SUMMABILITY $R_r$ FOR DOUBLE SERIES

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Let $r$ be a positive integer. A trigonometric series $T$ of a single variable is said to be summable $R_r$ at $\theta_0$ if the series obtained by $r$ times formally integrating $T$ has an $r$th symmetric derivative at $\theta_0$. For even values of $r$, summability $R_r$ has been applied to double trigonometric series. We study here summability $R_r$, for odd values of $r$, for double trigonometric series. We obtain a connection between Bochner-Riesz summable series and series which are summable $R_r$.

1. Let

(1.1) \[ \sum_{n=-\infty}^{\infty} c_n e^{in\theta} \]

be a trigonometric series of a single variable. Let $r$ be a positive integer. Suppose the series obtained by formally integrating (1.1) $r$ times

(1.2) \[ c_\theta \frac{\partial^r}{\partial^r} + \sum_{n=0}^{\infty} \frac{c_n}{(i\pi)^r} e^{in\theta} \]

converges to a function $F(\theta)$ in a neighborhood of $\theta_0 \in (0, 2\pi)$. We will say that the series (1.1) is at $\theta_0$ summable by the method $R_r$ to sum $s$ if $F(\theta)$ has at $\theta_0$ an $r$th symmetric derivative with value $s$. That is, if $r$ is even,

(1.3) \[ \frac{1}{2} \{F(\theta_0 + t) + F(\theta_0 - t)\} = a_0 + \frac{a_2}{2!} t^2 + \cdots + \frac{s}{r!} t^r + o(t^r) \]

as $t \to 0$, and if $r$ is odd,

(1.4) \[ \frac{1}{2} \{F(\theta_0 + t) - F(\theta_0 - t)\} = a_0 t + \frac{a_2}{3!} t^3 + \cdots + \frac{s}{r!} t^r + o(t^r) \]

as $t \to 0$.

The following result, see [8], p. 66, establishes a connection between summability $(C, \alpha)$ and summability $R_r$ for trigonometric series.

**Theorem A.** Let $\alpha > -1$ and assume the series (1.1) is summable $(C, \alpha)$ at $\theta_0$ to sum $s$. Let $r$ be an integer with $r > \alpha + 1$, and suppose the series (1.2) converges in a neighborhood of $\theta_0$. Then the series (1.1) is summable $R_r$ to $s$.

2. In two variables we will denote points $x \in E_2$ by $x = (x_1, x_2) = \ldots$
te^{i\theta}$ and integral lattice points by $n = (n_1, n_2)$. We write

$$|x| = \sqrt{x_1^2 + x_2^2}.$$ 

We will say a double trigonometric series

(2.1) \[ T: \sum_{n \in \mathbb{Z}^2} c_n e^{i\pi x \cdot n} \]

is Bochner-Riesz summable of order $\alpha$ at $x_0$ to sum $s_0$ if

$$\lim_{R \to \infty} \sum_{|n| < R} \left(1 - \left(\frac{|n|}{R}\right)^{\alpha}\right)c_n e^{i\pi x_0 \cdot n} = s_0.$$ 

Suppose $r$ is an even number, $r = 2s$. A two dimensional analogue of summability $R_r$ is given as follows, see [7], [4].

**DEFINITION.** Let $F(x)$ be defined in a neighborhood of $x_0 \in E_2$. $F$ has at $x_0$ a $s$th generalized Laplacian equal to $s_0$ if $F$ is integrable on each circle $|x - x_0| = t$ and

(2.2) \[ \frac{1}{2\pi} \int_0^{2\pi} F(x_0 + te^{i\theta}) d\theta = a_0 + \frac{a_2 t^2}{(2!)^2} + \cdots + \frac{s_0 t^{2s}}{(2^s!)^2} + o(t^{2s}) \]

as $t \to 0$.

**Theorem B.** Let the series $T$ of (2.1) be Bochner-Riesz-$m$ summable at $x_0$ to sum $s_0$, where $m$ is a nonnegative integer, and suppose the coefficients of $T$ satisfy

$$\sum_{n \in \mathbb{Z}^2} |n|^{-2s+\varepsilon}|c_n|^2 < \infty$$

for some $\varepsilon > 0$. Let $r = 2s$ be an even integer with $r \geq m + 2$. Set

(2.3) \[ F(x) = \frac{c_n (x_1 + x_2)^{2s}}{2^s (2s)!} + (-1)^s \sum_{n \in \mathbb{Z}^2} \frac{c_n}{|n|^{2s}} e^{i\pi x \cdot n}. \]

Then the generalized $s$th Laplacian of $F(x)$ exists at $x_0$ and is equal to $s_0$.

That is, if the series (2.1) is Bochner-Riesz-$m$ summable to $s_0$ and $r$ is an even number with $r \geq m + 2$, then the series is also summable $R_r$ to sum $s_0$.

