

Pacific Journal of Mathematics

**A GENERALIZATION OF CARISTI'S THEOREM WITH
APPLICATIONS TO NONLINEAR MAPPING THEORY**

DAVID DOWNING AND WILLIAM A. KIRK

A GENERALIZATION OF CARISTI'S THEOREM WITH APPLICATIONS TO NONLINEAR MAPPING THEORY

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Suppose X and Y are complete metric spaces, $g: X \rightarrow X$ an arbitrary mapping, and $f: X \rightarrow Y$ a closed mapping (thus, for $\{x_n\} \subset X$ the conditions $x_n \rightarrow x$ and $f(x_n) \rightarrow y$ imply $f(x) = y$). It is shown that if there exists a lower semicontinuous function φ mapping $f(X)$ into the nonnegative real numbers and a constant $c > 0$ such that for all x in X , $\max\{d(x, g(x)), cd(f(x), f(g(x)))\} \leq \varphi(f(x)) - \varphi(f(g(x)))$, then g has a fixed point in X . This theorem is then used to prove surjectivity theorems for nonlinear closed mappings $f: X \rightarrow Y$, where X and Y are Banach spaces.

1. Introduction. The following fact is well-known in the theory of linear operators;

(1.1) Let X and Y be Banach spaces with D a dense subspace of X , and let $T: D \rightarrow Y$ be a closed linear mapping with dual T' . Suppose the following two conditions hold:

(i) $N(T') = \{0\}$.

(ii) For fixed $c > 0$, $\text{dist}(x, N(T)) \leq c \|Tx\|$, $x \in D$.

Then $T(D) = Y$.

Proof. Because T is a closed mapping it routinely follows from (ii) that $T(D)$ is closed in Y (e.g., [15, p. 72]), whence it follows from the Hahn-Banach theorem (cf. [17, p. 205]) that $(N(T'))^\perp = T(D)$ where $(N(T'))^\perp$ denotes the annihilator in Y of the nullspace of T' . By (i), $(N(T'))^\perp = Y$.

It is our objective in this paper to give a nonlinear generalization of the above along with more technical related results. The key to our approach is an application of a new generalized version of Caristi's fixed point theorem. While our method parallels that of Kirk and Caristi [12], these new results differ from those of [12] and the earlier 'normal solvability' results of others, e.g., Altman [1], Browder [3-6], Pohozhayev [13, 14], and Zabreiko-Krasnoselskii [18], in that by using the improved fixed point theorem we are able to replace the usual closed range assumption with the assumption that the mapping be closed (in conjunction with a condition which in the linear case reduces to (ii)). Before doing this, however, we state and prove our fixed point theorem.

2. **The fixed point theorem.** The following theorem reduces to the theorem of Caristi [7, 8] in the case that $X = Y$, f is the identity mapping, and $c = 1$. (We should remark that Caristi's theorem is essentially equivalent to a theorem stated earlier by Ekeland [9]. A simple proof along the general lines below is implicit in Brøndsted [2]. A similar proof is given by Kasahara in [10], and in [16] Wong gives a simplified version of Caristi's original transfinite induction argument.)

THEOREM 2.1. *Let X and Y be complete metric spaces and $g: X \rightarrow X$ an arbitrary mapping. Suppose there exists a closed mapping $f: X \rightarrow Y$, a lower semicontinuous mapping $\varphi: f(X) \rightarrow R^+$, and a constant $c > 0$ such that for each $x \in X$,*

$$(*) \quad \begin{cases} d(x, g(x)) \leq \varphi(f(x)) - \varphi(f(g(x))), & \text{and} \\ cd(f(x), f(g(x))) \leq \varphi(f(x)) - \varphi(f(g(x))). \end{cases}$$

Then there exists $\bar{x} \in X$ such that $g(\bar{x}) = \bar{x}$.

