Pacific Journal of Mathematics

NOETHERIAN FIXED RINGS

DANIEL REUVEN FARKAS AND ROBERT L. SNIDER

Vol. 69, No. 2 June 1977

PACIFIC JOURNAL OF MATHEMATICS
Vol. 69, No. 2, 1977

NOETHERIAN FIXED RINGS

DANIEL R. FARKAS AND ROBERT L. SNIDER

One of the basic questions of noncommutative Galois theory is the relation between a ring R and the ring S fixed by a group of automorphisms of R. This paper explores what happens when the group is finite and the fixed ring S is assumed to be Noetherian. Easy examples show that R may not be Noetherian; however, in this paper it is shown that R is Noetherian with some rather natural assuptions. More precisely we prove the Theorem 2: Let S be a semiprime ring. Assume that G is a finite group of automorphisms of S and that S has no $|G|$-torsion. If S^G is left noetherian then S is left noetherian.

Theorem 2 answers a question raised by Fisher and Osterburg [4]. This result rests on calculations which can best be described as belonging to noncommutative Galois theory. The basic theorem here may be of independent interest.

THEOREM 1. *Let R be a semisimple artinian ring. If G is a finite group of automorphisms of R and $|G|$ is invertible in R then R is a finitely generated ring R^G-module.*

The proof of Theorem 1 follows the spirit of Karchenko's work on polynomial identity rings ([6]).

1. A proof of Theorem 1.

We will repeatedly need Levitzki's fixed ring theorem ([8]): Suppose R is a semisimple artinian ring. If G is a finite group of automorphisms of R with $|G|$ invertible in R then R^G is semisimple artinian.

LEMMA 1. *If Theorem 1 is true when G is a simple group then it is true for an arbitrary finite G.*

Proof. By induction on the length of a composition series for G. If G is not already simple choose $H \Delta G$ with $1 \neq H \neq G$. By Levitzki's theorem R^H is semisimple artinian. G/H acts on R^H and R^H has no $|G/H|$-torsion; by induction R^H is a finitely generated right R^G-module. Again, induction shows that R is a finitely generated right R^H-module. The lemma follows.

We eventually assume that G is simple. In that case either G consists entirely of outer automorphisms or entirely of inner automorphisms.

LEMMA 2. *Let B be a simple artinian ring and let G be a finite group of outer automorphisms of B. Then B is a finitely generated right B^G-module.*

Proof. By [1], B^G is a simple ring and B is a free module over B^G of rank $|G|$. (Cf. [5] for the case of a division ring.)

LEMMA 3. *Let B bė a simple artinian ring and let G be a finite group of inner automorphisms of B. Assume $|G|$ is invertible in B. Then B is a finitely generated right B^G-module.*

Proof. Let F be the center of B.

For each $g \in G$ pick one $x \in B$ such that ${}^g b = xbx^{-1}$ for all $b \in B$. Call the finite set so chosen, $\bar{G}$. Then collection of sums, $F\bar{G}$, is a finite dimensional algebra over F. Since $1/|G| \in F$, Maschke's theorem for twisted group algebras ([9]) states that $F\bar{G}$ is a separable algebra. Thus there is a finite extension field K of F such that K is a splitting field for each simple constituent of $F\bar{G}$.

$K \otimes_F B$ is a simple artinian ring with center K. G acts on $K \otimes_F B$ by

$$ {}^g(k \otimes b) = k \otimes {}^g b \ . $$

Obviously this action, too, is induced by inner automorphisms. A straight-forward calculation shows that $(K \otimes B)^G = K \otimes B^G$. Similarly, if $K \otimes B$ is a finitely generated right $(K \otimes B)^G$-module then B is a finitely generated B^G-module.

Thus we replace B with $K \otimes_F B$ and assume each simple constituent of $F\bar{G}$ is a total matrix ring with entires in F. Let $\mathscr{E}$ be the set of centrally primitive idempotents in $F\bar{G}$.

The crux of this lemma is to show that if $e \in \mathscr{E}$ then eBe is a finitely generated right B^G-module. An element of B^G commutes with elements of $F\bar{G}$ so it certainly commutes with e; hence eBe is a right B^G-module. Let ε_{ij} be a set of matrix units for $eF\bar{G}$. If x is in eBe, set

$$ \pi_{ij}(x) = \sum_k \varepsilon_{ki} x \varepsilon_{jk} $$

$\pi_{ij}(x)$ commutes with each of the matrix units. Since F is the center of B, it commutes with $eF\bar{G}$. Thus it commutes with $F\bar{G}$. In other words, $\pi_{ij}(x)$ is in B^G. The map $\pi_{ij}\colon eBe \rightarrow B^G$ is a right B^G-module map by the argument at the beginning of this paragraph. We claim that the map

$$ \sum_{i,j} \pi_{ij}\colon eBe \longrightarrow \bigoplus \sum_{i,j} B^G $$

is injective. For if $\sum_k \varepsilon_{ki}x\varepsilon_{jk} = 0$ for all i and j, multiple on the right by ε_{ij}:

$$\varepsilon_{ii}x\varepsilon_{jj} = 0 \quad \text{for all } i \text{ and } j.$$