3. The purpose of this paper is to derive a connection between Bochner-Riesz summability and summability $R_r$, for odd values of $r$. We use the following definition, from [5]. This definition extends the formula of (1.4) to two dimensions in a manner analogous to the extension of (1.3) to two variables by (2.2).
**DEFINITION.** Let \( r = 2s + 1 \) be an odd positive integer. Let 
\( L(x) \) be a function defined in a neighborhood of \( x_0 \in \mathbb{R}^2 \). We will say 
\( L(x) \) has at \( x_0 \) a generalized symmetric derivative of order \( r \) with 
value \( s_0 \) if \( L \) is integrable on each circle \(|x - x_0| = t\), for \( t \) small, and if 
\[
(3.1) \quad \frac{1}{2\pi} \int_0^{2\pi} L(x_0 + te^{i\theta})(\cos \theta + \sin \theta) d\theta \\
\quad = a_1 t + a_3 t^3 + \cdots + \frac{s_0}{2^{2s+s_0}!(s_0 + 1)!} t^{2s+s_0} + o(t^{2s+s_0})
\]
as \( t \to 0 \).

We are able to obtain the following results which, for odd values of \( r \), form a two dimensional version of Theorem A. We begin with 
the case of double trigonometric series which are Bochner-Riesz summable of integral order, since the statement and proof of our 
results are much simpler in this case.

**THEOREM 1.** Let \( m \) be a nonnegative integer. Suppose 
\[
(3.2) \quad T: \sum_{n \in \mathbb{Z}^2} c_n e^{i\mathbf{n} \cdot \mathbf{x}}
\]
is Bochner-Riesz-\( m \) summable at \( x_0 \) to finite sum \( s_0 \). Let \( r = 2s + 1 \) 
be an odd integer such that \( r \geq m + 1 \). Suppose the coefficients of 
\( T \) satisfy 
\[
(3.3) \quad \sum_{n_1 + n_2 = 0} |n|^{-2r+2s+1} |c_n|^2 + \sum_{n_1 + n_2 = 0} \sum_{n_0} |n_1 + n_2 - n_0|^{-2r+2s+1} |c_n|^2 < \infty
\]
for some \( \varepsilon > 0 \). Then the series 
\[
(3.4) \quad \frac{c_n(x_1 + x_2)^r}{(r)!((2r)!2^{2s+1})} + \frac{1}{2} (x_1 + x_2) \sum_{n_1 + n_2 = 0} c_n e^{i\mathbf{n} \cdot \mathbf{x}} \\
\quad + \sum_{n_1 + n_2 = 0} \frac{-ic_n}{n_1 + n_2} e^{i\mathbf{n} \cdot \mathbf{x}}
\]
converges spherically to a function \( L(x) \) which has at \( x_0 \) a generalized 
symmetric derivative of order \( r \) with value \( s_0 \).

We are able to extend Theorem 1 to include some, but not all, 
fractional orders of Bochner-Riesz summability. Let \( \beta \) be a nonnegative real number. We denote by \([\beta]\) the largest integer \( \leq \beta \) 
and by \( \langle \beta \rangle \) the fractional part of \( \beta \), \( \langle \beta \rangle = \beta - [\beta] \).

**THEOREM 2.** Let \( \beta \) be a nonnegative real number with \( \langle \beta \rangle < 1/2 \). Suppose the series (3.2) is summable Bochner-Riesz-\( \beta \) to finite 
sum \( s_0 \). Let \( r = 2s + 1 \) be an odd integer with \( r \geq [\beta] + 1 \). Suppose 
the coefficients of the series (3.2) satisfy formula (3.3) for some \( \varepsilon > 0 \).
Then the conclusion of Theorem 1 still holds.

In particular, in the two dimensional case, Bochner-Riesz summability of order $\beta$, for $\beta < 1/2$, is enough to imply summability $R_1$ (which is Lebesgue summability).

4. Although Theorem 1 is a special case of Theorem 2, we give its proof separately, since its proof is much easier than that of Theorem 2. We will assume, as we may, that $c_0 = 0$, $x_0 = 0$, and $s_0 = 0$. We set

$$S_R = S_R(0) = \sum_{|n| < R} c_n,$$

and for $\eta > 0$

$$S_R^\eta = \frac{1}{\Gamma(\eta)} \int_0^R (R - u)^{\eta-1} S_u du.$$  

Note that $S_R^\eta$, as a function of $R$, is the fractional integral of order $\eta$ of $f(R) = S_R$, see [6].

Hardy, see [2], has shown that a series $\sum c_n$ is Bochner-Riesz-$\eta$ summable to 0 if and only if

$$\sum_{|n| < R} c_n \left(1 - \frac{|n|}{R}\right)^\eta \to 0$$

as $R \to \infty$. Thus, for the proof of Theorem 1 we may assume

$$S_R^m = o(R^m)$$

as $R \to \infty$.

We will need the following lemmas. The first lemma has been adapted from [7].

