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LAPLACE-BELTRAMI OPERATOR ON THE FOLIATED  
MANIFOLD**

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## A FORMULA FOR THE NORMAL PART OF THE LAPLACE-BELTRAMI OPERATOR ON THE FOLIATED MANIFOLD

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**In this paper, we give a formula for the normal part of the Laplace-Beltrami operator with respect to the second connection on a foliated manifold with a bundle-like metric. This formula is analogous to the formula obtained by S. Helgason.**

1. Introduction. We shall be in  $C^\infty$ -category and manifolds are supposed<sup>f</sup> to be paracompact, connected Hausdorff spaces.

Let  $M$  be a complete  $(p + q)$ -dimensional Riemannian manifold and  $H$  a compact subgroup of the Lie group of all isometries of  $M$ . We suppose that all orbits of  $H$  have the same dimension  $p$ . Then  $H$  defines a  $p$ -dimensional foliation  $F$  whose leaves are orbits of  $H$ , and the Riemannian metric is a bundle-like metric with respect to the foliation  $F$ . A quotient space  $B = M/F$  is a Riemannian  $V$ -manifold [5]. Let  $L_D$  be the Laplace-Beltrami operator on  $M$  with respect to the second connection  $D$ [8], and let  $\Delta(L_D)$  denote the operator defined by (\*) in § 4. Our goal in this paper is the following theorem:

**THEOREM.** *Let  $L_D$  be the Laplace-Beltrami operator on  $M$  with respect to the second connection  $D$  and  $L_B$  the Laplace-Beltrami operator on  $B$  with respect to the Levi-Civita connection associated with the Riemannian metric defined by the normal component of the metric on  $M$ . Then*

$$\Delta(L_D) = \delta^{-1/2} L_B \circ \delta^{1/2} - \delta^{-1/2} L_B (\delta^{1/2})$$

where  $\delta$  is the function given by (\*\*) below.

This theorem is analogous to the following result obtained by S. Helgason [2]: Suppose  $V$  is a Riemannian manifold,  $H$  a closed unimodular subgroup of the Lie group of all isometries of  $V$  (with the compact open topology). Let  $W \subset V$  be a submanifold satisfying the condition: For each  $w \in W$ ,

$$(H \cdot w) \cap W = \{w\}, \quad V_w = (H \cdot w)_w \oplus W_w,$$

where  $\oplus$  denotes orthogonal direct sum. Let  $L_V$  and  $L_W$  denote the Laplace-Beltrami operators on  $V$  and  $W$ , respectively. Then

$$\Delta(L_V) = \delta^{-1/2} L_W \circ \delta^{1/2} - \delta^{-1/2} L_W (\delta^{1/2})$$

where  $\Delta(L_V)$  denotes the operator called the radial part of  $L_V$  and  $\delta$  is the function given by  $d\sigma_w = \delta(w)dh$  ( $d\sigma_w$  is the Riemannian volume element on the orbit  $H \cdot w$  and  $dh$  is an  $H$ -invariant measure on each orbit  $H \cdot w = H/(\text{the isotropy subgroup of } H \text{ at } w)$ ).

2. Definition of  $V$ -manifold [1, 6, 7]. The concept of  $V$ -manifold is defined by I. Satake. Let  $M$  be a Hausdorff space. A  $C^\infty$ -local uniformizing system  $\{\tilde{U}, G, \varphi\}$  for an open set  $U$  in  $M$  is a collection of the following objects:

$\tilde{U}$ : a connected open set in the  $m$ -dimensional Euclidean space (or  $C^\infty$ -manifold).

$G$ : a finite group of  $C^\infty$ -transformations of  $\tilde{U}$ .

$\varphi$ : a continuous map from  $\tilde{U}$  onto  $U$  such that  $\varphi \circ \sigma = \varphi$  for all  $\sigma \in G$ , inducing a homeomorphism from the quotient space  $\tilde{U}/G$  onto  $U$ .

Let  $\{\tilde{U}, G, \varphi\}$ ,  $\{\tilde{U}', G', \varphi'\}$  be local uniformizing systems for  $U$ ,  $U'$  respectively, and let  $U \subset U'$ . By a  $C^\infty$ -injection  $\lambda: \{\tilde{U}, G, \varphi\} \rightarrow \{\tilde{U}', G', \varphi'\}$  we mean a  $C^\infty$ -isomorphism from  $\tilde{U}$  onto an open subset of  $\tilde{U}'$  such that for any  $\sigma \in G$  there exists  $\sigma' \in G'$  satisfying relations  $\varphi = \varphi' \circ \lambda$  and  $\lambda \circ \sigma = \sigma' \circ \lambda$ .