Hence $exe = 0$. But $x \in eBe$ implies $exe = x$. We finish this paragraph by noticing that Levitzki's theorem says that B^G is right noetherian. Since eBe is isomorphic to a submodule of a finitely generated B^G-module, eBe is finitely generated.

Next we show that if e and f are different elements of $\mathscr{E}$ then fBe is a finitely generated right B^G-module. (Of course it is a B^G-module as above.) Since B is simple, $BeB = B$. Thus we can choose $v_i \in fBe$ and $u_i \in eBf$ so that

$$f = \sum_i v_i u_i \, .$$

Define $\varphi: fBe \to \bigoplus \sum_i eBe$ by $\varphi(y) = (u_i y)$, a right B^G-module map. $\varphi(y) = 0 \Rightarrow u_i y = 0$ for each $i \Rightarrow (\sum v_i u_i)y = 0 \Rightarrow fy = 0$. But $fy = y$. Hence φ is injective. Finish the argument as before.

Because $B = \sum_{e,f \in \mathscr{E}} fBe$, B is a finitely generated right B^G-module.

Proof of Theorem 1. Induct on the order of G. Assume G is simple.

Let e be a centrally primitive idempotent in R. eR is a simple artinian ring. Moreover the stabilizer $H = \mathrm{Stab}_G(e)$ acts on eR and $1/|H|e \in eR$. By Lemmas 2 and 3, eR is a finitely generated right $(eR)^H$-module.

Claim. $(eR)^H = e(R^G)$.

Certainly $e(R^G) \subseteqq (eR)^H$. Let $G = \bigcup_{\gamma \in \Gamma} \gamma H$ be a coset decomposition of G with $1 \in \Gamma$. G permutes the centrally primitive idempotents of R and for $\alpha \neq \beta$ in Γ, ${}^{\alpha}e \neq {}^{\beta}e$. Equivalently, if $\gamma \neq 1$ is in Γ, $e({}^{\gamma}e) = 0$. If $x \in (eR)^H$ define $t_\Gamma(x) = \sum_{\gamma \in \Gamma} ({}^{\gamma}x)$. If $g \in G$, $\{g\gamma \mid \gamma \in \Gamma\}$ are also coset representatives for H. Thus ${}^{g}t_\Gamma(x) = t_\Gamma(x)$. That is, $t_\Gamma(x) \in R^G$. But $et_\Gamma(x) = x$ by the remarks above about multiplying idempotents. Thus $(eR)^H \subseteqq (eR^G)$.

We now know that eR is a finitely generated right $e(R^G)$-module. That means eR is a finitely generated R^G-module. Since $R = \sum_e eR$, we are done.

2. Theorem 2 and its relatives.

LEMMA 4. *Let A be a semiprime ring. Assume G is a finite group of automorphisms of A and A has no $|G|$-torsion. Then tr_G does not vanish on any nonzero right ideal of A.*

$$(\text{Here } tr_G(a) = \sum_{g \in G} ({}^g a).)$$

Proof. Suppose I is a right ideal of A with $tr_G(I) = 0$. If $J = \sum_{g \in G} {}^g I$ then J is a G-invariant right ideal of A with $tr_G(J) = 0$. By [2], J is nilpotent. But the only nilpotent right ideal in a semiprime ring is 0.

Proof of Theorem 2. S^G is left Goldie, so according to [6], S is (semiprime) left Goldie. Let R be the left quotient ring for S; R is semisimple artinian. By Theorem 1 we can find a finite set of generators $x_1, \cdots, x_n$ for R as a right R^G-module. Choose a regular t and s_i both in S such that $x_i = t^{-1}s_i$.

$R = \sum_{i=1}^n t^{-1}s_i R^G \Rightarrow tR = \sum_i s_i R^G$. But $tR = R$ since t is invertible. Thus we assume $x_i \in S$.

Define $T: S \to \bigoplus \sum_{i=1}^n S^G$ by $T(a) = [tr_G(ax_i)]_{i=1}^n$. T is clearly a left S^G-module map. We will be done once we prove that T is injective.

$T(a) = 0$ implies $tr_G(ax_i) = 0$ for all i. But tr_G is a *right* R^G-module map. Thus $tr_G(aR) = 0$. By the previous lemma, $a = 0$.

