Pacific Journal of Mathematics

SUMMABILITY R_r FOR DOUBLE SERIES

MARVIN J. KOHN

Vol. 69, No. 2 June 1977

PACIFIC JOURNAL OF MATHEMATICS
Vol. 69, No. 2, 1977

SUMMABILITY R_r FOR DOUBLE SERIES

M. J. KOHN

Let r be a positive integer. A trigonometric series T of a single variable is said to be summable R_r at θ_0 if the series obtained by r times formally integrating T has an rth symmetric derivative at θ_0. For even values of r, summability R_r has been applied to double trigonometric series. We study here summability R_r, for odd values of r, for double trigonometric series. We obtain a connection between Bochner-Riesz summable series and series which are summable R_r.

1. Let

$$\sum_{-\infty}^{\infty} c_n e^{in\theta} \tag{1.1}$$

be a trigonometric series of a single variable. Let r be a positive integer. Suppose the series obtained by formally integrating (1.1) r times

$$c_0 \frac{\theta^r}{r!} + \sum_{n \neq 0} \frac{c_n}{(in)^r} e^{in\theta} \tag{1.2}$$

converges to a function $F(\theta)$ in a neighborhood of $\theta_0 \in (0, 2\pi)$. We will say that the series (1.1) is at θ_0 *summable by the method* R_r to sum s if $F(\theta)$ has at θ_0 an rth symmetric derivative with value s. That is, if r is even,

$$\frac{1}{2}\{F(\theta_0 + t) + F(\theta_0 - t)\} = a_0 + \frac{a_2}{2!}t^2 + \cdots + \frac{s}{r!}t^r + o(t^r) \tag{1.3}$$

as $t \to 0$, and if r is odd,

$$\frac{1}{2}\{F(\theta_0 + t) - F(\theta_0 - t)\} = a_1 t + \frac{a_3}{3!}t^3 + \cdots + \frac{s}{r!}t^r + o(t^r)\,, \tag{1.4}$$

as $t \to 0$.

The following result, see [8], p. 66, establishes a connection between summability (C, α) and summability R_r for trigonometric series.

THEOREM A. *Let $\alpha > -1$ and assume the series* (1.1) *is summable (C, α) at θ_0 to sum s. Let r be an integer with $r > \alpha + 1$, and suppose the series* (1.2) *converges in a neighborhood of θ_0. Then the series* (1.1) *is summable R_r to s.*

2. In two variables we will denote points $x \in E_2$ by $x = (x_1, x_2) =$

$te^{i\theta}$ and integral lattice points by $n = (n_1, n_2)$. We write

$$|x| = \sqrt{x_1^2 + x_2^2}\,.$$

We will say a double trigonometric series

$$T: \sum_{n \in Z_2} c_n e^{in\cdot x} \tag{2.1}$$

is *Bochner-Riesz summable* of order α at x_0 to sum s_0 if

$$\lim_{R\to\infty} \sum_{|n|<R} \left(1 - \left(\frac{|n|}{R}\right)^2\right)^\alpha c_n e^{in\cdot x_0} = s_0\,.$$

Suppose r is an even number, $r = 2s$. A two dimensional analogue of summability R_r is given as follows, see [7], [4].

DEFINITION. Let $F(x)$ be defined in a neighborhood of $x_0 \in E_2$. F has at x_0 a sth *generalized Laplacian* equal to s_0 if F is integrable on each circle $|x - x_0| = t$ and

$$\frac{1}{2\pi}\int_0^{2\pi} F(x_0 + te^{i\theta})d\theta = a_0 + \frac{a_2 t^2}{(2!)^2} + \cdots + \frac{s_0 t^{2s}}{(2^s s!)^2} + o(t^{2s}) \tag{2.2}$$

as $t \to 0$.

THEOREM B. *Let the series T of* (2.1) *be Bochner-Riesz-m summable at x_0 to sum s_0, where m is a nonnegative integer, and suppose the coefficients of T satisfy*

$$\sum_{n\in Z_2} |n|^{-3+\varepsilon}|c_n|^2 < \infty$$

for some $\varepsilon > 0$. Let $r = 2s$ be an even integer with $r \geqq m + 2$. Set

$$F(x) = \frac{c_0(x_1 + x_2)^{2s}}{2^s(2s)!} + (-1)^s \sum_{n\neq 0} \frac{c_n}{|n|^{2s}} e^{in\cdot x}\,. \tag{2.3}$$

Then the generalized sth *Laplacian of $F(x)$ exists at x_0 and is equal to s_0.*

That is, if the series (2.1) is Bochner-Riesz-m summable to s_0 and r is an even number with $r \geqq m + 2$, then the series is also summable R_r to sum s_0.

3. The purpose of this paper is to derive a connection between Bochner-Riesz summability and summability R_r, for *odd* values of r. We use the following definition, from [5]. This definition extends the formula of (1.4) to two dimensions in a manner analogous to the extension of (1.3) to two variables by (2.2).

DEFINITION. Let $r = 2s + 1$ be an *odd* positive integer. Let $L(x)$ be a function defined in a neighborhood of $x_o \in E_2$. We will say $L(x)$ has at x_o a *generalized symmetric derivative* of order r with value s_o if L is integrable on each circle $|x - x_o| = t$, for t small, and if

$$(3.1)\qquad \frac{1}{2\pi}\int_0^{2\pi} L(x_o + te^{i\theta})(\cos\theta + \sin\theta)d\theta = a_1 t + a_3 t^3 + \cdots + \frac{s_o}{2^{2s+1}s!(s+1)!}t^{2s+1} + o(t^{2s+1})$$

as $t \to 0$.