**Lemma 1.** Suppose $\sum_{n_1 + n_2 \neq 0} c_{n_1} e^{i(n_1 x + n_2 y)}$ is Bochner-Riesz-$(m + 1)$ summable to 0 at $x = 0$, and suppose the coefficients $c_n$ satisfy condition (3.3) of Theorem 1, with $r \geq m + 1$. Then

$$S_R^k = o(R^{r+1/2}),$$

as $R \to \infty$, for $k = 0, 1, \ldots, m + 1$.

**Proof.** We first note that for $n_1 + n_2 \neq 0$,

$$\sum_{n_1 + n_2 \neq 0} (n_1 + n_2)^{-2} |n|^{-2r+3+\epsilon} |c_n|^2$$

$$\geq \frac{1}{4} \sum_{n_1 + n_2 \neq 0} |n|^{-2} |n|^{-2r+3+\epsilon} |c_n|^2$$

$$= \frac{1}{4} \sum_{n_1 + n_2 \neq 0} |n|^{-2r+1+\epsilon} |c_n|^2.$$
Thus, from (3.3),
\[ \sum_{n_1, n_2 = 0} |n|^{-2r+1+\varepsilon} |c_n|^2 < \infty, \]
and therefore
\[ \sum_{n \in \mathbb{Z}^2} |n|^{-2r+1+\varepsilon} |c_n|^2 < \infty. \]

Using Schwartz's inequality,
\[ \sum_{|n| < R} |c_n|^2 = \sum_{|n| < R} (|n|^{-1/2} |c_n|) (\sum_{|n| < R} |n|^{-1/2} |c_n|)^{1/2} \]
\[ \leq \left( \sum_{n \in \mathbb{Z}^2} |n|^{-2r+1+\varepsilon} |c_n|^2 \right)^{1/2} \left( \sum_{|n| < R} |n|^{-2r+1+\varepsilon} \right)^{1/2} \]
\[ = C_s (R^{2r+1-\varepsilon})^{1/2} \]
\[ = o(R^{r+1/2}) \]
as \( R \to \infty \).

Now fix an integer \( j \).
\[ \sum_{|i| < R} c_i (R - |i| + j)^{m+1} = \sum_{|i| < R+j} c_i (R - |i| + j)^{m+1} \]
\[ - \sum_{R \leq |i| < R+j} c_i (R - |i| + j)^{m+1}. \]

Since \( \sum c_n e^{in\cdot z} \) is Bochner-Riesz-(\( m + 1 \)) summable to 0 at 0,
\[ \sum_{|i| < R+j} c_i (R - |i| + j)^{m+1} = o(R^{m+1}) \]
as \( R \to \infty \).

\[ \sum_{R \leq |i| < R+j} c_i (R - |i| + j)^{m+1} = o(R^{r+1/2}), \]
because of (4.4). Thus,
\[ \sum_{|i| < R} c_i (R - |i| + j)^{m+1} = o(R^{m+1}) + o(R^{r+1/2}) \]
\[ = o(R^{r+1/2}), \]
as \( R \to \infty \).

We next use the fact, see [7], that there are number \( C_{jk} \), for \( j = 1, \ldots, m + 2, k = 0, \ldots, m + 1 \) such that for all complex numbers \( z \),
\[ \sum_{j=1}^{m+2} C_{jk} (z + j)^{m+1} = z^k. \]

Thus, for \( 0 \leq k \leq m + 1 \),
\[ S_k^R = \frac{1}{k!} \sum_{|i| < R} c_i (R - |i|)^k \]
\[ = \frac{1}{k!} \sum_{|i| < R} c_i \sum_{j=1}^{m+2} C_{jk} (R - |i| + j)^{m+1} \]
= \sum_{\ell = 1}^{\infty} \frac{1}{\ell!} C_{j \ell} \sum_{|i| < R} c_i (R - |i| + j)^{m+1} \\
= \sum_{\ell = 1}^{\infty} \frac{1}{\ell!} C_{j \ell} o(R^{r+1/2}) \\
= o(R^{r+1/2}),

by (4.5). This proves Lemma 1.

**Lemma 2.** Let \( x = (x_1, x_2) = te^{i\theta} \in E_2 \) and \( n = (n_1, n_2) \in \mathbb{Z}_2 \), with \( |n| \neq 0 \). Define

\[
g_s(x) = \begin{cases} 
\frac{1}{2} (x_1 + x_2) e^{i n_1 z} & \text{if } n_1 + n_2 = 0 \\
-\frac{i e^{i n_2 z}}{n_1 + n_2} & \text{if } n_1 + n_2 \neq 0.
\end{cases}
\]

Then

\[
\frac{1}{2\pi} \int_0^{2\pi} g_s(te^{i\theta})(\cos \theta + \sin \theta) d\theta = \frac{J_i(|n|t)}{|n|},
\]

where \( J_i(z) \) is the Bessel function of the first kind of order 1.