Proof. We introduce a partial order \leq in X as follows. For $x, y \in X$ define $x \leq y$ provided

$$\begin{cases} d(x, y) \leq \varphi(f(x)) - \varphi(f(y)), & \text{and} \\ cd(f(x), f(y)) \leq \varphi(f(x)) - \varphi(f(y)). \end{cases}$$

Let $\{x_\alpha\}_{\alpha \in I}$ be any chain in X , i.e., suppose (I, \leq) is a totally ordered set with $x_\alpha \leq x_\beta$ iff $\alpha \leq \beta$. Then $\{\varphi(f(x_\alpha))\}_{\alpha \in I}$ is a decreasing net in R^+ so there exists $r \geq 0$ such that $\varphi(f(x_\alpha)) \downarrow r$. Let $\varepsilon > 0$. Then there exists $\alpha_0 \in I$ such that $\alpha \geq \alpha_0$ implies

$$r \leq \varphi(f(x_\alpha)) \leq r + \varepsilon$$

and so for $\beta \geq \alpha$,

$$\begin{aligned} d(x_\alpha, x_\beta) &\leq \varphi(f(x_\alpha)) - \varphi(f(x_\beta)) \leq \varepsilon, & \text{and} \\ cd(f(x_\alpha), f(x_\beta)) &\leq \varphi(f(x_\alpha)) - \varphi(f(x_\beta)) \leq \varepsilon. \end{aligned}$$

Thus $\{f(x_\alpha)\}$ is a Cauchy net in Y while $\{x_\alpha\}$ is a Cauchy net in X . By completeness there exist $\bar{y} \in Y$ and $\bar{x} \in X$ such that $f(x_\alpha) \rightarrow \bar{y}$ and $x_\alpha \rightarrow \bar{x}$. Since f is a closed mapping, $f(\bar{x}) = \bar{y}$ and lower-semicontinuity of φ yields $\varphi(f(\bar{x})) \leq r$. Moreover, if $\alpha, \beta \in I$ with $\alpha \leq \beta$, then

$$\begin{aligned} d(x_\alpha, x_\beta) &\leq \varphi(f(x_\alpha)) - \varphi(f(x_\beta)) \leq \varphi(f(x_\alpha)) - r; \\ cd(f(x_\alpha), f(x_\beta)) &\leq \varphi(f(x_\alpha)) - r. \end{aligned}$$

Taking limits with respect to β yields

$$\begin{aligned} d(x_\alpha, \bar{x}) &\leq \varphi(f(x_\alpha)) - r \leq \varphi(f(x_\alpha)) - \varphi(f(\bar{x})); \\ cd(f(x_\alpha), f(\bar{x})) &\leq \varphi(f(x_\alpha)) - \varphi(f(\bar{x})). \end{aligned}$$

This proves that $x_\alpha \leq \bar{x}$, $\alpha \in I$.

Having thus shown that every totally ordered set in (X, \leq) has an upper bound we apply Zorn's lemma to obtain maximal element $x \in X$. By (*), $x \leq g(x)$; hence $x = g(x)$.

3. Applications. If X and Y are topological vector spaces and $f: X \rightarrow Y$, then f is said to be *Gâteaux differentiable* at $x \in X$ if there exists a (possibly unbounded) linear operator $L: X \rightarrow Y$ such that for each $w \in X$,

$$t^{-1}(f(x + tw) - f(x)) \longrightarrow Lw \quad \text{as } t \longrightarrow 0^+.$$

The operator $L = df_x$ is called the Gâteaux derivative of f at x and we use df'_x to denote the dual of df_x in the usual sense (e.g., [17, p. 194]).

We now state a theorem which is an immediate generalization of the theorem of the introduction. Notationally, we let $B_\delta(\cdot)$ denote the closed ball centered at (\cdot) with radius δ . Also, $N(df'_x)$ denotes the nullspace of df'_x in Y^* , the space of all continuous linear functionals on Y , and $(N(df'_x))^\perp$ denotes its annihilator in Y .

