

# Pacific Journal of Mathematics

**MODULES WHOSE QUOTIENTS HAVE FINITE GOLDIE  
DIMENSION**

VICTOR P. CAMILLO

## MODULES WHOSE QUOTIENTS HAVE FINITE GOLDIE DIMENSION

V. P. CAMILLO

If  $M$  is a module and  $N$  is a submodule of  $M$ , then  $N$  is irreducible in  $M$  if  $N$  cannot be written as a proper intersection of two submodules of  $M$ . The purpose of this note is to study modules whose submodules can be written as a finite intersection of irreducible submodules. Such modules are characterized by the fact that their quotients all have finite Goldie dimension, so they are called q.f.d. modules.

The main result is: A module  $M$  is q.f.d. if and only if every submodule  $N$  has a finitely generated submodule  $T$  such that  $N/T$  has no maximal submodules. Because  $T$  is finitely generated this generalizes a theorem of Shock (using his ideas), who showed a q.f.d. module  $M$  having the property that every subquotient of  $M$  has a maximal submodule must be noetherian (and conversely, of course).

The q.f.d. condition also arises in the study of Krull dimension because a module with Krull dimension must be q.f.d. [1].

First, a remark, which we isolate as a lemma.

LEMMA. A module  $M$  is q.f.d. if and only if each quotient of  $M$  has finite dimensional socle (possibly zero).

*Proof.* Every nonzero module has a quotient with nonzero socle. (Take  $A \subset M$  to be maximal with respect to not containing  $m$ ,  $0 \neq m \in M$ .) So an infinite direct sum of modules has a quotient with infinite dimensional socle.

THEOREM. A module  $M$  is q.f.d. if and only if every submodule  $N$  contains a finitely generated submodule  $T$ , such that  $N/T$  has no maximal submodules.

*Proof.* Suppose that  $X$  is a module such that every finitely generated submodule of  $X$  is contained in a maximal submodule of  $X$ . Having chosen maximal submodules  $M_1, \dots, M_n$  and elements  $x_1, \dots, x_n$  such that  $x_i \notin M_i$ , but  $x_i \in M_j$  for  $j > i$ , choose a maximal submodule  $M_{n+1}$  containing  $x_1R + \dots + x_nR$  and an element  $x_{n+1}$  not in  $M_{n+1}$ .

Let  $\bar{X} = X / \bigcap_{i=1}^{\infty} M_i$ . Then,  $\bar{X} = \bigcap_{i=1}^n M_i \oplus \bigcap_{i=n+1}^{\infty} M_i$ , because, if we denote the right hand summand by  $M$ , we have a strict descending chain  $M \supset M \cap M_n \supset M \cap M_n \cap M_{n-1} \dots$  so that  $M$  has the same

composition length as  $X/\bigcap_{i=1}^n M_i$ .

Thus,  $\bar{X}$  has direct sums of arbitrary size, and so is infinite dimensional.

Conversely, we wish to show that if for every  $N \subset M$ , we can always find such a  $T$ , then  $M$  has no quotient which contains an infinite direct sum. If  $M/K$  does, then by the lemma we can find a  $K'$  with  $M/K'$  having an infinite direct sum of simple submodules. Let  $K' \subset S \subset M$  be such that  $S/K'$  is this infinite direct sum. Choose  $T$  finitely generated such that  $S/T$  has no maximal submodules. Then  $S/T + K'$  has no maximal submodules. But  $S/T + K'$  is a homomorphic image of the semisimple module  $S/K'$  so  $S/T + K'$  is semisimple, and always has maximal submodules if it is not zero. So, it must be zero and  $S = T + K'$ . Since  $T$  is finitely generated  $S/K'$  must be, a clear impossibility.