**Proof.** This is the lemma from [5].

5. Proof of Theorem 1. Let

\[
T_R(x) = \sum_{|n| < R} \frac{1}{2} (x_1 + x_2) \frac{c_n}{|n|^{2s}} e^{i n_1 z} + \sum_{|n| < R} \frac{-i c_n}{(n_1 + n_2)|n|^{2s}} e^{i n_2 z}.
\]

The hypothesis (3.3) insures that

\[
L(x) = \lim_{R \to \infty} T_R(x)
\]

exists a.e. on each circle \( |x| = t \), see [3], Theorem 1. Also, by Theorem 2 of [3],

\[
\int_0^{2\pi} \sup_R |T_R(te^{i\theta})| d\theta < \infty,
\]

so, using Lebesgue's Dominated Convergence Theorem,

\[
\frac{1}{2\pi} \int_0^{2\pi} L(te^{i\theta})(\cos \theta + \sin \theta) d\theta
\]

\[
= \lim_{R \to \infty} \frac{1}{2\pi} \int_0^{2\pi} T_R(te^{i\theta})(\cos \theta + \sin \theta) d\theta
\]

\[
= \lim_{R \to \infty} \sum_{|n| < R} \frac{c_n}{|n|^{2s}} \frac{1}{2\pi} \int_0^{2\pi} g_s(te^{i\theta})(\cos \theta + \sin \theta) d\theta
\]
where \( g_n(x) \) is defined by (4.6). Using Lemma 2 we get
\[
\frac{1}{2\pi} \int_0^{2\pi} L(te^{i\theta})(\cos \theta + \sin \theta) d\theta
\]
\[
= \lim_{R \to \infty} \sum |n| < R \frac{c_n}{|n|^{2r}} J_n(\frac{nt}{n})
\]
\[
= \lim_{R \to \infty} \sum |n| < R \frac{c_n}{|n|^r} J_n(\frac{nt}{n})
\]
\[
= t^r \lim_{R \to \infty} \sum |n| < R c_n \gamma(\frac{nt}{n})
\]
where \( \gamma(t) = z^{-r} J_n(z) \).

We express the last sum as an integral and integrate by parts \( m + 1 \) times.
\[
\sum |n| < R c_n \gamma(\frac{nt}{n}) = S_R \gamma(Rt) - \int_0^R S_u \frac{d}{du} \gamma(ut) du
\]
\[
= S_R \gamma(Rt) - S_R \frac{d}{dR} \gamma(Rt) + \int_0^R S_u \frac{d^2}{du^2} \gamma(ut) du
\]
\[
+ \cdots + (-1)^m S_R \frac{d^m}{dR^m} \gamma(Rt)
\]
\[
+ (-1)^{m+1} \int_0^R S_u \frac{d^{m+1}}{du^{m+1}} \gamma(ut) du
\]

From Lemma 1,
\[
S_R^k = o(R^{r+1/2}) \quad \text{for} \quad k = 0, \cdots, m.
\]
Repeatedly using the relations from [1],
\[
\frac{d}{dz} (z^{-n} J_n(z)) = z^{-n} J_{n+1}(z)
\]
and
\[
J_n(z) = o(z^{-1/2})
\]
as \( z \to \infty \), we get
\[
\frac{d^k}{dz^k} \gamma(z) = o(z^{-r-1/2})
\]
as \( z \to \infty \). So, for \( k = 0, \cdots, m \)
\[
S_R^k \frac{d^k}{dR^k} \gamma(Rt) = o(R^{r+1/2}) o(R^{-r-1/2})
\]
\[
= o(1)
\]
as \( R \to \infty \). Thus, returning to (5.2),

\[
\lim_{R \to \infty} \sum_{|n| < R} c_n \gamma(|n| t) = (-1)^{m+1} \int_0^\infty S_n^m \frac{d^{m+1}}{du^{m+1}} \gamma(ut) du ,
\]

and (5.1) becomes,

\[
\frac{1}{2\pi} \int_0^{2\pi} L(te^{i\theta})(\cos \theta + \sin \theta) d\theta
\]

\[= t^r \lim_{R \to \infty} \sum_{|n| < R} c_n \gamma(|n| t)\]

\[= t^r (-1)^{m+1} \int_0^\infty S_n^m \frac{d^{m+1}}{du^{m+1}} \gamma(ut) du .\]

Now we make use of the series expansion for \( J_r(z) \), [1], p. 4.

\[
J_r(z) = \sum_{k=0}^\infty \frac{(-1)^k (\frac{1}{2} z)^{k+r}}{k! (k + 1)!}
\]

\[= a_r z + a_{r-1} z^3 + \cdots .\]

Then,

\[
\gamma(z) = z^{-r} J_r(z)
\]

\[= z^{-r} (a_r z + a_{r-1} z^3 + \cdots + a_1 z^r + a_0 z + \cdots) .\]

We define a polynomial \( P(z) \) as follows. If \( r = 1 \), let \( P(z) \equiv 0 \). Otherwise, let

\[
P(z) = a_r z + a_{r-1} z^3 + \cdots + a_1 z^r .
\]

where the \( a_i \)'s are given by (5.7). Now we let

\[
\lambda(z) = \gamma(z) - z^{-r} P(z) .
\]

