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### SUMMABILITY $R_r$ FOR DOUBLE SERIES

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Let r be a positive integer. A trigonometric series T of a single variable is said to be summable  $R_r$  at  $\theta_o$  if the series obtained by r times formally integrating T has an rth symmetric derivative at  $\theta_o$ . For even values of r, summability  $R_r$  has been applied to double trigonometric series. We study here summability  $R_r$ , for odd values of r, for double trigonometric series. We obtain a connection between Bochner-Riesz summable series and series which are summable  $R_r$ .

1. Let

(1.1) 
$$\sum_{-\infty}^{\infty} c_n e^{in\theta}$$

be a trigonometric series of a single variable. Let r be a positive integer. Suppose the series obtained by formally integrating (1.1) r times

(1.2) 
$$c_{\circ}\frac{\theta^{r}}{r!} + \sum_{n \neq 0} \frac{c_{n}}{(in)^{r}} e^{in\theta}$$

converges to a function  $F(\theta)$  in a neighborhood of  $\theta_o \in (0, 2\pi)$ . We will say that the series (1.1) is at  $\theta_o$  summable by the method  $R_r$  to sum s if  $F(\theta)$  has at  $\theta_o$  an rth symmetric derivative with value s. That is, if r is even,

$$(1.3) \quad \frac{1}{2} \left\{ F(\theta_o + t) + F(\theta_o - t) \right\} = a_o + \frac{a_2}{2!} t^2 + \cdots + \frac{s}{r!} t^r + o(t^r)$$

as  $t \mapsto 0$ , and if r is odd,

$$(1.4) \quad \frac{1}{2} \{ F(\theta_o + t) - F(\theta_o - t) \} = a_1 t + \frac{a_3}{3!} t^3 + \cdots + \frac{s}{r!} t^r + o(t^r) ,$$

as  $t \rightarrow 0$ .

The following result, see [8], p. 66, establishes a connection between summability  $(C, \alpha)$  and summability  $R_r$  for trigonometric series.

THEOREM A. Let  $\alpha > -1$  and assume the series (1.1) is summable (C,  $\alpha$ ) at  $\theta_o$  to sum s. Let r be an integer with  $r > \alpha + 1$ , and suppose the series (1.2) converges in a neighborhood of  $\theta_o$ . Then the series (1.1) is summable  $R_r$  to s.

2. In two variables we will denote points  $x \in E_2$  by  $x = (x_1, x_2) =$ 

 $te^{i\theta}$  and integral lattice points by  $n = (n_1, n_2)$ . We write

$$|x| = \sqrt{x_1^2 + x_2^2}$$
.

We will say a double trigonometric series

$$(2.1) T: \sum_{n \in \mathbb{Z}_2} c_n e^{in \cdot x}$$

is Bochner-Riesz summable of order  $\alpha$  at  $x_o$  to sum  $s_o$  if

$$\lim_{R o\infty}\sum_{|n|< R} \Big(1-\Big(rac{|n|}{R}\Big)^{\!\!\!2}\Big)^{\!\!\!lpha} c_n e^{in\cdot x_o} = s_o$$
 .

Suppose r is an even number, r = 2s. A two dimensional analogue of summability  $R_r$  is given as follows, see [7], [4].

DEFINITION. Let F(x) be defined in a neighborhood of  $x_o \in E_2$ . F has at  $x_o$  a sth generalized Laplacian equal to  $s_o$  if F is integrable on each circle  $|x - x_o| = t$  and

$$(2.2) \quad \frac{1}{2\pi} \int_0^{2\pi} F(x_o + te^{i\theta}) d\theta = a_o + \frac{a_2 t^2}{(2!)^2} + \cdots + \frac{s_o t^{2s}}{(2^s s!)^2} + o(t^{2s})$$

as  $t \rightarrow 0$ .

THEOREM B. Let the series T of (2.1) be Bochner-Riesz-m summable at  $x_o$  to sum  $s_o$ , where m is a nonnegative integer, and suppose the coefficients of T satisfy

$$\sum_{n \in \mathbb{Z}_2} |n|^{-3+\varepsilon} |c_n|^2 < \infty$$

for some  $\varepsilon > 0$ . Let r = 2s be an even integer with  $r \ge m + 2$ . Set

(2.3) 
$$F(x) = \frac{c_s(x_1 + x_2)^{2s}}{2^s(2s)!} + (-1)^s \sum_{n \neq 0} \frac{c_n}{|n|^{2s}} e^{in \cdot x}$$

Then the generalized sth Laplacian of F(x) exists at  $x_o$  and is equal to  $s_o$ .

That is, if the series (2.1) is Bochner-Riesz-*m* summable to  $s_o$  and *r* is an even number with  $r \ge m + 2$ , then the series is also summable  $R_r$  to sum  $s_o$ .

3. The purpose of this paper is to derive a connection between Bochner-Riesz summability and summability  $R_r$ , for odd values of r. We use the following definition, from [5]. This definition extends the formula of (1.4) to two dimensions in a manner analogous to the extension of (1.3) to two variables by (2.2).