REMARK. The above naturally raises the question, when are finitely generated modules finite dimensional? We observe the following:

PROPOSITION. *If cyclic modules are finite dimensional then finitely generated modules are.*

*Proof.* Let  $E(M)$  denote the injective hull of the module  $M$ . Let  $M$  be generated by  $\{m_1, \dots, m_r\}$ . Let  $E(M) = E(m_1R) \oplus K_1$ , and write  $m_2 = a_1 + k_1$ . Then  $K_1 = E(k_1R) \oplus K_2$ . So  $E(M) = E(m_1R) \oplus E(k_1R) \oplus K_2$ , and  $m_1R + m_2R \subset E(m_1R) \oplus E(k_1R)$ . Continue in this fashion to get  $E(M)$  as a finite direct sum of injective hulls of cyclic modules. These are finite dimensional, so  $M$  is.

Since there are non-noetherian valuation rings (ideals are linearly ordered), there are non-noetherian rings whose finitely generated modules are finite dimensional. The result cited in the first paragraph is 4.10 of [4].

#### REFERENCES

1. Robert Gordon and J. C. Robson, *Krull Dimension*, Amer. Math. Soc. Memoir 133.
2. ———, *The Gabriel dimension of a module*, J. Algebra, **29**, (1974), 459-573.
3. Robert Shock, *Dual generalizations of the Artinian and noetherian conditions*, Pacific J. Math., **54** (1974), 227-235.
4. Sharpe and Vámos, *Injective Modules*, Cambridge University Press, 1972.

Received July 22, 1976.

UNIVERSITY OF IOWA  
IOWA CITY, IA 52242

# PACIFIC JOURNAL OF MATHEMATICS

## EDITORS

RICHARD ARENS (Managing Editor)

University of California  
Los Angeles, CA 90024

R. A. BEAUMONT

University of Washington  
Seattle, WA 98105

C. C. MOORE

University of California  
Berkeley, CA 94720

J. DUGUNDJI

Department of Mathematics  
University of Southern California  
Los Angeles, CA 90007

R. FINN and J. MILGRAM

Stanford University  
Stanford, CA 94305

## ASSOCIATE EDITORS

E. F. BECKENBACH

B. H. NEUMANN

F. WOLF

K. YOSHIDA

## SUPPORTING INSTITUTIONS

UNIVERSITY OF BRITISH COLUMBIA  
CALIFORNIA INSTITUTE OF TECHNOLOGY  
UNIVERSITY OF CALIFORNIA  
MONTANA STATE UNIVERSITY  
UNIVERSITY OF NEVADA  
NEW MEXICO STATE UNIVERSITY  
OREGON STATE UNIVERSITY  
UNIVERSITY OF OREGON  
OSAKA UNIVERSITY

UNIVERSITY OF SOUTHERN CALIFORNIA  
STANFORD UNIVERSITY  
UNIVERSITY OF HAWAII  
UNIVERSITY OF TOKYO  
UNIVERSITY OF UTAH  
WASHINGTON STATE UNIVERSITY  
UNIVERSITY OF WASHINGTON  
\* \* \*  
AMERICAN MATHEMATICAL SOCIETY

---

The Supporting Institutions listed above contribute to the cost of publication of this Journal, but they are not owners or publishers and have no responsibility for its content or policies.

---

Mathematical papers intended for publication in the *Pacific Journal of Mathematics* should be in typed form or offset-reproduced, (not dittoed), double spaced with large margins. Please do not use built up fractions in the text of your manuscript. You may however, use them in the displayed equations. Underline Greek letters in red, German in green, and script in blue. The first paragraph or two must be capable of being used separately as a synopsis of the entire paper. Items of the bibliography should not be cited there unless absolutely necessary, in which case they must be identified by author and Journal, rather than by item number. Manuscripts, in triplicate, may be sent to any one of the editors. Please classify according to the scheme of Math. Reviews, Index to Vol. **39**. All other communications should be addressed to the managing editor, or Elaine Barth, University of California, Los Angeles, California, 90024.

The Pacific Journal of Mathematics expects the author's institution to pay page charges, and reserves the right to delay publication for nonpayment of charges in case of financial emergency

100 reprints are provided free for each article, only if page charges have been substantially paid. Additional copies may be obtained at cost in multiples of 50.

