ where

\[
\begin{align*}
y_1 &= 0, \\ y_2 &= -y_2 = \alpha_1 x_1, \\ y_{2n} &= y_{2n+1} = \frac{\alpha_n}{2}(x_{2n-1} - x_{2n}), & n &= 2, 3, \ldots.
\end{align*}
\]

Since $f$ is linear, $f$ has fixed point 0, and it suffices to show $f$ is contractive at 0; if

$\{z_n\} \in c$, $\{x_n\} \neq 0$, $d(f([x_n]), 0) = \sup \{|y_n|\} = |y_{n_0}|$,

since $y_n \to 0$. If $n_0 = 1$ or 2 then it is easy to see that

$d(f([x_n]), 0) < d([y_n], 0)$.

If $n_0 = 2k(k > 1)$, we have

\[
|y_{n_0}| = \frac{\alpha_k}{2} |x_{n_0-2} - x_{n_0-1}| \leq \frac{\alpha_k}{2} \{ |x_{n_0-2}| + |x_{n_0-1}| \}
\leq \alpha_k d([x_n], 0) < d([x_n], 0).
\]

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Let \( e_k \) be the sequence \( \{d_{kn}\} = \{0, 0, 0, \ldots, 1, 0, \ldots\} \) (1 in the \( k \)th coordinate). We have

\[
 f^i(e_k) = \left( \prod_{i=1}^{j} \alpha_i \right) (e_{2j} - e_{2j+1}) .
\]

In particular, \( d(f^i(e_k), 0) = \prod_{i=1}^{j} \alpha_i \rightarrow 1/2 \), and so \( \{f^i(e_k)\} \) does not converge. (If \( \{f^i(e_k)\} \) converges, then, since \( f \) is contractive, \( f^i(e_k) \) must converge to the fixed point 0 of \( f \).)

It is of definite interest that the map \( f : c \rightarrow c \) constructed above is linear. It would seem to be easier to solve Nadler's Problem 1 (if a Banach space has ITC then it is finite dimensional) when restricted to linear maps.

**Lemma 2.** Let \( Y \) be a normed space and let \( X \) be a subspace of \( Y \). Let \( P \) be a projection of norm 1 from \( Y \) onto \( X \). Then if the iterative test is not conclusive for \( X \), it is not conclusive for \( Y \).

**Proof.** Let \( f : X \rightarrow X \) be a contractive map with fixed point such that, for some \( x_0 \), \( \{f^n(x_0)\} \) does not converge. Define \( g : Y \rightarrow Y \) by \( g = f \circ P \). Since \( f \) is contractive and \( \|P\| = 1 \), then \( g \) is contractive. Also, \( g^n(x_0) = f^n(x_0) \) (since \( P(x_0) = x_0 \)), and so \( \{g^n(x_0)\} \) does not converge.

If \( X \) is a compact Hausdorff space with a convergent sequence of distinct points, a projection \( P \) of norm 1 can be constructed from \( C(X) \) onto a subspace that is linearly isometric to \( c \).

Let \( \{x_n\} \) be any sequence of distinct points of \( X \) that converges and furthermore \( x_n \rightarrow \bar{x} \). Let \( P_1 : C(X) \rightarrow c \) be defined as follows: if \( f \in C(X) \), \( P_1(f) = \{y_n\} \) where \( y_n = f(x_n) \).

Since \( f \) is continuous \( y_n \rightarrow f(x) \) and \( P_1(f) \in c \). \( P_1 \) is nonexpansive for

\[
 \|P_1(f)\| = \sup_n |f(x_n)| \leq \sup_{x \in \bar{x}} |f(x)| = ||f|| .
\]

An isometric linear map \( Q \) is now constructed from \( c \) into \( C(X) \) such that \( P_1 \circ Q(x) = x \). Let \( \{U_i\} \) be a sequence of open sets such that \( x_i \in U_i, U_i \cap U_j = \emptyset \) if \( i \neq j \), and \( \bar{x} \in U_i \) for all \( i \). For each \( i \) define \( f_i \) to be a function such that \( f_i(x_i) = 1, f_i(X - U_i) = 0 \) and \( 0 \leq f_i(x) \leq 1 \) for all \( x \). If \( \{y_n\} \in c \) and \( y_n \rightarrow y \) then define \( Q(\{y_n\}) = f \) where

\[
 f(x) = \sum_{n=1}^{\infty} f_n(x)(y_n - y) + y .
\]

It is easily verified that \( f \) is continuous, \( f(x_i) = y_i \) and \( ||f|| = ||\{y_n\}|| \). Hence \( Q : c \rightarrow C(X) \) is a linear isometry and
Define \( P: C(X) \to C(X) \) as \( P = Q \circ P_1 \). Since \( P_1 \) is onto and \( Q \) is an isometry then \( ||P|| = 1 \) and \( P \) is a projection, for

\[
P^2 = Q \circ P_1 \circ Q \circ P_1 = Q \circ P_1 = P.
\]

Thus \( P \) is a projection of norm 1 from \( C(X) \) onto \( Q(c) \).

Combining this construction with Lemmas 1 and 2 we have:

**Theorem.** Let \( X \) be a compact Hausdorff space that contains an infinite sequence of distinct points that converge. Then the iterative test is not conclusive for \( C(X) \).

In each of the above, there is a linear selfmap for which the iterative test fails.

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