We have actually proved that S is a finitely generated S^G-module!

One might well ask whether the requirement that S have no $|G|$-torsion can be dropped. Consider the following counterexample. Let F be a field of characteristic $p > 2$ and let Φ be the free group on x and y. If S denotes the ring of two-by-two matrices over the group algebra $F[\Phi]$ then S is semiprime but not noetherian. Let G be the multiplicative subgroup of S generated by

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix}, \text{ and } \begin{bmatrix} 1 & y \\ 0 & 1 \end{bmatrix}.$$

G is isomorphic to the semidirect product of $Z/p \oplus Z/p \oplus Z/p$ with $Z/2$. Since char $F \neq 2$, $S^{\left[\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right]}$ is the collection of diagonal matrices. The only diagonal matrices fixed by $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ are the scalar matrices. Now a simple calculation shows that S^G consists of those scalars in the center of $F[\Phi]$. But it is well known that the center is F, a patently noetherian ring.

However, the $|G|$-torsion restriction is not needed when S is (semiprime) commutative or, more generally, when S has no nilpotent elements. There are several difficulties in proving the last statement along the lines of Theorem 2. First, there are division rings on which tr_G vanishes. Even if this objection is met, our induction and restriction techniques all ignore the question of fidelity of action. Reconsider, for instance, Lemma 4. The Bergman-Isaacs theorem states that if H is a group of *automorphisms* of J and $tr_H(J) = 0$

then $J = 0$. Thus implicit in our argument is the proposition that $tr_G(J) = 0 \Rightarrow tr_{G/K}(J) = 0$ where K is the kernel of the action of G on J. The implication is true because J has no $|K|$-torsion.

We avoid these complications (and, of course, replace them with other complications) by refining the notion of trace. Let G be a finite group acting on a ring R. If $\wedge$ is a subset of G define $t_\wedge$: $R \to R$ by

$$t_\wedge(r) = \sum_{\lambda \in \wedge} ({}^\lambda r) .$$

$t_\wedge$ is an R^G-bimodule map. Notice that $tr_G \equiv t_G$.

LEMMA 5. *Let G be a finite group acting on the division ring D. Then there is a subset $\wedge \subseteqq G$ such that $t_\wedge$ is a mapping from D onto D^G.*

Proof. Suppose we can find $\wedge$ such that $t_\wedge$ is a nonzero function from D into D^G. Say $d \in D$ such that $t_\wedge(d) = w \neq 0$. If $x \in D^G$, $t_\wedge(dw^{-1}x) = t_\wedge(d)w^{-1}x = x$. Thus $t_\wedge$ is surjective.

We argue by induction on the length of a composition series for G. If G is simple and does not act faithfully then G acts trivially; choose $\wedge = \{1\}$. If G is simple group of automorphisms, a result of Faith ([3]) shows that t_G is not identically zero.

When G is not simple choose $H \Delta G$ with $H \neq 1$ and $H \neq G$. By induction there is a subset $A \subseteqq H$ such that t_A: $D \to D^H$ is surjective. G/H acts on D^H, so we can find $C \subseteqq G/H$ such that t_C: $D^H \to D^G$ is surjective. If B consists of representatives in G for elements of C then $t_C = t_B$. Now $t_{B \cdot A} = t_B \cdot t_A$ is the desired map.

Let S be a ring without nilpotent elements. Suppose G is a finite group of automorphisms of S such that S^G is left noetherian. By [7] S is a semiprime left Goldie ring. By the Faith-Utumi theorem the quotient ring, R, of S has no nilpotent elements. Let e be a centrally primitive idempotent of R.

LEMMA 6. *$S \cap eR$ is a finitely generated left S^G-module.*

Proof. We first observe that the left quotient ring of $S \cap eR$ in eR is the entire division ring eR. Choose z and s in S with z regular such that $e = z^{-1}s$. Then $s = ze \in S \cap eR$. If $x \in eR$ choose q and w in S with q regular such that $qx = w$. Then $(sq)x = sw$. But sq and sw are in $S \cap eR$ with sq regular when considered as an element in eR.

$H = \mathrm{Stab}_G(e)$ is a group which acts on $S \cap eR$. Pick a transversal, $G = \Gamma \cdot H$. As in Theorem 1, if $a \in S^H \cap eR$ then

$$t_\Gamma(a) \in S^G \quad \text{and} \quad e \cdot t_\Gamma(a) = a .$$

Thus t_Γ is an injective left S^G-module map from $S^H \cap eR$ into S^G.