Then \( \lambda(z) \) is an entire function in the plane and

\[
\gamma(z) = z^{-r} P(z) + \lambda(z) .
\]

Returning to (5.6),

\[
\frac{1}{2\pi} \int_0^{2\pi} L(te^{i\theta})(\cos \theta + \sin \theta) d\theta
\]

\[= t^r (-1)^{m+1} \int_0^\infty S_n^m \frac{d^{m+1}}{du^{m+1}} \gamma(ut) du
\]

\[= t^r (-1)^{m+1} \int_0^\infty S_n^m \frac{d^{m+1}}{du^{m+1}} [(ut)^{-r} P(ut) + \lambda(ut)] du
\]

\[= t^r (-1)^{m+1} \int_0^\infty S_n^m \frac{d^{m+1}}{du^{m+1}} [(ut)^{-r} P(ut)] du
\]

\[+ t^r (-1)^{m+1} \int_0^\infty S_n^m \frac{d^{m+1}}{du^{m+1}} \lambda(ut) du
\]

\[= A + t^r B(t) .\]
Since \( c_0 = 0 \), therefore \( S_u^m = 0 \) for \( 0 \leq u < 1 \). Thus we may replace the interval of integration of the integral involving \( A \) by the interval \((1/2, \infty)\).

\[
A = t'(-1)^{m+1} \int_{1/2}^\infty S_u^m \frac{d^{m+1}}{du^{m+1}}((ut)^{-r}P(ut)) \, du \\
= t'(-1)^{m+1} \int_{1/2}^\infty S_u^m \frac{d^{m+1}}{du^{m+1}}(\sum_{k=1}^{r-2} \alpha_k(ut)^{k-r}) \, du \\
= \sum_{k=1}^{r-2} \left[ t^{r-k-r} \alpha_k(-1)^{m+1} \int_{1/2}^\infty S_u^m \frac{d^{m+1}}{du^{m+1}} u^{k-r} \, du \right] \\
= \sum_{k=1}^{r-2} t^{r-k-r} \alpha_k(-1)^{m+1} \int_{1/2}^\infty o(u^m)O(u^{k-r-1}) \, du \\
= \sum_{k=1}^{r-2} \left[ b_k t^k \right].
\]

Returning to (5.9),

\[
\frac{1}{2\pi} \int_0^{2\pi} L(te^{i\theta})(\cos \theta + \sin \theta) \, d\theta = A + t' B(t) = b_1 t + b_2 t^2 + \cdots + b_{r-2} t^{r-2} + 0 \cdot t' + t' B(t).
\]

The proof of Theorem 1 will be complete when we establish \( B(t) \to 0 \) as \( t \to 0 \).

\[
B(t) = (-1)^{m+1} \int_0^\infty S_u^m \frac{d^{m+1}}{du^{m+1}} \lambda(ut) \, du \\
= \int_0^{1/t} + \int_{1/t}^\infty \\
= B_1(t) + B_2(t).
\]

(5.10)

To estimate \( B_2(t) \) we use the fact that \( \lambda(z) \) is entire, so for \( |z| \leq 1 \),

\[
\left| \frac{d^k}{dz^k} \lambda(z) \right| < K.
\]

Since \( |ut| \leq 1 \) in the interval of integration involving \( B_1(t) \),

\[
\left| \frac{d^{m+1}}{du^{m+1}} \lambda(ut) \right| \leq t^{m+1} K
\]

in this interval.
\( B_1(t) = (-1)^{m+1} \int_0^{1/t} o(u^m)u^{m+1}Kdu \)
\[= o(t^{m+1}) \int_0^{1/t} u^m du \]
\[= o(t^{m+1}) \left( \frac{1}{t} \right)^{m+1} \]
\[= o(1) \]

as \( t \to 0 \).

For the estimate of \( B_2(t) \) we use the decomposition
\[\lambda(z) = \gamma(z) - z^{-r}P(z).\]

Clearly, as \( z \to \infty \)
\[\frac{d^{m+1}}{dz^{m+1}} z^{-r}P(z) = O(z^{-m-2}),\]
and by (5.4),
\[\frac{d^{m+1}}{dz^{m+1}} \gamma(z) = O(z^{-r-1/2}).\]

Thus, for \( z \to \infty \)
(5.11)
\[\frac{d^{m+1}}{dz^{m+1}} \lambda(z) = O(z^{-r-1/2}),\]

and
\[ B_2(t) = (-1)^{m+1} \int_{1/t}^{\infty} S_u \frac{d^{m+1}}{du^{m+1}} \lambda(ut) du \]
\[= (-1)^{m+1} \int_{1/t}^{\infty} o(u^m)u^{m+1}O(ut)^{-r-1/2} du \]
\[= o(t^{m+1-r-1/2}) \int_{1/t}^{\infty} o(u)^{m-r-1/2} du \]
\[= o(t^{m-r+1/2}) o \left( \frac{1}{t} \right)^{m-r+1/2} \]
\[= o(1).\]

(Note we needed \( m - r - 1/2 < -1 \) to perform the last integration.)
Thus \( B_2(t) \to 0 \) as \( t \to 0 \), and returning to (5.10), the proof of Theorem 1 is complete.