We are able to obtain the following results which, for odd values of r, form a two dimensional version of Theorem A. We begin with the case of double trigonometric series which are Bochner-Riesz summable of integral order, since the statement and proof of our results are much simpler in this case.

THEOREM 1. *Let m be a nonnegative integer. Suppose*

$$(3.2)\qquad T\colon \sum_{n \in Z_2} c_n e^{in\cdot x}$$

is Bochner-Riesz-m summable at x_o to finite sum s_o. Let $r = 2s + 1$ be an odd integer such that $r \geqq m + 1$. Suppose the coefficients of T satisfy

$$(3.3)\qquad \sum_{n_1+n_2=0} |n|^{-2r+3+\varepsilon}|c_n|^2 + \sum_{n_1+n_2\neq 0}(n_1+n_2)^{-2}|n|^{-2r+3+\varepsilon}|c_n|^2 < \infty$$

for some $\varepsilon > 0$. Then the series

$$(3.4)\qquad \frac{c_o(x_1+x_2)^r}{(r)!(2r)!2^{s+1}} + \frac{1}{2}(x_1+x_2)\sum_{n_1+n_2=0}{}' \frac{c_n}{|n|^{2s}}e^{in\cdot x} + \sum_{n_1+n_2\neq 0}\frac{-ic_n}{(n_1+n_2)|n|^{2s}}e^{in\cdot x}$$

converges spherically to a function $L(x)$ which has at x_o a generalized symmetric derivative of order r with value s_o.

We are able to extend Theorem 1 to include some, but not all, fractional orders of Bochner-Riesz summability. Let β be a nonnegative real number. We denote by $[\beta]$ the largest integer $\leqq \beta$ and by $\langle\beta\rangle$ the fractional part of β, $\langle\beta\rangle = \beta - [\beta]$.

THEOREM 2. *Let β be a nonnegative real number with $\langle\beta\rangle < 1/2$. Suppose the series* (3.2) *is summable Bochner-Riesz-β to finite sum s_o. Let $r = 2s + 1$ be an odd integer with $r \geqq [\beta] + 1$. Suppose the coefficients of the series* (3.2) *satisfy formula* (3.3) *for some $\varepsilon > 0$.*

Then the conclusion of Theorem 1 *still holds.*

In particular, in the two dimensional case, Bochner-Riesz summability of order β, for $\beta < 1/2$, is enough to imply summability R_1 (which is Lebesgue summability).

4. Although Theorem 1 is a special case of Theorem 2, we give its proof separately, since its proof is much easier than that of Theorem 2. We will assume, as we may, that $c_0 = 0$, $x_0 = 0$, and $s_0 = 0$. We set

$$S_R = S_R(0) = \sum_{|n|<R} c_n ,$$

and for $\eta > 0$

$$S_R^\eta = \frac{1}{\Gamma(\eta)}\int_0^R (R-u)^{\eta-1} S_u du . \tag{4.1}$$

Note that S_R^η, as a function of R, is the fractional integral of order η of $f(R) = S_R$, see [6].

Hardy, see [2], has shown that a series $\sum c_n$ is Bochner-Riesz-η summable to 0 if and only if

$$\sum_{|n|<R} c_n \left(1 - \frac{|n|}{R}\right)^\eta \longrightarrow 0$$

as $R \longrightarrow \infty$. Thus, for the proof of Theorem 1 we may assume

$$S_R^m = o(R^m) \tag{4.2}$$

as $R \longrightarrow \infty$.

We will need the following lemmas. The first lemma has been adapted from [7].

LEMMA 1. *Suppose* $\sum_{n\in Z_2} c_n e^{in\cdot x}$ *is Bochner-Riesz-*$(m+1)$ *summable to* 0 *at* $x = 0$, *and suppose the coefficients* c_n *satisfy condition* (3.3) *of Theorem* 1, *with* $r \geqq m+1$. *Then*

$$S_R^k = o(R^{r+1/2}) , \tag{4.3}$$

as $R \longrightarrow \infty$, *for* $k = 0, 1, \cdots, m+1$.

Proof. We first note that for $n_1 + n_2 \neq 0$,

$$\begin{aligned}\sum_{n_1+n_2\neq 0} (n_1+n_2)^{-2} |n|^{-2r+3+\varepsilon} |c_n|^2 &\geqq \frac{1}{4} \sum_{n_1+n_2\neq 0} |n|^{-2} |n|^{-2r+3+\varepsilon} |c_n|^2 \\ &= \frac{1}{4} \sum_{n_1+n_2\neq 0} |n|^{-2r+1+\varepsilon} |c_n|^2 .\end{aligned}$$

Thus, from (3.3),

$$\sum_{n_1+n_2\neq 0} |n|^{-2r+1+\varepsilon} |c_n|^2 < \infty ,$$

and therefore

$$\sum_{n\in Z_2} |n|^{-2r+1+\varepsilon} |c_n|^2 < \infty .$$

Using Schwartz's inequality,

$$\begin{aligned}\sum_{|n|<R} |c_n| &= \sum_{|n|<R} (|n|^{1/2(-2r+1+\varepsilon)} |c_n|)(|n|^{-1/2(-2r+1+\varepsilon)}) \\ &\leqq (\sum_{n\in Z_2} |n|^{-2r+1+\varepsilon} |c_n|^2)^{1/2} (\sum_{|n|<R} |n|^{2r-1-\varepsilon})^{1/2} \\ &= C\cdot(R^{2r+1-\varepsilon})^{1/2} \\ &= o(R^{r+1/2})\end{aligned} \tag{4.4}$$

as $R \to \infty$.