A  $C^\infty$ - $V$ -manifold consists of a connected Hausdorff space  $M$  and a family  $\mathcal{F}$  of  $C^\infty$ -local uniformizing systems for open subsets in  $M$  satisfying the following conditions:

(I) If  $\{\tilde{U}, G, \varphi\}$ ,  $\{\tilde{U}', G', \varphi'\} \in \mathcal{F}$  and  $U \subset U'$ , then there exists a  $C^\infty$ -injection  $\lambda: \{\tilde{U}, G, \varphi\} \rightarrow \{\tilde{U}', G', \varphi'\}$ .

(II) The open sets  $U$ , for which there exists a local uniformizing system  $\{\tilde{U}, G, \varphi\} \in \mathcal{F}$ , form a basis of open sets in  $M$ .

The set  $R$  of all real numbers is regarded as a  $V$ -manifold defined by a single local uniformizing system  $\{R, \{1\}, 1\}$ , then a  $C^\infty$ -function on a  $V$ -manifold  $(M, \mathcal{F})$  is defined as a  $C^\infty$ -map  $M \rightarrow R$  defined by a  $C^\infty$ - $V$ -manifold map  $(M, \mathcal{F}) \rightarrow (R, \{R, \{1\}, 1\})$ .

A  $C^\infty$ - $V$ -bundle over  $C^\infty$ - $V$ -manifold is also defined, and in particular the tangent bundle  $(TM, \mathcal{F}^*)$  of a  $C^\infty$ - $V$ -manifold  $(M, \mathcal{F})$  is defined. Let  $(M, \mathcal{F})$  be a  $C^\infty$ - $V$ -manifold, then an  $h$ -form  $\omega$  on  $(M, \mathcal{F})$  is a collection of  $h$ -forms  $\{\omega_{\tilde{v}}\}$ , where  $\omega_{\tilde{v}}$  is a  $G$ -invariant  $h$ -form on  $\tilde{U}$  such that  $\omega_{\tilde{v}} = \omega_{\tilde{v} \circ \lambda}$  for any injection  $\lambda: \{\tilde{U}, G, \varphi\} \rightarrow \{\tilde{U}', G', \varphi'\} (\{\tilde{U}, G, \varphi\}, \{\tilde{U}', G', \varphi'\} \in \mathcal{F})$ , and if the support of  $\omega$  is contained in  $U = \varphi(\tilde{U})$ ,

$$\int_M \omega := \frac{1}{N_G} \int_{\tilde{v}} \omega_{\tilde{v}},$$

where  $N_G$  denotes the order of  $G$ . A Riemannian metric  $g$  on  $(M, \mathcal{F})$  is a collection of Riemannian metrics  $\{g_{\tilde{v}}\}$ , where  $g_{\tilde{v}}$  is a  $G$ -invariant Riemannian metric on  $\tilde{U}$  satisfying some condition with

any injection  $\lambda: \{\tilde{U}, G, \varphi\} \rightarrow \{\tilde{U}', G', \varphi'\}$ .

3. Review of the results from [4, 5]. Let  $M$  be a complete  $(p + q)$ -dimensional manifold with a “bundle-like metric” with respect to a  $p$ -dimensional foliation  $F$ . We suppose that each leaf of the foliation  $F$  is closed.

The quotient space  $B = M/F$  is the space formed from  $M$  by identifying each leaf to a point, and let  $\pi: M \rightarrow B$  denote the identification map.  $H(S)$  denotes the holonomy group of a leaf  $S$ . Since  $M$  has the bundle-like metric with respect to  $F$  and all leaves are closed,  $H(S)$  is a finite group for any  $S$  and  $B$  is a metric space defining the distance between two points of  $B$  to be the minimum distance between them considered as leaves in  $M$ .  $B$  is a connected Hausdorff space, since it is metric space and is the continuous image of  $M$  under  $\pi$ . Given any point  $b \in B$ , let  $S = \pi^{-1}(b)$ . Let  $U$  be a flat coordinate neighborhood of some point of  $S$ . Since  $H(S)$  may be considered as a group of isometries of the sphere of unit vectors orthogonal to the leaf  $S$  at some arbitrary point of  $S$ ,  $H(S)$  operates the  $q$ -ball orthogonal to  $S$ . Thus we may consider that  $H(S)$  operates on  $U$  such a manner that  $\{U, H(S), \pi\}$  is a local uniformizing system for the neighborhood  $\pi(U)$  in  $B$ . The natural injection map of two such local uniformizing systems are of  $C^\infty$ . Thus  $B$  is a  $C^\infty$ - $V$ -manifold. Since  $H(S)$  is an isometry on the normal vectors at a point of  $S$ , the normal component of the metric of  $M$  defines a Riemannian structure on  $B$ . Thus  $B$  is a Riemannian  $V$ -manifold.

