MODULES WHOSE QUOTIENTS HAVE FINITE GOLDIE DIMENSION

VICTOR P. CAMILLO
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If $M$ is a module and $N$ is a submodule of $M$, then $N$ is irreducible in $M$ if $N$ cannot be written as a proper intersection of two submodules of $M$. The purpose of this note is to study modules whose submodules can be written as a finite intersection of irreducible submodules. Such modules are characterized by the fact that their quotients all have finite Goldie dimension, so they are called q.f.d. modules.

The main result is: A module $M$ is q.f.d. if and only if every submodule $N$ has a finitely generated submodule $T$ such that $N/T$ has no maximal submodules. Because $T$ is finitely generated this generalizes a theorem of Shock (using his ideas), who showed a q.f.d. module $M$ having the property that every subquotient of $M$ has a maximal submodule must be noetherian (and conversely, of course).

The q.f.d. condition also arises in the study of Krull dimension because a module with Krull dimension must be q.f.d. [1].

First, a remark, which we isolate as a lemma.

**Lemma.** A module $M$ is q.f.d. if and only if each quotient of $M$ has finite dimensional socle (possible zero).

**Proof.** Every nonzero module has a quotient with nonzero socle. (Take $A \subset M$ to be maximal with respect to not containing $m, 0 \neq m \in M$.) So an infinite direct sum of modules has a quotient with infinite dimensional socle.

**Theorem.** A module $M$ is q.f.d. if and only if every submodule $N$ contains a finitely generated submodule $T$, such that $N/T$ has no maximal submodules.

**Proof.** Suppose that $X$ is a module such that every finitely generated submodule of $X$ is contained in a maximal submodule of $X$. Having chosen maximal submodules $M_i$, $\cdots$, $M_n$ and elements $x_i, \cdots, x_n$ such that $x_i \notin M_i$ but $x_i \in M_j$ for $j > i$, choose a maximal submodule $M_{n+1}$ containing $x_n R + \cdots + x_n R$ and an element $x_{n+1}$ not in $M_{n+1}$.

Let $\bar{X} = X/\bigcap_{i=1}^{\infty} M_i$. Then, $\bar{X} = \bigcap_{i=1}^{\infty} M_i \oplus \bigcap_{i=n+1}^{\infty} M_i$, because, if we denote the right hand summand by $M$, we have a strict descending chain $M \supset M \cap M_n \supset M_n \cap M_{n-1} \cdots$ so that $M$ has the same
composition length as $X/\bigcap_{i=1}^n M_i$.

Thus, $\tilde{X}$ has direct sums of arbitrary size, and so is infinite dimensional.

Conversely, we wish to show that if for every $N \subseteq M$, we can always find such a $T$, then $M$ has no quotient which contains an infinite direct sum. If $M/K$ does, then by the lemma we can find a $K'$ with $M/K'$ having an infinite direct sum of simple submodules. Let $K' \subseteq S \subseteq M$ be such that $S/K'$ is this infinite direct sum. Choose $T$ finitely generated such that $S/T$ has no maximal submodules. Then $S/T + K'$ has no maximal submodules. But $S/T + K'$ is a homomorphic image of the semisimple module $S/K'$ so $S/T + K'$ is semisimple, and always has maximal submodules if it is not zero. So, it must be zero and $S = T + K'$. Since $T$ is finitely generated $S/K'$ must be, a clear impossibility.

REMARK. The above naturally raises the question, when are finitely generated modules finite dimensional? We observe the following:

PROPOSITION. If cyclic modules are finite dimensional then finitely generated modules are.

Proof. Let $E(M)$ denote the injective hull of the module $M$. Let $M$ be generated by $\{m_1, \ldots, m_r\}$. Let $E(M) = E(m_i R) \oplus K_i$, and write $m_2 = a_i + k_i$. Then $K_i = E(k_i R) \oplus K_i$. So $E(M) = E(m_i R) \oplus E(k_i R) \oplus K_i$, and $m_i R + m_i R \subseteq E(m_i R) \oplus E(k_i R)$. Continue in this fashion to get $E(M)$ as a finite direct sum of injective hulls of cyclic modules. These are finite dimensional, so $M$ is.

Since there are non-noetherian valuation rings (ideals are linearly ordered), there are non-noetherian rings whose finitely generated modules are finite dimensional. The result cited in the first paragraph is 4.10 of [4].

REFERENCES


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