THEOREM 3.1. *Let X and Y be Banach spaces and $f: X \rightarrow Y$ a (nonlinear) closed mapping which is Gâteaux differentiable at each $x \in X$ with derivative df_x . Let df'_x denote the dual of df_x , and suppose for each $x \in X$ and fixed $c > 0$:*

$$(i)' \quad N(df'_x) = \{0\}.$$

(ii)' *There exists $\delta = \delta(x) > 0$ such that if $y \in B_\delta(f(x)) \cap f(X)$, then for some $v \in f^{-1}(y)$,*

$$\|x - v\| \leq c \|f(x) - y\|.$$

Then $f(X) = Y$.

It is obvious that (i)' reduces to (i) in the linear case and it is a routine matter to show that (ii)' similarly reduces to (ii). In contrast with the linear case, however, we do not show directly that (ii)' implies closedness of the range of f . Instead we derive Theorem 3.1 from the following more general result which follows quite easily from Theorem 2.1.

THEOREM 3.2. *Suppose X is a complete metric space, Y a Banach space, and $f: X \rightarrow Y$ a closed mapping. Suppose for $y_0 \in Y$ there exist constants $c > 0$, $p < 1$ such that:*

(a) Corresponding to each $x \in X$ there exists $\delta = \delta(x) > 0$ such that if $y \in B_\delta(f(x)) \cap f(X)$, then

$$d(x, v) \leq c \|f(x) - y\|$$

for some $v \in f^{-1}(y)$.

(b) For each $y \in f(X)$ there exists a sequence $\{y_j\}$ in $f(X)$ with $y_j \neq y$ for each j such that $y_j \rightarrow y$ and a sequence $\{\xi_j\}$ of nonnegative real numbers such that for each j

$$\|\xi_j(y_j - y) - (y_0 - y)\| \leq p \|y_0 - y\|.$$

Then $y_0 \in f(X)$.

The following geometric lemma, implicit in [12], will facilitate the proof of Theorem 3.2.

LEMMA. Let Y be a normed linear space with $a, b, c \in Y$. Suppose for $\xi \geq 1$ and $p < 1$,

$$(*) \quad \|\xi(a - b) - (c - b)\| \leq p \|c - b\|.$$

Then

$$\|a - b\| \leq (1 + p)(1 - p)^{-1} [\|b - c\| - \|a - c\|].$$

Proof.

$$\begin{aligned} & \|\xi(a - c)\| - \|(1 - \xi)(b - c)\| \\ & \leq \|\xi(a - c) + (1 - \xi)(b - c)\| \\ & = \|\xi(a - b) - (c - b)\| \\ & \leq p \|b - c\|. \end{aligned}$$

Thus $\|\xi(a - c)\| \leq (\xi - 1 + p) \|b - c\|$, i.e.,

$$\|a - c\| \leq [1 - \xi^{-1}(1 - p)] \|b - c\|$$

from which (using $(*)$ and the triangle inequality)

$$\begin{aligned} \|b - c\| - \|a - c\| & \geq \{1 - [1 - \xi^{-1}(1 - p)]\} \|b - c\| \\ & = \xi^{-1}(1 - p) \|b - c\| \\ & \geq \xi^{-1}(1 - p) \xi (1 + p)^{-1} \|a - b\| \\ & = (1 - p)(1 + p)^{-1} \|a - b\|. \end{aligned}$$

Proof of Theorem 3.2. Suppose $y_0 \in f(X)$. Let $x \in X$ and $y = f(x)$, and let $\{y_j\}$ be the sequence defined by (b). Since $y_j \rightarrow y$, j may be chosen so large that $\|y_j - y\| \leq \delta(x)$. We also assume $\xi_j \geq 1$. (Note that since $y_0 \neq y$, (b) implies $\xi_j \rightarrow +\infty$.) With j thus fixed we apply the lemma to the inequality in (b) and obtain

$$(1) \quad 0 < \|y - y_j\| \leq (1 + p)(1 - p)^{-1}[\|y - y_0\| - \|y_j - y_0\|].$$