Now fix an integer j.

$$\begin{aligned}\sum_{|i|<R} c_i(R-|i|+j)^{m+1} &= \sum_{|i|<R+j} c_i(R-|i|+j)^{m+1} \\ &\quad - \sum_{R\leqq |i|<R+j} c_i(R-|i|+j)^{m+1} .\end{aligned}$$

Since $\sum c_n e^{in\cdot x}$ is Bochner-Riesz-$(m+1)$ summable to 0 at 0,

$$\sum_{|i|<R+j} c_i(R-|i|+j)^{m+1} = o(R^{m+1})$$

as $R \to \infty$.

$$\sum_{R\leqq |i|<R+j} c_i(R-|i|+j)^{m+1} = o(R^{r+1/2}) ,$$

because of (4.4). Thus,

$$\begin{aligned}\sum_{|i|<R} c_i(R-|i|+j)^{m+1} &= o(R^{m+1}) + o(R^{r+1/2}) \\ &= o(R^{r+1/2}) ,\end{aligned} \tag{4.5}$$

as $R \to \infty$.

We next use the fact, see [7], that there are number C_{jk}, for $j = 1, \cdots, m+2$, $k = 0, \cdots, m+1$ such that for all complex numbers z,

$$\sum_{j=1}^{m+2} C_{jk}(z+j)^{m+1} = z^k .$$

Thus, for $0 \leqq k \leqq m+1$,

$$\begin{aligned}S_R^k &= \frac{1}{k!} \sum_{|i|<R} c_i(R-|i|)^k \\ &= \frac{1}{k!} \sum_{|i|<R} c_i \sum_{j=1}^{m+2} C_{jk}(R-|i|+j)^{m+1}\end{aligned}$$

$$= \sum_{j=1}^{m+2} \frac{1}{k!} C_{jk} \sum_{|i|<R} c_i(R - |i| + j)^{m+1}$$

$$= \sum_{j=1}^{m+2} \frac{1}{k!} C_{jk} o(R^{r+1/2})$$

$$= o(R^{r+1/2}) ,$$

by (4.5). This proves Lemma 1.

LEMMA 2. *Let* $x = (x_1, x_2) = te^{i\theta} \in E_2$ *and* $n = (n_1, n_2) \in \boldsymbol{Z}_2$, *with* $|n| \neq 0$. *Define*

$$g_n(x) = \begin{cases} \frac{1}{2}(x_1 + x_2)e^{in\cdot x} & \text{if} \quad n_1 + n_2 = 0 \\ \frac{-ie^{in\cdot x}}{n_1 + n_2} & \text{if} \quad n_1 + n_2 \neq 0 . \end{cases} \tag{4.6}$$

Then

$$\frac{1}{2\pi}\int_0^{2\pi} g_n(te^{i\theta})(\cos\theta + \sin\theta)d\theta = \frac{J_1(|n|t)}{|n|} ,$$

where $J_1(z)$ *is the Bessel function of the first kind of order* 1.

Proof. This is the lemma from [5].

5. Proof of Theorem 1. Let

$$T_R(x) = \sum_{\substack{|n|<R \\ n_1+n_2=0}} \frac{1}{2}(x_1 + x_2)\frac{c_n}{|n|^{2s}}e^{in\cdot x} + \sum_{\substack{|n|<R \\ n_1+n_2\neq 0}} \frac{-ic_n}{(n_1 + n_2)|n|^{2s}}e^{in\cdot x} .$$

The hypothesis (3.3) insures that

$$L(x) = \lim_{R\to\infty} T_R(x)$$

exists a.e. on each circle $|x| = t$, see [3], Theorem 1. Also, by Theorem 2 of [3],

$$\int_0^{2\pi} \sup_R |T_R(te^{i\theta})|d\theta < \infty ,$$

so, using Lebesgue's Dominated Convergence Theorem,

$$\frac{1}{2\pi}\int_0^{2\pi} L(te^{i\theta})(\cos\theta + \sin\theta)d\theta$$

$$= \lim_{R\to\infty} \frac{1}{2\pi}\int_0^{2\pi} T_R(te^{i\theta})(\cos\theta + \sin\theta)d\theta$$

$$= \lim_{R\to\infty} \sum_{|n|<R} \frac{c_n}{|n|^{2s}} \frac{1}{2\pi}\int_0^{2\pi} g_n(te^{i\theta})(\cos\theta + \sin\theta)d\theta$$

where $g_n(x)$ is defined by (4.6). Using Lemma 2 we get

$$\begin{aligned}(5.1)\qquad &\frac{1}{2\pi}\int_0^{2\pi} L(te^{i\theta})(\cos\theta + \sin\theta)d\theta \\ &= \lim_{R\to\infty}\sum_{|n|<R}\frac{c_n}{|n|^{2s}}\frac{J_1(|n|t)}{|n|} \\ &= \lim_{R\to\infty}\sum_{|n|<R} c_n\frac{J_1(|n|t)}{|n|^r} \\ &= t^r\lim_{R\to\infty}\sum_{|n|<R} c_n\gamma(|n|t)\ ,\end{aligned}$$

where $\gamma(t) = z^{-r}J_1(z)$.