4. Laplace-Beltrami operator with respect to the second connection. Let  $M$  be a  $(p + q)$ -dimensional manifold with a Riemannian metric  $\langle , \rangle$  and a  $p$ -dimensional foliation  $F$ . Let  $(U, (x^1, \dots, x^p, y^1, \dots, y^q))$  be a flat coordinate neighborhood system, that is, in  $U$ , the foliation  $F$  is defined by  $dy^\alpha = 0$  for  $1 \leq \alpha \leq q$ . Hereafter we will agree on the following ranges of indices:  $1 \leq i, j, k \leq p, 1 \leq \alpha, \beta, \gamma, \delta \leq q$ .

We may choose in each flat coordinate neighborhood system  $(U, (x^1, \dots, x^p, y^1, \dots, y^q))$  1-forms  $w^1, \dots, w^p$  such that  $\{w^1, \dots, w^p, dy^1, \dots, dy^q\}$  is a basis for the cotangent space, and vectors  $v_1, \dots, v_q$  such that  $\{\partial/\partial x^1, \dots, \partial/\partial x^p, v_1, \dots, v_q\}$  is the dual base for the tangent space. Then we may get

$$w^i = dx^i + A_\alpha^i dy^\alpha, \quad v_\alpha = \frac{\partial}{\partial y^\alpha} - A_\alpha^i \frac{\partial}{\partial x^i}.$$

We may choose  $A_\alpha^i$  such that  $\langle \partial/\partial x^i, v_\alpha \rangle = 0$ , then the metric has the local expression

$$ds^2 = g_{ij}(x, y)w^i w^j + g_{\alpha\beta}(x, y)dy^\alpha dy^\beta$$

where

$$g_{ij} := \left\langle \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j} \right\rangle, \quad g_{\alpha\beta} := \langle v_\alpha, v_\beta \rangle$$

and  $x := (x^1, \dots, x^n)$ ,  $y := (y^1, \dots, y^q)$ .

We may uniquely define the “second connection”  $D$  on  $M$  as follows (cf. [8]);

$$\begin{aligned} \text{(a)} \quad & D_{\partial/\partial x^i} \frac{\partial}{\partial x^j} = \Gamma_{ji}^k \frac{\partial}{\partial x^k}, \quad D_{v_\alpha} \frac{\partial}{\partial x^j} = \Gamma_{\alpha j}^k \frac{\partial}{\partial x^k}, \\ & D_{\partial/\partial x^i} v_\beta = \Gamma_{i\beta}^\gamma v_\gamma, \quad D_{v_\alpha} v_\beta = \Gamma_{\alpha\beta}^\gamma v_\gamma, \\ \text{(b)} \quad & \frac{\partial}{\partial x^i} \left\langle \frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^k} \right\rangle = \left\langle D_{\partial/\partial x^i} \frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^k} \right\rangle + \left\langle \frac{\partial}{\partial x^j}, D_{\partial/\partial x^i} \frac{\partial}{\partial x^k} \right\rangle, \\ & v_\alpha \langle v_\beta, v_\gamma \rangle = \langle D_{v_\alpha} v_\beta, v_\gamma \rangle + \langle v_\beta, D_{v_\alpha} v_\gamma \rangle, \\ \text{(c)} \quad & T\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right) = T_{ij}^\gamma v_\gamma, \quad T\left(\frac{\partial}{\partial x^i}, v_\beta\right) = 0, \\ & T\left(v_\alpha, \frac{\partial}{\partial x^j}\right) = 0, \quad T(v_\alpha, v_\beta) = T_{\alpha\beta}^k \frac{\partial}{\partial x^k}, \end{aligned}$$