The Galois theory for division rings ([5]) as applied to eR implies that eR is a finite dimensional right $(eR)^H$-vector space. As in the proof of Theorem 2 we can choose a basis $x_1, \cdots, x_n$ in $S \cap eR$. Use Lemma 5 to find $\wedge \subseteqq H$ so that $t_\wedge$ is nondegenerate on eR. Define $T: S \cap eR \to \bigoplus \sum_{i=1}^n S^G$ by

$$T(a) = [t_{\Gamma \cdot \wedge}(ax_i)]_{i=1}^n .$$

It is easy to check that T is a well defined left S^G-module map. The lemma is completed by showing that T is injective. Suppose $a \neq 0$ and $T(a) = 0$. Then $t_\Gamma \cdot t_\wedge(ax_i) = 0$ for each i. Since t_Γ is injective, $t_\wedge(ax_i) = 0$ for each i. That is, $t_\wedge(a \cdot eR) = 0$. But eR is a division ring: $a \cdot eR = eR$. We have contradicted the nonvanishing of $t_\wedge$.

THEOREM 3. *Let S be a ring without nilpotent elements. If G is a finite group of automorphisms of S and S^G is left noetherian then S is left noetherian* (in fact, is finitely generated as an S^G-module).

Proof. So far we have proved that $\sum_e (S \cap eR)$ is a finitely generated left S^G-module, where the sum is taken over the centrally primitive idempotents of R.

As observed in the first paragraph of Lemma 6, $S \cap eR$ contains an element invertible in eR. Consequently there is an element $d \in \Sigma(S \cap eR)$ which is invertible in R. Define $f: S \to \Sigma(S \cap eR)$ by $f(s) = sd$. Since f is an injective left S^G-module map, S is a finitely generated left S^G-module.

REFERENCES

1. G. Azumaya and T. Nakayama, *On irreducible rings*, Ann. of Math., **48** (1947), 949–965.
2. G. M. Bergman and I. M. Isaacs, *Rings with fixed-point-free group actions*, Proc. London Math. Soc., **27** (1973), 69–87.
3. C. Faith, *Galois subrings of Ore domains are Ore domains*, BAMS, **78** (1972), 1077–1080.
4. J. W. Fisher and J. Osterburg, *Semiprime ideals in rings with finite group actions*, to appear.
5. N. Jacobson, *Structure of rings*, Amer. Math. Soc. Colloq. Publ., **37** (1964).
6. V. K. Kharchenko, *Galois extensions and quotient rings*, Algebra and Logic (transl.), Nov. 1975, 265–281.
7. V. K. Kharchenko, *Generalized identities with automorphisms*, Algebra and Logic (transl.), March 1976, 132–148.
8. J. Levitzki, *On automorphisms of certain rings*, Ann. of Math., **36** (1935), 984–992.

9. D. S. Passman, *Radicals of twisted group rings*, Proc. London Math. Soc., (3), **20** (1970), 409-37.

Received October 13, 1976. Farkas was partially supported by NSF grant MCS76-06010 and Snider by NSF grant MCS76-05991.

VIRGINIA POLYTECHNIC INSTITUTE AND STATE UNIVERSITY
BLACKSBURG, VA 24061

PACIFIC JOURNAL OF MATHEMATICS

Mathematical papers intended for publication in the *Pacific Jaurnal of Mathematics* should be in typed form or offset-reproduced, (not dittoed), double spaced with large margins. Please do not use built up fractions in the text of your manuscript. You may however, use them in the displayed equations. Underline Greek letters in red, German in green, and script in blue. The first paragraph or two must be capable of being used separately as a synopsis of the entire paper. Items of the bibliography should not be cited there unless absolutely necessary, in which case they must be identified by author and Journal, rather than by item number. Manuscripts, in triplicate, may be sent to any one of the editors. Please classify according to the scheme of Math. Reviews, Index to Vol. **39**. All other communications should be addressed to the managing editor, or Elaine Barth, University of California, Los Angeles, California, 90024.

The Pacific Journal of Mathematics expects the author's institution to pay page charges, and reserves the right to delay publication for nonpayment of charges in case of financial emergency

100 reprints are provided free for each article, only if page charges have been substantially paid. Additional copies may be obtained at cost in multiples of 50.

The *Pacific Journal of Mathematics* is issued monthly as of January 1966. Regular subscription rate: $72 00 a year (6 Vols., 12 issues). Special rate: $36.00 a year to individual members of supporting institutions.

Subscriptions, orders for back numbers, and changes of address should be sent to Pacific Journal of Mathematics, 103 Highland Boulevard, Berkeley, California, 94708.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION

Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.).
8-8, 3-chome, Takadanobaba, Shinjuku-ku, Tokyo 160, Japan.

Pacific Journal of Mathematics

Vol. 69, No. 2 June, 1977