We express the last sum as an integral and integrate by parts $m+1$ times.

$$\begin{aligned}\sum_{|n|<R} c_n\gamma(|n|t) &= S_R\gamma(Rt) - \int_0^R S_u\frac{d}{du}\gamma(ut)du \\ &= S_R\gamma(Rt) - S_R^1\frac{d}{dR}\gamma(Rt) + \int_0^R S_u^1\frac{d^2}{du^2}\gamma(ut)du \\ (5.2)\qquad &\vdots \\ &= S_R\gamma(Rt) - S_R^1\frac{d}{dR}\gamma(Rt) + \cdots + (-1)^m S_R^m\frac{d^m}{dR^m}\gamma(Rt) \\ &\quad + (-1)^{m+1}\int_0^R S_u^m\frac{d^{m+1}}{du^{m+1}}\gamma(ut)du\ .\end{aligned}$$

From Lemma 1,

$$S_R^k = o(R^{r+1/2}) \quad \text{for} \quad k = 0, \cdots, m\ .$$

Repeatedly using the relations from [1],

$$(5.3)\qquad \frac{d}{dz}(z^{-n}J_n(z)) = z^{-n}J_{n+1}(z)$$

and

$$J_\nu(z) = o(z^{-1/2})\ ,$$

as $z \to \infty$, we get

$$(5.4)\qquad \frac{d^k}{dz_k}\gamma(z) = o(z^{-r-1/2})$$

as $z \to \infty$. So, for $k = 0, \cdots, m$

$$\begin{aligned}(5.5)\qquad S_R^k\frac{d^k}{dR^k}\gamma(Rt) &= o(R^{r+1/2})o(R^{-r-1/2}) \\ &= o(1)\ ,\end{aligned}$$

as $R \to \infty$. Thus, returning to (5.2),

$$\lim_{R\to\infty} \sum_{|n|<R} c_n \gamma(|n|t) = (-1)^{m+1} \int_0^\infty S_u^m \frac{d^{m+1}}{du^{m+1}} \gamma(ut)du ,$$

and (5.1) becomes,

$$\begin{aligned} &\frac{1}{2\pi}\int_0^{2\pi} L(te^{i\theta})(\cos\theta + \sin\theta)d\theta \\ &\quad = t^r \lim_{R\to\infty} \sum_{|n|<R} c_n \gamma(|n|t) \\ &\quad = t^r(-1)^{m+1} \int_0^\infty S_u^m \frac{d^{m+1}}{du^{m+1}} \gamma(ut)du . \end{aligned} \tag{5.6}$$

Now we make use of the series expansion for $J_1(z)$, [1], p. 4.

$$\begin{aligned} J_1(z) &= \sum_{k=0}^\infty \frac{(-1)^k(\frac{1}{2}z)^{1+2k}}{k!(k+1)!} \\ &= a_1 z + a_3 z^3 + \cdots . \end{aligned} \tag{5.7}$$

Then,

$$\begin{aligned} \gamma(z) &= z^{-r} J_1(z) \\ &= z^{-r}(a_1 z + a_3 z^3 + \cdots + a_{r-2} z^{r-2} + a_r z^r + \cdots) . \end{aligned}$$

We define a polynomial $P(z)$ as follows. If $r = 1$, let $P(z) \equiv 0$. Otherwise, let

$$P(z) = a_1 z + a_3 z^3 + \cdots + a_{r-2} z^{r-2}$$

where the a_i's are given by (5.7). Now we let

$$\lambda(z) = \gamma(z) - z^{-r}P(z) . \tag{5.8}$$

Then $\lambda(z)$ is an entire function in the plane and

$$\gamma(z) = z^{-r}P(z) + \lambda(z) .$$

Returning to (5.6),

$$\begin{aligned} &\frac{1}{2\pi}\int_0^{2\pi} L(te^{i\theta})(\cos\theta + \sin\theta)d\theta \\ &\quad = t^r(-1)^{m+1} \int_0^\infty S_u^m \frac{d^{m+1}}{du^{m+1}} \gamma(ut)du \\ &\quad = t^r(-1)^{m+1} \int_0^\infty S_u^m \frac{d^{m+1}}{du^{m+1}} \{(ut)^{-r}P(ut) + \lambda(ut)\}du \\ &\quad = t^r(-1)^{m+1} \int_0^\infty S_u^m \frac{d^{m+1}}{du^{m+1}} \{(ut)^{-r}P(ut)\}du \\ &\qquad + t^r(-1)^{m+1} \int_0^\infty S_u^m \frac{d^{m+1}}{du^{m+1}} \lambda(ut)du \\ &\quad = A + t^r B(t) . \end{aligned} \tag{5.9}$$

Since $c_0 = 0$, therefore $S_u^m = 0$ for $0 \leqq u < 1$. Thus we may replace the interval of integration of the integral involving A by the interval $(1/2, \infty)$.

$$\begin{aligned}
A &= t^r(-1)^{m+1}\int_{1/2}^{\infty} S_u^m \frac{d^{m+1}}{du^{m+1}}\{(ut)^{-r}P(ut)\}du \\
&= t^r(-1)^{m+1}\int_{1/2}^{\infty} S_u^m \frac{d^{m+1}}{du^{m+1}}\Big(\sum_{\substack{k=1\\ k \text{ odd}}}^{r-2} a_k(ut)^{k-r}\Big)du \\
&= \sum_{\substack{k=1\\ k \text{ odd}}}^{r-2} t^{r+k-r}a_k(-1)^{m+1}\int_{1/2}^{\infty} S_u^m \frac{d^{m+1}}{du^{m+1}}u^{k-r}du \\
&= \sum_{\substack{k=1\\ k \text{ odd}}}^{r-2} t^k a_k(-1)^{m+1}\int_{1/2}^{\infty} o(u^m)O(u^{k-r-m-1})du \\
&= \sum_{\substack{k=1\\ k \text{ odd}}}^{r-2} t^k a_k(-1)^{m+1}\int_{1/2}^{\infty} o(u^{k-r-1})du \\
&= \sum_{\substack{k=1\\ k \text{ odd}}}^{r-2} b_k t^k .
\end{aligned}$$