6. Proof of Theorem 2. We may assume that the fractional part of \( \beta \) is not zero. Otherwise Theorem 2 reduces to Theorem 1. Write \( \beta = m + \alpha \), where \( m \) is an integer and \( 0 < \alpha < 1/2 \).

We again assume \( c_0 = 0, x_0 = 0, s_0 = 0 \). We proceed as in the beginning of the proof of Theorem 1.
\[
\frac{1}{2\pi} \int_0^{2\pi} L(te^{i\theta})(\cos \theta + \sin \theta) d\theta
= t^r \lim_{R \to \infty} \sum_{n < R} c_n \gamma(|n| t),
\]
with \(\gamma(z) = z^{-r\lambda}(z)\).

As in the proof of Theorem 1 we integrate the last sum by parts. We now integrate by parts \(m + 2\) times.

\[
\sum_{n < R} c_n \gamma(|n| t) = S_R^\gamma(Rt) - \int_0^R S_u \frac{d}{du} \gamma(ut) du
\]

\[
\cdots
\]

(6.1)

\[
= S_R^\gamma(Rt) - \frac{d}{dR} \gamma(Rt) + \cdots + (-1)^{m+1} S_R^{m+1} \frac{d^{m+1}}{dR^{m+1}} \gamma(Rt)
\]

\[
+ (-1)^{m+2} \int_0^R S_u^{m+1} \frac{d^{m+2}}{du^{m+2}} \gamma(ut) du.
\]

We are now assuming the series (3.1) is summable Bochner-Riesz-\(\beta\) to 0 at \(x_0 = 0\), so it is also summable Bochner-Riesz-\((m + 1)\) to 0 at \(x_0 = 0\). Therefore we may again apply Lemma 1. For \(\gamma = k\),

\[
S_R^k \frac{d^k}{dR^k} \gamma(Rt) = o(R^{\gamma + 1/\gamma})O(R^{-\gamma - 1/\gamma})
= o(1),
\]
as \(R \to \infty\), so

\[
\frac{1}{2\pi} \int_0^{2\pi} L(te^{i\theta})(\cos \theta + \sin \theta) d\theta
= t^r \lim_{R \to \infty} \sum_{n < R} c_n \gamma(|n| t)
\]

\[
= t^r(-1)^{m+2} \int_0^\infty S_u^{m+1} \frac{d^{m+2}}{du^{m+2}} \gamma(ut) du.
\]

We define \(P(z)\) and \(\lambda(z)\) as in the proof of Theorem 1:

\[
P(z) = \begin{cases} 
0 & \text{if } r = 1 \\
\alpha_1 z + \alpha_2 z^2 + \cdots + \alpha_r z^{r-2} & \text{if } r \neq 1
\end{cases}
\]

and

\[
\lambda(z) = \gamma(z) - z^{-r}P(z).
\]

Then (6.2) becomes,

\[
\frac{1}{2\pi} \int_0^{2\pi} L(te^{i\theta})(\cos \theta + \sin \theta) d\theta
= t^r(-1)^{m} \int_0^\infty S_u^{m+1} \frac{d^{m+2}}{du^{m+2}} [(ut)^{-r}P(ut) + \lambda(ut)] du
\]
\[
= t^r(-1)^m \int_0^\infty S_u^{m+1} \frac{d^{m+2}}{du^{m+2}} [(ut)^{-r} P(ut)] du
+ t^r(-1)^m \int_0^\infty S_u^{m+1} \frac{d^{m+2}}{du^{m+2}} \lambda(ut) du
= A(t) + t^r B(t).
\]

Hence,
\[
(6.3) \quad \frac{1}{2\pi} \int_0^{2\pi} L(te^{i\theta})(\cos \theta + \sin \theta) d\theta = \sum_{k=1 \atop k \text{ odd}}^{r-2} b_k t^k + t^r B(t)
\]
where
\[
(6.4) \quad B(t) = (-1)^m \int_0^\infty S_u^{m+1} \frac{d^{m+2}}{du^{m+2}} \lambda(ut) du.
\]

To complete the proof of Theorem 2 we must show \(B(t) \to 0\) as \(t \to 0\).