where  $T$  denotes the torsion of  $D$ , that is, for any vector fields  $X, Y$  on  $M$ ,  $T(X, Y) := D_X Y - D_Y X - [X, Y]$  ( $[ , ]$  denotes the usual bracket operator). Note that, in general, the torsion of  $D$  doesn't vanish. If the metric has the local expression

$$ds^2 = g_{ij}(x, y)w^i w^j + g_{\alpha\beta}(y)dy^\alpha dy^\beta,$$

the metric is called a “bundle-like metric” with respect to the foliation  $F$ . Hereafter we suppose that  $M$  has a bundle-like metric with respect to  $F$ . Then we get

$$\frac{\partial}{\partial x^i} \langle v_\alpha, v_\beta \rangle = \langle D_{\partial/\partial x^i} v_\alpha, v_\beta \rangle + \langle v_\alpha, D_{\partial/\partial x^i} v_\beta \rangle.$$

For a vector field  $X$  on  $M$ ,  $\text{div}_D X$  is defined by

$$\text{div}_D X := \text{Trace} (Y \longrightarrow D_Y X),$$

for any vector field  $Y$  on  $M$ . For a function  $f$  on  $M$ ,  $\text{grad}_D f$  is defined by

$$\begin{aligned} \text{grad}_D f &:= (\tilde{g}^{ij} D_{\partial/\partial x^j} f) \frac{\partial}{\partial x^i} + (\tilde{g}^{\alpha\beta} D_{v_\beta} f) v_\alpha \\ &= \left( \tilde{g}^{ij} \frac{\partial}{\partial x^j} (f) \right) \frac{\partial}{\partial x^i} + (\tilde{g}^{\alpha\beta} v_\beta (f)) v_\alpha \end{aligned}$$

where  $(\tilde{g}^{ij})$  and  $(\tilde{g}^{\alpha\beta})$  are inverse matrices of  $(g_{ij})$  and  $(g_{\alpha\beta})$  respectively. We define the Laplace-Beltrami operator  $L_D$  with respect to the second connection  $D$  by

$$L_D(f) := \operatorname{div}_D \operatorname{grad}_D f ,$$

that is,

$$\begin{aligned} L_D(f) &= \tilde{g}^{ij} \frac{\partial}{\partial x^i} \left( \frac{\partial}{\partial x_j} (f) \right) - \tilde{g}^{ij} \Gamma_{ij}^k \frac{\partial}{\partial x^k} (f) \\ &\quad + \tilde{g}^{\alpha\beta} v_\alpha (v_\beta(f)) - \tilde{g}^{\alpha\beta} \Gamma_{\alpha\beta}^\gamma v_\gamma(f) . \end{aligned}$$

Let  $B$  be the  $C^\infty$ - $V$ -manifold  $M/F$ . Let  $\mathcal{E}(B)$  (resp.  $\mathcal{D}(B)$ ) be the space of  $C^\infty$ -functions (resp.  $C^\infty$ -functions of compact support) on  $B$ , and let  $\mathcal{E}_s(M)$  be the space of  $C^\infty$ -functions on  $M$  which are constants on leaves. We may define a map  $\Phi: \mathcal{E}_s(M) \rightarrow \mathcal{E}(B)$  by  $\Phi(f)(\pi(m)) := f(m)$  where  $f \in \mathcal{E}_s(M)$ ,  $m \in M$  and  $\pi: M \rightarrow B$ , then  $\Phi$  is of one-to-one. Let  $\mathcal{E}_s^0(M) := \Phi^{-1}(\mathcal{D}(B))$ .

It is clear that  $f \in \mathcal{E}_s(M)$  if and only if  $\partial/\partial x^i(f) = 0$  for  $1 \leq i \leq p$ .

LEMMA. *If  $f \in \mathcal{E}_s(M)$ , then  $L_D(f) \in \mathcal{E}_s(M)$ .*

*Proof.* For  $f \in \mathcal{E}_s(M)$ , we get

$$L_D(f) = \tilde{g}^{\alpha\beta} v_\alpha (v_\beta(f)) - \tilde{g}^{\alpha\beta} \Gamma_{\alpha\beta}^\gamma v_\gamma(f) .$$

Since  $g_{\alpha\beta} = g_{\alpha\beta}(y)$  and  $\Gamma_{\alpha\beta}^\gamma = (1/2)\tilde{g}^{\gamma\delta}\{v_\alpha(g_{\delta\beta}) + v_\beta(g_{\alpha\delta}) - v_\delta(g_{\alpha\beta})\}$ , we get  $\tilde{g}^{\alpha\beta} = \tilde{g}^{\alpha\beta}(y)$  and so  $\partial/\partial x^i(L_D(f)) = 0$ . Thus we get  $L_D(f) \in \mathcal{E}_s(M)$ .