Returning to (5.9),

$$\begin{aligned}
&\frac{1}{2\pi}\int_0^{2\pi} L(te^{i\theta})(\cos\theta + \sin\theta)d\theta \\
&\quad = A + t^r B(t) \\
&\quad = b_1 t + b_3 t^3 + \cdots + b_{r-2}t^{r-2} + 0\cdot t^r + t^r B(t) .
\end{aligned}$$

The proof of Theorem 1 will be complete when we establish $B(t) \to 0$ as $t \to 0$.

$$\begin{aligned}
B(t) &= (-1)^{m+1}\int_0^{\infty} S_u^m \frac{d^{m+1}}{du^{m+1}}\lambda(ut)du \\
&= \int_0^{1/t} + \int_{1/t}^{\infty} \qquad (5.10)\\
&= B_1(t) + B_2(t) .
\end{aligned}$$

To estimate $B_1(t)$ we use the fact that $\lambda(z)$ is entire, so for $|z| \leqq 1$,

$$\left|\frac{d^k}{dz^k}\lambda(z)\right| < K .$$

Since $|ut| \leqq 1$ in the interval of integration involving $B_1(t)$,

$$\left|\frac{d^{m+1}}{du^{m+1}}\lambda(ut)\right| \leqq t^{m+1}K$$

in this interval.

$$\begin{aligned}B_1(t) &= (-1)^{m+1}\int_0^{1/t} o(u^m)t^{m+1}K du\\ &= o(t^{m+1})\int_0^{1/t} u^m du\\ &= o(t^{m+1})\left(\frac{1}{t}\right)^{m+1}\\ &= o(1)\end{aligned}$$

as $t \to 0$.

For the estimate of $B_2(t)$ we use the decomposition

$$\lambda(z) = \gamma(z) - z^{-r}P(z) .$$

Clearly, as $z \to \infty$

$$\frac{d^{m+1}}{dz^{m+1}} z^{-r}P(z) = O(z^{-m-3}) ,$$

and by (5.4),

$$\frac{d^{m+1}}{dz^{m+1}}\gamma(z) = O(z^{-r-1/2}) .$$

Thus, for $z \to \infty$

$$\frac{d^{m+1}}{dz^{m+1}}\lambda(z) = O(z^{-r-1/2}) , \tag{5.11}$$

and

$$\begin{aligned}B_2(t) &= (-1)^{m+1}\int_{1/t}^{\infty} S_u^m \frac{d^{m+1}}{du^{m+1}}\lambda(ut)du\\ &= (-1)^{m+1}\int_{1/t}^{\infty} o(u^m)t^{m+1}O(ut)^{-r-1/2}du\\ &= o(t^{m+1-r-1/2})\int_{1/t}^{\infty} o(u)^{m-r-1/2}du\\ &= o(t^{m-r+1/2})\, o\left(\frac{1}{t}\right)^{m-r+1/2}\\ &= o(1) .\end{aligned}$$

(Note we needed $m - r - 1/2 < -1$ to perform the last integration.) Thus $B_2(t) \to 0$ as $t \to 0$, and returning to (5.10), the proof of Theorem 1 is complete.

6. Proof of Theorem 2. We may assume that the fractional part of β is not zero. Otherwise Theorem 2 reduces to Theorem 1. Write $\beta = m + \alpha$, where m is an integer and $0 < \alpha < 1/2$.

We again assume $c_0 = 0$, $x_0 = 0$, $s_0 = 0$. We proceed as in the beginning of the proof of Theorem 1.

$$\frac{1}{2\pi}\int_0^{2\pi} L(te^{i\theta})(\cos\theta + \sin\theta)d\theta$$
$$= t^r \lim_{R\to\infty} \sum_{|n|<R} c_n \gamma(|n|t) ,$$

with $\gamma(z) = z^{-r}J_1(z)$.

As in the proof of Theorem 1 we integrate the last sum by parts. We now integrate by parts $m + 2$ times.

$$\sum_{|n|<R} c_n\gamma(|n|t) = S_R\gamma(Rt) - \int_0^R S_u \frac{d}{du}\gamma(ut)du$$
$$\vdots$$
$$= S_R\gamma(Rt) - S_R^1\frac{d}{dR}\gamma(Rt) + \cdots + (-1)^{m+1}S_R^{m+1}\frac{d^{m+1}}{dR^{m+1}}\gamma(Rt) \tag{6.1}$$
$$+ (-1)^{m+2}\int_0^R S_u^{m+1}\frac{d^{m+2}}{du^{m+2}}\gamma(ut)du .$$

We are now assuming the series (3.1) is summable Bochner-Riesz-β to 0 at $x_0 = 0$, so it is also summable Bochner-Riesz-$(m + 1)$ to 0 at $x_0 = 0$. Therefore we may again apply Lemma 1. For $0 \leqq k \leqq m + 1$,

$$S_R^k\frac{d^k}{dR^k}\gamma(Rt) = o(R^{r+1/2})O(R^{-r-1/2})$$
$$= o(1) ,$$

as $R \to \infty$, so

$$\frac{1}{2\pi}\int_0^{2\pi} L(te^{i\theta})(\cos\theta + \sin\theta)d\theta$$
$$= t^r \lim_{R\to\infty} \sum_{|n|<R} c_n\gamma(|n|t) \tag{6.2}$$
$$= t^r(-1)^m\int_0^\infty S_u^{m+1}\frac{d^{m+2}}{du^{m+2}}\gamma(ut)du .$$