---

The *Pacific Journal of Mathematics* is issued monthly as of January 1966. Regular subscription rate: \$72.00 a year (6 Vols., 12 issues). Special rate: \$36.00 a year to individual members of supporting institutions.

Subscriptions, orders for back numbers, and changes of address should be sent to Pacific Journal of Mathematics, 103 Highland Boulevard, Berkeley, California, 94708.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION

Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.).  
8-8, 3-chome, Takadanobaba, Shinjuku-ku, Tokyo 160, Japan.

Copyright © 1975 by Pacific Journal of Mathematics  
Manufactured and first issued in Japan

# Pacific Journal of Mathematics

Vol. 69, No. 2

June, 1977

|  |     |
|--|-----|
| Carol Alf and Thomas Alfonso O'Connor, <i>Unimodality of the Lévy spectral function</i> .....  | 285 |
| S. J. Bernau and Howard E. Lacey, <i>Bicontractive projections and reordering of <math>L_p</math>-spaces</i> .....                       | 291 |
| Andrew J. Berner, <i>Products of compact spaces with bi-k and related spaces</i> .....   | 303 |
| Stephen Richard Bernfeld, <i>The extendability and uniqueness of solutions of ordinary differential equations</i> .....                  | 307 |
| Marilyn Breen, <i>Decompositions for nonclosed planar <math>m</math>-convex sets</i> .....   | 317 |
| Robert F. Brown, <i>Cohomology of homomorphisms of Lie algebras and Lie groups</i> .....   | 325 |
| Jack Douglas Bryant and Thomas Francis McCabe, <i>A note on Edelstein's iterative test and spaces of continuous functions</i> .....      | 333 |
| Victor P. Camillo, <i>Modules whose quotients have finite Goldie dimension</i> .....   | 337 |
| David Downing and William A. Kirk, <i>A generalization of Caristi's theorem with applications to nonlinear mapping theory</i> .....      | 339 |
| Daniel Reuven Farkas and Robert L. Snider, <i>Noetherian fixed rings</i> .....   | 347 |
| Alessandro Figà-Talamanca, <i>Positive definite functions which vanish at infinity</i> .....   | 355 |
| Josip Globevnik, <i>The range of analytic extensions</i> .....   | 365 |
| André Goldman, <i>Mesures cylindriques, mesures vectorielles et questions de concentration cylindrique</i> .....                         | 385 |
| Richard Grassl, <i>Multisectioned partitions of integers</i> .....   | 415 |
| Haruo Kitahara and Shinsuke Yorozu, <i>A formula for the normal part of the Laplace-Beltrami operator on the foliated manifold</i> ..... | 425 |
| Marvin J. Kohn, <i>Summability <math>R_r</math> for double series</i> .....  | 433 |
| Charles Philip Lanski, <i>Lie ideals and derivations in rings with involution</i> .....  | 449 |
| Solomon Leader, <i>A topological characterization of Banach contractions</i> .....   | 461 |
| Daniel Francis Xavier O'Reilly, <i>Cobordism classes of fiber bundles</i> .....  | 467 |
| James William Pendergrass, <i>The Schur subgroup of the Brauer group</i> .....   | 477 |
| Howard Lewis Penn, <i>Inner-outer factorization of functions whose Fourier series vanish off a semigroup</i> .....                       | 501 |
| William T. Reid, <i>Some results on the Floquet theory for disconjugate definite Hamiltonian systems</i> .....                           | 505 |
| Caroll Vernon Riecke, <i>Complementation in the lattice of convergence structures</i> .....  | 517 |
| Louis Halle Rowen, <i>Classes of rings torsion-free over their centers</i> .....   | 527 |
| Manda Butchi Suryanarayana, <i>A Sobolev space and a Darboux problem</i> .....   | 535 |
| Charles Thomas Tucker, II, <i>Riesz homomorphisms and positive linear maps</i> .....   | 551 |
| William W. Williams, <i>Semigroups with identity on Peano continua</i> .....   | 557 |
| Yukinobu Yajima, <i>On spaces which have a closure-preserving cover by finite sets</i> .....   | 571 |