By (a) there exists $v \in f^{-1}(y_j)$ such that

$$(2) \quad d(x, v) \leq c \|y - y_j\|.$$

Define $g: X \rightarrow X$ by taking $g(x) = v$ with v obtained as above, and define $\varphi: f(X) \rightarrow R^+$ by

$$\varphi(f(x)) = c(1 + p)(1 - p)^{-1} \|f(x) - y_0\|.$$

Then clearly φ is continuous on $f(X)$ and together (1) and (2) yield

$$\begin{cases} d(x, g(x)) \leq \varphi(f(x)) - \varphi(f(g(x))), & \text{and} \\ c \|f(x) - f(g(x))\| \leq \varphi(f(x)) - \varphi(f(g(x))). \end{cases}$$

By Theorem 2.1 there exists $\bar{x} \in X$ such that $g(\bar{x}) = \bar{x}$, contradicting (1).

In order to derive Theorem 3.1 from Theorem 3.2 we need an elementary fact from linear algebra. Let X and Y be locally convex topological vector spaces and suppose $L: X \rightarrow Y$ is a linear operator. The dual L' of L (cf. [17, p. 194]) is defined on a subset D of Y^* by the relation

$$\langle x, L'y' \rangle = \langle Lx, y' \rangle, \quad y' \in D, \quad x \in X$$

where X^* and Y^* denote respectively the spaces of continuous linear functionals on X and Y and where by assumption $\langle \cdot, L'y' \rangle \in X^*$. If $(N(L'))^\perp$ denotes the annihilator of $N(L')$ in Y it routinely follows from the Hahn-Banach theorem that $(N(L'))^\perp \subset \overline{L(X)}$. (For, suppose there exists $y_0 \in (N(L'))^\perp$ with $y_0 \notin \overline{L(X)}$. Then there exists $y' \in Y^*$ such that $\langle y_0, y' \rangle \neq 0$ while $\langle z, y' \rangle = 0$ for all $z \in \overline{L(X)}$; hence $\langle Lu, y' \rangle = \langle u, L'y' \rangle = 0$ for all $u \in X$ yielding $L'y' = 0$, i.e., $y' \in N(L')$. Since $y_0 \in (N(L'))^\perp$ implies $\langle y_0, y' \rangle = 0$, we have a contradiction.)

We now follow an approach of Browder [4, 6]. With X as above and Y a Banach space the asymptotic direction set of the mapping $f: X \rightarrow Y$ in the direction $x \in X$ is the set

$$D_x(f) = \bigcap_{\epsilon > 0} c1(\{y \in Y \mid y = \xi(f(u) - f(x)), \xi \geq 0, u \in X, \|f(u) - f(v)\| < \epsilon\}).$$

The following is a minor variant of Proposition 1 of [4, 6]. We include the proof to show that continuity of df_x is not essential.

PROPOSITION 3.1. *Let X be a locally convex topological vector space, Y a Banach space, and suppose f is a mapping of X into Y which is Gâteaux differentiable at $x \in X$ with derivative df_x . If*

$N(df'_x)$ denotes the nullspace in Y' of the dual of df_x and if $(N(df'_x))^\perp$ denotes its annihilator in Y , then

$$(N(df'_x))^\perp = \overline{df_x(X)} \subset D_x(f).$$

Proof. The equality is immediate from observations above. To see that $\overline{df_x(X)} \subset D_x(f)$ we follow [6]: Let $\varepsilon > 0$ and $y \in df_x(X)$. Then $y = df_x(w)$ for some $w \in X$ and by differentiability

$$(\#) \quad t^{-1}(f(x + tw) - f(x)) \longrightarrow y \quad \text{as } t \longrightarrow 0^+.$$

Letting $x_t = x + tw$ we have for $t > 0$ sufficiently small, $\|f(x_t) - f(x)\| < \varepsilon$. It follows from this and (#) that

$$y \in \text{cl}\{\xi(f(u) - f(x)) \mid \xi \geq 0, u \in X, \|f(u) - f(x)\| < \varepsilon\};$$

i.e., $y \in D_x(f)$. Since $D_x(f)$ is closed, $\overline{df_x(X)} \subset D_x(f)$.