We define $P(z)$ and $\lambda(z)$ as in the proof of Theorem 1:

$$P(z) = \begin{cases} 0 & \text{if} \quad r = 1 \\ a_1z + a_3z^3 + \cdots + a_{r-2}z^{r-2} & \text{if} \quad r \neq 1 \end{cases}$$

and

$$\lambda(z) = \gamma(z) - z^{-r}P(z) .$$

Then (6.2) becomes,

$$\frac{1}{2\pi}\int_0^{2\pi} L(te^{i\theta})(\cos\theta + \sin\theta)d\theta$$
$$= t^r(-1)^m\int_0^\infty S_u^{m+1}\frac{d^{m+2}}{du^{m+2}}[(ut)^{-r}P(ut) + \lambda(ut)]du$$

$$= t^r(-1)^m \int_0^\infty S_u^{m+1} \frac{d^{m+2}}{du^{m+2}} [(ut)^{-r} P(ut)] du$$
$$+ t^r(-1)^m \int_0^\infty S_u^{m+1} \frac{d^{m+2}}{du^{m+2}} \lambda(ut) du$$
$$= A(t) + t^r B(t) .$$

$$A = t^r(-1)^m \int_{1/2}^\infty S_u^{m+1} \frac{d^{m+2}}{du^{m+2}} \Big(\sum_{\substack{k=1 \\ k \text{ odd}}}^{r-2} a_k (ut)^{k-r} \Big) du$$
$$= \sum_{\substack{k=1 \\ k \text{ odd}}}^{r-2} t^{r+k-r} a_k (-1)^m \int_{1/2}^\infty o(u)^{m+1} \frac{d^{m+2}}{du^{m+2}} u^{k-r} du$$
$$= \sum_{\substack{k=1 \\ k \text{ odd}}}^{r-2} b_k t^k .$$

Hence,

$$\frac{1}{2\pi} \int_0^{2\pi} L(te^{i\theta})(\cos\theta + \sin\theta) d\theta = \sum_{\substack{k=1 \\ k \text{ odd}}}^{r-2} b_k t^k + t^r B(t) \tag{6.3}$$

where

$$B(t) = (-1)^m \int_0^\infty S_u^{m+1} \frac{d^{m+2}}{du^{m+2}} \lambda(ut) du . \tag{6.4}$$

To complete the proof of Theorem 2 we must show $B(t) \to 0$ as $t \to 0$.

If $f(u)$ is a function defined for $u > 0$ and η is a positive real number, denote by

$$I^\eta f(z) = \frac{1}{\Gamma(\eta)} \int_0^z (z-u)^{\eta-1} f(u) du ,$$

the fractional integral of order η, see [6]. Now if we set

$$f(u) = S_u = \sum_{|n|<u} c_n ,$$

then by (4.1),

$$S_u^\eta = I^\eta S_u ,$$

so

$$S_u^{m+1} = I^{m+1} S_u$$
$$= I^{1-\alpha} I^{m+\alpha} S_u$$
$$= I^{1-\alpha} S_u^{m+\alpha} .$$

Thus,

$$S_u^{m+1} = \frac{1}{\Gamma(1-\alpha)} \int_0^u (u-z)^{1-\alpha-1} S_z^{m+\alpha} dz$$
$$= \frac{1}{\Gamma(1-\alpha)} \int_0^u (u-z)^{-\alpha} S_z^{m+\alpha} dz .$$

Returning to (6.4)

$$\begin{aligned}
B(t) &= (-1)^m \int_0^\infty S_u^{m+1} \frac{d^{m+2}}{du^{m+2}} \lambda(ut)du \\
&= \lim_{R\to\infty} (-1)^m \int_0^R \frac{1}{\Gamma(1-\alpha)} \int_0^u (u-z)^{-\alpha} S_z^{m+\alpha} dz \, \frac{d^{m+2}}{du^{m+2}} \lambda(ut)du \\
&= \lim_{R\to\infty} \frac{(-1)^m}{\Gamma(1-\alpha)} \int_0^R S_z^{m+\alpha} \int_z^R (u-z)^{-\alpha} \frac{d^{m+2}}{du^{m+2}} \lambda(ut)dudz \\
&= \lim_{R\to\infty} \frac{(-1)^m}{\Gamma(1-\alpha)} \int_0^R S_z^{m+\alpha} H(z, t, R)dz \, ,
\end{aligned}$$

where

$$H(z, t, R) = \int_z^R (u-z)^{-\alpha} \frac{d^{m+2}}{du^{m+2}} \lambda(ut)du \, .$$

$$\begin{aligned}
B(t) &= \lim_{R\to\infty} \frac{(-1)^m}{\Gamma(1-\alpha)} \int_0^{1/t} S_z^{m+\alpha} H(z, t, R)dz \\
&\quad + \lim_{R\to\infty} \frac{(-1)^m}{\Gamma(1-\alpha)} \int_{1/t}^R S_z^{m+\alpha} H(z, t, R)dz \\
&= B_1(t) + B_2(t) \, .
\end{aligned}$$

We will make separate estimates of $H(z, t, R)$ for $B_1(t)$ and for $B_2(t)$.
First, in the interval of integration involving $B_1(t)$, $0 \leqq z \leqq 1/t$.

$$\begin{aligned}
H(z, t, R) &= \int_z^R (u-z)^{-\alpha} \frac{d^{m+2}}{du^{m+2}} \lambda(ut)du \\
&= \int_z^{1/t} + \int_{1/t}^R \qquad (6.5) \\
&= H_1 + H_2 \, .
\end{aligned}$$

Using the fact that λ is entire,

$$\begin{aligned}
|H_1| &\leqq \int_z^{1/t} (z-u)^{-\alpha} t^{m+2} \cdot K du \\
&\leqq K t^{m+2} \int_z^{1/t} (z-u)^{-\alpha} du \\
&= O(t^{m+2}) \left(\frac{1}{t} - z \right)^{1-\alpha} \, .
\end{aligned}$$