REMARK. Let  $L$  be the Laplace-Beltrami operator with respect to the Levi-Civita connection associated with the bundle-like metric. In general  $L(f) \notin \mathcal{E}_s(M)$  for  $f \in \mathcal{E}_s(M)$ .

For  $L_D$  and  $\underline{f} \in \mathcal{E}(B)$ , we define  $\Delta(L_D)$  by

$$(*) \quad \Delta(L_D)(\underline{f})(b) := L_D(\Phi^{-1}(\underline{f}))(\pi^{-1}(b)) , \quad b \in B .$$

This is well-defined by lemma. Roughly speaking,  $\Delta(L_D)$  seems to be an operator projected on  $B$  of the normal part of  $L_D$ .

5. Proof of theorem. Using the same notations as above sections, we give a proof of our theorem.

The isotropy subgroup  $H_m$  at each point  $m \in M$  is compact and the orbit  $H \cdot m$  is compact. We fix a Haar measure on  $H$  and a Haar measure on  $H_m$ , we get an  $H$ -invariant measure  $d\hat{h}$  on each orbit  $H \cdot m = H/H_m$ . Since  $M$  has the bundle-like metric,  $ds^2 = g_{ij}(x, y)w^i w^j + g_{\alpha\beta}(y)dy^\alpha dy^\beta$ , the volume element  $dM$  of  $M$  is given by

$$\begin{aligned} dM &= G(x, y)dx^1 \wedge \cdots \wedge dx^p \wedge dy^1 \wedge \cdots \wedge dy^q \\ &= G(x, y)w^1 \wedge \cdots \wedge w^p \wedge dy^1 \wedge \cdots \wedge dy^q \end{aligned}$$

where

$$G(x, y) = \sqrt{\det \begin{pmatrix} g_{\alpha\beta} & \mathbf{0} \\ \mathbf{0} & g_{\alpha\beta} \end{pmatrix}}.$$

For a flat coordinate system  $(U, (x^1, \dots, x^p, y^1, \dots, y^q))$  and the projection  $\pi: M \rightarrow B$ ,

$$d\sigma = G'(y)dy^1 \wedge \cdots \wedge dy^q,$$

where  $G'(y) = \sqrt{|\det(g_{\alpha\beta})|}$ , is regarded as the volume element  $dB$  of  $B$ , since  $\{U, H(S), \pi\}$  is a local uniformizing system for  $\pi(U)$  in  $B$ . Also we get

$$G(x, y) = \sqrt{|\det(g_{ij}(x, y))|} \cdot G'(y).$$

However

$$\sqrt{|\det(g_{ij}(x, y))|} w^1 \wedge \cdots \wedge w^p$$

is the volume element  $dS_m$  on the leaf  $S_m$  through a point  $m=(x, y)$  (that is, on the orbit  $H \cdot m$ ). Thus, if  $f \in \mathcal{E}_s^0(M)$  we get from the Fubini's theorem that

$$\int_M f dM = \int_B \left[ \int_{H \cdot m} f dS_m \right] dB(\pi(m))$$

where “ $\underline{\quad}$ ” denotes the image under  $\Phi$ .  $dS_m$  is invariant under  $H$ , so it must be a scalar multiple of  $d\dot{h}$ ,

$$dS_m = \bar{\delta}(m)d\dot{h}.$$

Then the function  $\bar{\delta}$  belongs to  $\mathcal{E}_s(M)$ . We put

$$(**) \quad \delta := \Phi(\bar{\delta}).$$

Thus we get

$$\int_M f dM = \int_B \left[ \int_{H \cdot m} f(h \cdot m) d\dot{h} \right] \delta(\pi(m)) dB(\pi(m)).$$

The normal component of the bundle-like metric  $ds^2 = g_{ij}(x, y)w^i w^j + g_{\alpha\beta}(y)dy^\alpha dy^\beta$  is  $ds_N^2 = g_{\alpha\beta}(y)dy^\alpha dy^\beta$ , thus  $L_B$  is defined by the Levi-Civita connection associated with the metric defined from  $ds_N^2$ . Thus we observe that

$$A(L_D) = L_B + \text{lower order terms}.$$

The operator  $L_D$  restricted to  $\mathcal{E}_s^0(M)$  is symmetric with respect to  $dM$  (cf. [8]), that is,

$$(***) \quad \int_M L_D(f_1)f_2dM = \int_M f_1L_D(f_2)dM$$

for  $f_1, f_2 \in \mathcal{E}_s^0(M)$ .