If \(f(u)\) is a function defined for \(u > 0\) and \(\eta\) is a positive real number, denote by
\[
I^\eta f(z) = \frac{1}{\Gamma(\eta)} \int_0^z (z - u)^{\eta-1} f(u) du,
\]
the fractional integral of order \(\eta\), see [6]. Now if we set
\[
f(u) = S_u = \sum_{|s| < u} c_s,
\]
then by (4.1),
\[
S_u^\eta = I^\eta S_u,
\]
so
\[
S_u^{m+1} = I^{m+1} S_u
= I^{m+1} I^{-\alpha} S_u
= I^{m+1} S_u^{m+\alpha}.
\]

Thus,
\[
S_u^{m+1} = \frac{1}{\Gamma(1 - \alpha)} \int_0^u (u - z)^{-\alpha-1} S_z^{m+\alpha} dz
= \frac{1}{\Gamma(1 - \alpha)} \int_0^u (u - z)^{-\alpha} S_z^{m+\alpha} dz.
\]
Returning to (6.4)

\[ B(t) = (-1)^m \int_0^\infty S_u^{m+1} \frac{d^{m+2}}{du^{m+2}} \lambda(ut)du \]

\[ = \lim_{R \to \infty} \frac{1}{\Gamma(1 - \alpha)} \int_0^R (u - z)^{-\alpha} S_z^{m+\alpha} du \frac{d^{m+2}}{du^{m+2}} \lambda(ut)du \]

\[ = \lim_{R \to \infty} \frac{(-1)^m}{\Gamma(1 - \alpha)} \int_0^R S_z^{m+\alpha} \int_z^R (u - z)^{-\alpha} \frac{d^{m+2}}{du^{m+2}} \lambda(ut)du dz \]

\[ = \lim_{R \to \infty} \frac{(-1)^m}{\Gamma(1 - \alpha)} \int_0^R S_z^{m+\alpha} H(z, t, R)dz \]

where

\[ H(z, t, R) = \int_z^R (u - z)^{-\alpha} \frac{d^{m+2}}{du^{m+2}} \lambda(ut)du . \]

\[ B(t) = \lim_{R \to \infty} \frac{(-1)^m}{\Gamma(1 - \alpha)} \int_0^{1/t} S_z^{m+\alpha} H(z, t, R)dz \]

\[ + \lim_{R \to \infty} \frac{(-1)^m}{\Gamma(1 - \alpha)} \int_{1/t}^R S_z^{m+\alpha} H(z, t, R)dz \]

\[ = B_1(t) + B_2(t) . \]

We will make separate estimates of \( H(z, t, R) \) for \( B_1(t) \) and for \( B_2(t) \).

First, in the interval of integration involving \( B_1(t) \), \( 0 \leq z \leq 1/t \).

\[ H(z, t, R) = \int_z^R (u - z)^{-\alpha} \frac{d^{m+2}}{du^{m+2}} \lambda(ut)du \]

(6.5)

Using the fact that \( \lambda \) is entire,

\[ |H_1| \leq \int_z^{1/t} (z - u)^{-\alpha} t^{m+2} \cdot Kdu \]

\[ \leq Kt^{m+2} \int_z^{1/t} (z - u)^{-\alpha} du \]

\[ = O(t^{m+2}) \left( \frac{1}{t} - z \right)^{-\alpha} . \]

We estimate \( H_2 \) by employing (5.11)

\[ H_2 = \int_{1/t}^R (u - z)^{-\alpha} \frac{d^{m+2}}{du^{m+2}} \lambda(ut)du \]

\[ = \int_{1/t}^\infty (u - z)^{-\alpha} t^{m+2} O(ut)^{-r-1/2} du \]
Returning to (6.5),
\[ H(z, t, R) = O(t^{m+\frac{3}{2}}) \left( \frac{1}{t} - z \right)^{1-\alpha} + O(t^{m+1}) \left( \frac{1}{t} - z \right)^{-\alpha} . \]

and

\[
B_2(t) = \frac{(-1)^m}{\Gamma(1-\alpha)} \int_0^{1/t} S_z^{m+\alpha} H(z, t, R) dz
\]

\[
= \int_0^{1/t} o(z^{m+\alpha}) \left\{ O(t^{m+\frac{3}{2}}) \left( \frac{1}{t} - z \right)^{1-\alpha} + O(t^{m+1}) \left( \frac{1}{t} - z \right)^{-\alpha} \right\} dz
\]

\[
= o\left( \frac{1}{t} \right)^{m+\alpha} \left\{ O(t^{m+\frac{3}{2}}) \int_0^{1/t} \left( \frac{1}{t} - z \right)^{1-\alpha} dz + O(t^{m+1}) \int_0^{1/t} \left( \frac{1}{t} - z \right)^{-\alpha} dz \right\}
\]

\[
= o\left( \frac{1}{t} \right)^{m+\alpha} \left\{ O(t^{m+\frac{3}{2}}) \left( \frac{1}{t} \right)^{2-\alpha} + O(t^{m+1}) \left( \frac{1}{t} \right)^{1-\alpha} \right\}
\]

\[
= o(1),
\]
as \( t \to 0 \).