We estimate H_2 by employing (5.11)

$$\begin{aligned}
H_2 &= \int_{1/t}^R (u-z)^{-\alpha} \frac{d^{m+2}}{du^{m+2}} \lambda(ut)du \\
&= \int_{1/t}^\infty (u-z)^{-\alpha} t^{m+2} O(ut)^{-r-1/2} du
\end{aligned}$$

$$= O(t^{m-r+3/2})\left(\frac{1}{t} - z\right)^{-\alpha}\int_{1/t}^{\infty} u^{-r-1/2}du$$
$$= O(t^{m-r+3/2})\left(\frac{1}{t} - z\right)^{-\alpha}\left(\frac{1}{t}\right)^{-r+1/2}$$
$$= O(t^{m+1})\left(\frac{1}{t} - z\right)^{-\alpha} .$$

Returning to (6.5),

$$H(z, t, R) = O(t^{m+2})\left(\frac{1}{t} - z\right)^{1-\alpha} + O(t^{m+1})\left(\frac{1}{t} - z\right)^{-\alpha} .$$

and

$$\begin{aligned}
B_1(t) &= \frac{(-1)^m}{\Gamma(1-\alpha)}\int_0^{1/t} S_z^{m+\alpha}H(z, t, R)dz \\
&= \int_0^{1/t} o(z^{m+\alpha})\left\{O(t^{m+2})\left(\frac{1}{t} - z\right)^{1-\alpha} + O(t^{m+1})\left(\frac{1}{t} - z\right)^{-\alpha}\right\}dz \\
&= o\left(\frac{1}{t}\right)^{m+\alpha}\left\{O(t^{m+2})\int_0^{1/t}\left(\frac{1}{t} - z\right)^{1-\alpha}dz + O(t^{m+1})\int_0^{1/t}\left(\frac{1}{t} - z\right)^{-\alpha}dz\right\} \\
&= o\left(\frac{1}{t}\right)^{m+\alpha}\left\{O(t^{m+2})\left(\frac{1}{t}\right)^{2-\alpha} + O(t^{m+1})\left(\frac{1}{t}\right)^{1-\alpha}\right\} \\
&= o(1) ,
\end{aligned}$$

as $t \to 0$.

It remains to be shown that $B_2(t) \to 0$. In the interval of integration for B_2, $1/t \leqq z \leqq R$, and

$$\begin{aligned}
H(z, t, R) &= \int_z^R (u - z)^{-\alpha}\frac{d^{m+2}}{du^{m+2}}\lambda(ut)du \\
&= \int_z^R (u - z)^{-\alpha}\frac{d^{m+2}}{du^{m+2}}\left(\frac{-P(ut)}{(ut)^r}\right)du \\
&\quad + \int_z^R (u - z)^{-\alpha}\frac{d^{m+2}}{du^{m+2}}\gamma(ut)du \\
&= H_a + H_b .
\end{aligned}$$

$$\begin{aligned}
H_a &= -\int_z^R (u - z)^{-\alpha}\frac{d^{m+2}}{du^{m+2}}\Big(\sum_{\substack{k=1 \\ k \text{ odd}}}^{r-2} a_k(ut)^{k-r}\Big)du \\
&= \int_z^R (u - z)^{-\alpha}t^{m+2}O(ut)^{-m-4}du \\
&= t^{-2}\left\{\int_z^{2z}(u - z)^{-\alpha}O(u)^{-m-4}du + \int_{2z}^{\infty}(u - z)^{-\alpha}O(u)^{-m-4}du\right\} \\
&= t^{-2}\{O(z)^{1-\alpha}z^{-m-4} + O(z^{-\alpha})z^{-m-3}\} \\
&= t^{-2}O(z^{-m-\alpha-3}) .
\end{aligned}$$

We change variables in the interval for H_b by letting $x = ut$.

$$\begin{aligned} H_b(z, t, R) &= \int_z^R (u - z)^{-\alpha} \frac{d^{m+2}}{du^{m+2}} \gamma(ut) du \\ &= \int_{tz}^{tR} \left(\frac{x}{t} - z\right)^{-\alpha} t^{m+2} \frac{d^{m+2}}{du^{m+2}} \gamma(x) \frac{dx}{t} \\ &= t^{m+1+\alpha} \int_{tz}^{tR} (x - tz)^{-\alpha} \gamma^{(m+2)}(x) dx \\ &= t^{m+1+\alpha} \left\{ \int_{tz}^{tz+1} + \int_{tz+1}^{tR} \right\} \\ &= H_b' + H_b'' . \end{aligned}$$

Recall that $1/t \leqq z$, so in the interval of integration for H_b, $x > tz \geqq 1$. Thus, by (5.11)

$$|\gamma^{(m+2)}(x)| \leqq Cx^{-r-1/2} ,$$

and

$$\begin{aligned} H_b' &= t^{m+1+\alpha} \int_{tz}^{tz+1} (x - tz)^{-\alpha} \gamma^{(m+2)}(x) dx \\ &= t^{m+1+\alpha} O(tz)^{-r-1/2} \int_{tz}^{tz+1} (x - tz)^{-\alpha} dx \\ &= t^{m+1+\alpha} O(tz)^{-r-1/2} . \end{aligned}$$

We estimate H_b'' by integrating by parts.