For  $f \in \mathcal{E}_s(M)$  and  $m \in M$ , we get

$$\int_{H \cdot m} f d\dot{h} = \underline{f}(\pi(m))c$$

where  $c$  denotes a nonzero constant  $\int_{H \cdot m} d\dot{h}$ . Putting  $\underline{f}_1 = \Phi(f_1)$ ,  $\underline{f}_2 = \Phi(f_2)$  for  $f_1, f_2 \in \mathcal{E}_s^0(M)$ , we get

$$\begin{aligned} \int_M L_D(f_1)f_2dM &= \int_B \left[ \int_{H \cdot m} L_D(f_1)f_2d\dot{h} \right] \delta dB \\ &= \int_B \left[ \int_{H \cdot m} L_D(f_1)d\dot{h} \right] c \delta \underline{f}_2 dB \\ &= c^2 \int_B \underline{L_D}(f_1)\underline{f}_2 \delta dB. \end{aligned}$$

Thus we get from (\*\*\*)

$$\int_B \underline{L_D}(f_1)\underline{f}_2 \delta dB = \int_B \underline{f}_1 \underline{L_D}(f_2) \delta dB$$

for  $f_1, f_2 \in \mathcal{E}_s^0(M)$ . By the definition of  $\Delta(L_D)$  we get  $\underline{L_D}(f) = \Delta(L_D)(\underline{f})$  for  $f \in \mathcal{E}_s(M)$ , so

$$\int_B \Delta(L_D)(\underline{f}_1)\underline{f}_2 \delta dB = \int_B \underline{f}_1 \Delta(L_D)(\underline{f}_2) \delta dB.$$

This expression implies that  $\Delta(L_D)$  is symmetric with respect to  $\delta dB$ . Since  $L_B$  is symmetric with respect to  $dB$ ,  $\delta^{-1/2}L_B \circ \delta^{1/2}$  is symmetric with respect to  $\delta dB$  and it clearly agrees with  $L_B$  up to lower order terms. The symmetric operators  $\Delta(L_D)$  and  $\delta^{-1/2}L_B \circ \delta^{1/2}$  agree up to an operator of order  $\leq 1$ , thus this operator, being symmetric, must be a function. By applying the operators to the constant function 1, we get

$$\Delta(L_D)(1) - \delta^{-1/2}L_B \circ \delta^{1/2}(1) = -\delta^{-1/2}L_B(\delta^{1/2}).$$

Thus

$$\Delta(L_D) = \delta^{-1/2}L_B \circ \delta^{1/2} - \delta^{-1/2}L_B(\delta^{1/2}).$$

This completes the proof of our theorem.

REMARK. The example of “RS-manifold of almost fibered type”



given by S. Kashiwabara (Appendix 5 in [3]) is a foliated manifold with a 1-dimensional foliation and bundle-like metric. Each leaf of the foliation is a "S-geodesic." This example is constructed from the space  $D$  which consists of all points  $x_1e_1 + x_2e_2 + x_3e_3 + te_4$  such that  $|x_i| \leq 1 (i = 1, 2, 3)$ ,  $0 \leq t \leq 1$ , where  $(e_1, e_2, e_3, e_4)$  denotes an orthonormal frame with origin  $o$  in Euclidean 4-space. If S-geodesics are of direction of  $e_4$ , a leaf through the origin  $o$  has nontrivial holonomy group. Then  $\delta = 1$ .

REMARK. The semi-reducible Riemannian space are a special class of foliated manifolds with bundle-like metrics. The metric of such a space has the local expression

$$d s^2 = \sigma(y)q_{ij}(x)dx^i dx^j + g_{\alpha\beta}(y)dy^\alpha dy^\beta$$

(cf. [4]). Then  $\delta$  is defined from  $\sigma$ .

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