Proof of Theorem 3.1. Let $y_0 \in Y$, $p \in (0, 1)$. It suffices to establish (b) of Theorem 3.2. Suppose $y = f(x) \in f(X)$, $y \neq y_0$. Since $N(df'_x) = \{0\}$, $(N(df'_x))^\perp = Y$ and by Proposition 3.1

$$y_0 - f(x) \in D_x(f).$$

Choose $\varepsilon_j > 0$ with $\varepsilon_j \rightarrow 0$. For each j there exists $z_j \in X$ and $\xi_j \geq 0$ such that

$$(3) \quad \|\xi_j(f(z_j) - f(x)) - (y_0 - f(x))\| \leq p \|y_0 - f(x)\|$$

and

$$(4) \quad \|f(z_j) - f(x)\| < \varepsilon_j.$$

Letting $y_j = f(z_j)$, since $p < 1$ (3) implies $y_j \neq y$ for all j . By (4) $y_j \rightarrow y$ as $j \rightarrow \infty$ and rewriting (3) we have

$$\|\xi_j(y_j - y) - (y_0 - y)\| \leq p \|y_0 - y\|.$$

This completes the proof.

Finally we note that if $\text{int } \overline{f(X)} \neq \emptyset$, it is not necessary in Theorem 3.1 to assume f is differentiable at each $x \in X$.

THEOREM 3.3. *Let X and Y be Banach spaces and $f: X \rightarrow Y$ a closed mapping. Let $N = \{x \in X \mid f(x) \in \text{int } \overline{f(X)}\}$ and suppose for $x \in X \setminus N$, f is Gâteaux differentiable with derivative df_x where $N(df'_x) = \{0\}$. Suppose also that there exists $c > 0$ such that condition (ii)' of Theorem 3.1 holds for all $x \in X$. Then $f(X) = Y$.*

Proof. Let $y_0 \in Y$ and suppose $y_0 \notin f(X)$. Fix $p \in (0, 1)$ and let $x \in X$. If $x \in X \setminus N$, then $(N(df'_x))^{\perp} = Y$ and by Proposition 3.1, $y_0 - f(x) \in D_x(f)$. But also if $x \in N$, i.e., if $f(x) \in \text{int } \overline{f(X)}$, then for $\varepsilon > 0$ chosen so that $B_\varepsilon(f(x)) \subset \overline{f(X)}$ it is possible to select $w \in \text{seg}[f(x), y_0]$ so that $w \neq y_0$ and $0 < \|f(x) - w\| < \varepsilon$, and because $w \in \overline{f(X)}$ there exists $\{w_j\} \subset f(X)$ such that $w_j \rightarrow w$. Since $y_0 - f(x) = \xi(w - f(x))$ for $\xi > 0$ it thus follows that $\xi(w_j - f(x)) \rightarrow y_0 - f(x)$ with $\|w_j - f(x)\| < \varepsilon$ for j sufficiently large proving $y_0 - f(x) \in D_x(f)$. Since $y_0 - f(x) \in D_x(f)$, the proof now follows the proof of Theorem 3.1.

We remark that as a consequence of the above theorem, if $f: X \rightarrow Y$ is a closed mapping with range dense in Y , then (ii)' of Theorem 3.1 implies $f(X) = Y$.

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Received July 16, 1976, Research supported in part by a National Science Foundation grant.

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Subscriptions, orders for back numbers, and changes of address should be sent to Pacific Journal of Mathematics, 103 Highland Boulevard, Berkeley, California, 94708.

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Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.).
8-8, 3-chome, Takadanobaba, Shinjuku-ku, Tokyo 160, Japan.

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