$$\begin{aligned} H_b'' &= t^{m+1+\alpha} \int_{tz+1}^{tR} (x - tz)^{-\alpha} \gamma^{(m+2)}(x) dx \\ &= t^{m+1+\alpha} (x - tz)^{-\alpha} \gamma^{)m+1)}(x) \Big|_{tz+1}^{tR} \\ &\quad + t^{m+1+\alpha} \alpha \int_{tz+1}^{tR} (x - tz)^{-\alpha-1} \gamma^{(m+1)}(x) dx \\ &= t^{m+1+\alpha} (x - tz)^{-\alpha} \gamma^{(m+1)}(x) \Big|_{tz+1}^{tR} \\ &\quad + t^{m+1+\alpha} O(tz)^{-r-1/2} \int_{tz+1}^{tR} (x - tz)^{-\alpha-1} dx \\ &= t^{m+1+\alpha} (tR - tz)^{-\alpha} \gamma^{(m+1)}(tR) - t^{m+1+\alpha} \gamma^{(m+1)}(tz + 1) \\ &\quad + t^{m+1+\alpha} O(tz)^{-r-1/2} \left(\frac{1}{-\alpha}\right) \{(tR - tz)^{-\alpha} - 1\} \\ &= t^{m+1+\alpha} (tR - tz)^{-\alpha} O(tz)^{-r-1/2} + t^{m+1+\alpha} O(tz)^{-r-1/2} \\ &= t^{m+1+\alpha} O(tz)^{-r-1/2} . \end{aligned}$$

Hence, in the interval of integration for B_2,

$$\begin{aligned} H_b(z, t, R) &= H_b' + H_b'' \\ &= t^{m+1+\alpha} O(tz)^{-r-1/2} , \end{aligned}$$

and

$$\begin{aligned} H(z, t, R) &= H_a + H_b \\ &= t^{-2}O(z^{-m-\alpha-3}) + t^{m+1+\alpha}O(tz)^{-r-1/2} . \end{aligned}$$

So,

$$\begin{aligned} B_2(t) &= \lim_{R\to\infty} \frac{(-1)^m}{\Gamma(1-\alpha)} \int_{1/t}^{R} S_z^{m+\alpha} H(z, t, R)dz \\ &= \lim_{R\to\infty} \frac{(-1)^m}{\Gamma(1-\alpha)} \int_{1/t}^{R} o(z)^{m+\alpha}\{t^{-2}O(z^{-m-\alpha-3}) + t^{m+1+\alpha}O(tz)^{-r-1/2}\}dz \\ &= t^{-2}\int_{1/t}^{\infty} o(z^{m+\alpha-m-\alpha-3})dz + t^{m+1+\alpha-r-1/2}\int_{1/t}^{\infty} o(z^{m+\alpha-r-1/2})dz \\ &= t^{-2}o(z^{-2})\Big|_{1/t}^{\infty} + t^{m+1/2+\alpha-r}o(z^{m+\alpha-r+1/2})\Big|_{1/t}^{\infty} \\ &= o(1) . \end{aligned}$$

(Note that the hypothesis $\alpha < 1/2$ is necessary here to insure that the last integral converge.) This completes the proof of Theorem 2.

References

1. Erdelyi et al, *Higher Transcendental Functions*, vol. II, McGraw-Hill, New York, 1953.
2. G. H. Hardy, *The second theorem of consistency for summable series*, Proc. London Math. Soc., (2), **15** (1916), 72-88.
3. M. Kohn, *Spherical convergence and integrability of multiple trigonometric series on hypersurfaces*, Studia Math., **44** (1972), 345-354.
4. M. Kohn, *Riemann summability of multiple trigonometric series*, Indiana Univ. Math. J., **24** (1975), 813-823.
5. M. Kohn, *Lebesgue summability of double trigonometric series*, Trans. Amer. Math. Soc., to appear.
6. M. Riesz, *L'integrale de Riemann-Liouville et le problem de Cauchy*, Acta Math., **81** (1949), 1-233.
7. V. Shapiro, *Circular summability C of double trigonometric series*, Trans. Amer. Math. Soc., **76** (1954), 223-233.
8. A. Zygmund, *Trigonometric Series*, Vol. II, Cambridge University Press, Cambridge, 1968.

Received March 30, 1976 and in revised form October 6, 1976. Research partially sponsored by the City University of New York Research Foundation.

BROOKLYN COLLEGE OF THE CITY UNIVERSITY OF NEW YORK
BROOKLYN, NY 11210

PACIFIC JOURNAL OF MATHEMATICS

Mathematical papers intended for publication in the *Pacific Jaurnal of Mathematics* should be in typed form or offset-reproduced, (not dittoed), double spaced with large margins. Please do not use built up fractions in the text of your manuscript. You may however, use them in the displayed equations. Underline Greek letters in red, German in green, and script in blue. The first paragraph or two must be capable of being used separately as a synopsis of the entire paper. Items of the bibliography should not be cited there unless absolutely necessary, in which case they must be identified by author and Journal, rather than by item number. Manuscripts, in triplicate, may be sent to any one of the editors. Please classify according to the scheme of Math. Reviews, Index to Vol. **39**. All other communications should be addressed to the managing editor, or Elaine Barth, University of California, Los Angeles, California, 90024.

The Pacific Journal of Mathematics expects the author's institution to pay page charges, and reserves the right to delay publication for nonpayment of charges in case of financial emergency

100 reprints are provided free for each article, only if page charges have been substantially paid. Additional copies may be obtained at cost in multiples of 50.

The *Pacific Journal of Mathematics* is issued monthly as of January 1966. Regular subscription rate: $72 00 a year (6 Vols., 12 issues). Special rate: $36.00 a year to individual members of supporting institutions.

Subscriptions, orders for back numbers, and changes of address should be sent to Pacific Journal of Mathematics, 103 Highland Boulevard, Berkeley, California, 94708.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION
Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.).
8-8, 3-chome, Takadanobaba, Shinjuku-ku, Tokyo 160, Japan.

Pacific Journal of Mathematics
Vol. 69, No. 2 June, 1977