It remains to be shown that \( B_2(t) \to 0 \). In the interval of integration for \( B_2 \), \( 1/t \leq z \leq R \), and

\[
H(z, t, R) = \int_z^R (u - z)^{-\alpha} \frac{d^{m+2}}{du^{m+2}} \lambda(ut) du
\]

\[
= \int_z^R (u - z)^{-\alpha} \frac{d^{m+2}}{du^{m+2}} \left( \frac{-P(ut)}{(ut)^r} \right) du
\]

\[
+ \int_z^R (u - z)^{-\alpha} \frac{d^{m+2}}{du^{m+2}} \gamma(ut) du
\]

\[
= H_a + H_b .
\]

\[
H_a = -\int_z^R (u - z)^{-\alpha} \frac{d^{m+2}}{du^{m+2}} \left( \sum_{k=1}^{r-2} a_k (ut)^k - r \right) du
\]

\[
= \int_z^R (u - z)^{-\alpha} t^{m+2} O(ut)^{-m+4} du
\]

\[
= t^{-2} \left\{ \int_z^R (u - z)^{-\alpha} O(ut)^{-m+4} du + \int_{2z}^\infty (u - z)^{-\alpha} O(ut)^{-m+4} du \right\}
\]

\[
= t^{-2} \left( O(t)^{-\alpha} z^{-m+4} + O(z^{-\alpha}) z^{-m+3} \right)
\]

\[
= t^{-2} O(z^{-m-\alpha+3}) .
\]
We change variables in the interval for $H_b$ by letting $x = ut$.

\[
H_3(z, t, R) = \int_0^z \frac{d^{m+1}}{du} \gamma(ut) du
\]

\[
= \int_0^1 \left( \frac{x}{t} - 1 \right)^{-(m+1)} \frac{d^{m+1}}{du} \gamma(x) \frac{dx}{t}
\]

\[
= t^{m+1} \int_0^1 (x - tz)^{-(m+1)} \gamma(x) dx
\]

\[
= t^{m+1} \{ \int_{t^2}^{t_{s+1}} + \int_{t_{s+1}}^{t_R} \}
\]

\[
= H'_b + H''_b.
\]

Recall that $1/t \leq z$, so in the interval of integration for $H_b$, $x > tz \geq 1$. Thus, by (5.11)

\[
|\gamma^{(m+1)}(x)| \leq Cx^{-r-1/2},
\]

and

\[
H'_b = t^{m+1} \int_{t^2}^{t_{s+1}} (x - tz)^{-(m+1)} \gamma^{(m+1)}(x) dx
\]

\[
= t^{m+1} O(tz)^{-r-1/2} \int_{t^2}^{t_{s+1}} (x - tz)^{-1} dx
\]

\[
= t^{m+1} O(tz)^{-r-1/2}.
\]

We estimate $H''_b$ by integrating by parts.

\[
H''_b = t^{m+1} \int_{t_{s+1}}^{t_R} (x - tz)^{-(m+1)} \gamma^{(m+1)}(x) dx
\]

\[
= t^{m+1} (x - tz)^{-(m+1)} \gamma^{(m+1)}(x) \bigg|_{t_{s+1}}^{t_R}
\]

\[
+ t^{m+1} \int_{t_{s+1}}^{t_R} (x - tz)^{-(m+1)} \gamma^{(m+1)}(x) dx
\]

\[
= t^{m+1} (tR - tz)^{-(m+1)} \gamma^{(m+1)}(tR) - t^{m+1} \gamma^{(m+1)}(tR - 1)
\]

\[
+ t^{m+1} O(tz)^{-r-1/2} \left( \frac{1}{-\alpha} \right) (tR - tz)^{-1} - 1
\]

\[
= t^{m+1} (tR - tz)^{-r-1/2} + t^{m+1} O(tz)^{-r-1/2}
\]

\[
= t^{m+1} O(tz)^{-r-1/2}.
\]

Hence, in the interval of integration for $B_3$,

\[
H_3(x, t, R) = H'_b + H''_b
\]

\[
= t^{m+1} O(tz)^{-r-1/2}.
\]
and
\[ H(z, t, R) = H_a + H_b = t^{-2}O(z^{-m-a-3}) + t^{m+1+a}O(tz)^{-r-1/2}. \]

So,
\[
B_s(t) = \lim_{R \to \infty} \frac{(-1)^m}{l(1 - \alpha)} \int_{1/t}^{R} S_{z}^{m+a} H(z, t, R) dz
\]
\[
= \lim_{R \to \infty} \frac{(-1)^m}{l(1 - \alpha)} \int_{1/t}^{R} o(z)^{m+a(1-t^{-2}O(z^{-m-a-3}) + t^{m+1+a}O(tz)^{-r-1/2})} dz
\]
\[
= t^{-2} \int_{1/t}^{\infty} o(z^{-a}) dz + t^{m+1+a-r-1/2} \int_{1/t}^{\infty} o(z^{a+r-1/2}) dz
\]
\[
= t^{-2} o(z^{-a}) \bigg|_{1/t}^{\infty} + t^{m+1/2+a-r} o(z^{a+r+1/2}) \bigg|_{1/t}^{\infty}
\]
\[
= o(1).
\]

(Note that the hypothesis \( \alpha < 1/2 \) is necessary here to insure that the last integral converge.) This completes the proof of Theorem 2.

REFERENCES


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