DEFINITION. Let r = 2s + 1 be an odd positive integer. Let L(x) be a function defined in a neighborhood of  $x_o \in E_2$ . We will say L(x) has at  $x_o$  a generalized symmetric derivative of order r with value  $s_o$  if L is integrable on each circle  $|x - x_o| = t$ , for t small, and if

(3.1) 
$$\frac{1}{2\pi} \int_{0}^{2\pi} L(x_{o} + te^{i\theta}) (\cos \theta + \sin \theta) d\theta$$
$$= a_{1}t + a_{3}t^{3} + \cdots + \frac{s_{o}}{2^{2s+1}s!(s+1)!}t^{2s+1} + o(t^{2s+1})$$

as  $t \rightarrow 0$ .

We are able to obtain the following results which, for odd values of r, form a two dimensional version of Theorem A. We begin with the case of double trigonometric series which are Bochner-Riesz summable of integral order, since the statement and proof of our results are much simpler in this case.

**THEOREM 1.** Let m be a nonnegative integer. Suppose

$$(3.2) T: \sum_{n \in \mathbb{Z}_2} c_n e^{in \cdot x}$$

is Bochner-Riesz-m summable at  $x_o$  to finite sum  $s_o$ . Let r = 2s + 1be an odd integer such that  $r \ge m + 1$ . Suppose the coefficients of T satisfy

$$(3.3) \qquad \sum_{n_1+n_2=0} |n|^{-2r+3+\varepsilon} |c_n|^2 + \sum_{n_1+n_2\neq 0} (n_1 + n_2)^{-2} |n|^{-2r+3+\varepsilon} |c_n|^2 < \infty$$

for some  $\varepsilon > 0$ . Then the series

(3.4) 
$$\frac{c_{o}(x_{1} + x_{2})^{r}}{(r)!(2r)!2^{s+1}} + \frac{1}{2}(x_{1} + x_{2})\sum_{n_{1}+n_{2}=0}^{r}\frac{c_{n}}{|n|^{2s}}e^{in\cdot x} + \sum_{n_{1}+n_{2}\neq0}\frac{-ic_{n}}{(n_{1} + n_{2})|n|^{2s}}e^{in\cdot x}$$

converges spherically to a function L(x) which has at  $x_o$  a generalized symmetric derivative of order r with value  $s_o$ .

We are able to extend Theorem 1 to include some, but not all, fractional orders of Bochner-Riesz summability. Let  $\beta$  be a non-negative real number. We denote by  $[\beta]$  the largest integer  $\leq \beta$  and by  $\langle \beta \rangle$  the fractional part of  $\beta$ ,  $\langle \beta \rangle = \beta - [\beta]$ .

THEOREM 2. Let  $\beta$  be a nonnegative real number with  $\langle \beta \rangle < 1/2$ . Suppose the series (3.2) is summable Bochner-Riesz- $\beta$  to finite sum  $s_o$ . Let r = 2s + 1 be an odd integer with  $r \ge [\beta] + 1$ . Suppose the coefficients of the series (3.2) satisfy formula (3.3) for some  $\varepsilon > 0$ . M. J. KOHN

Then the conclusion of Theorem 1 still holds.

In particular, in the two dimensional case, Bochner-Riesz summability of order  $\beta$ , for  $\beta < 1/2$ , is enough to imply summability  $R_1$  (which is Lebesgue summability).

4. Although Theorem 1 is a special case of Theorem 2, we give its proof separately, since its proof is much easier than that of Theorem 2. We will assume, as we may, that  $c_o = 0$ ,  $x_o = 0$ , and  $s_o = 0$ . We set

$$S_{\scriptscriptstyle R} = S_{\scriptscriptstyle R}(0) = \sum_{|n| < R} c_n$$
 ,

and for  $\eta > 0$ 

(4.1) 
$$S_R^{\gamma} = \frac{1}{\Gamma(\gamma)} \int_0^R (R-u)^{\gamma-1} S_u du .$$

Note that  $S_R^{\gamma}$ , as a function of R, is the fractional integral of order  $\gamma$  of  $f(R) = S_R$ , see [6].

Hardy, see [2], has shown that a series  $\sum c_n$  is Bochner-Riesz- $\eta$  summable to 0 if and only if

$$\sum_{|n|\leq R} c_n \left(1 - \frac{|n|}{R}\right)^n \to 0$$

as  $R \rightarrow \infty$ . Thus, for the proof of Theorem 1 we may assume

$$(4.2) S_R^m = o(R^m)$$

as  $R \rightarrow \infty$ .

We will need the following lemmas. The first lemma has been adapted from [7].

LEMMA 1. Suppose  $\sum_{n \in \mathbb{Z}_2} c_n e^{in \cdot x}$  is Bochner-Riesz-(m + 1) summable to 0 at x = 0, and suppose the coefficients  $c_n$  satisfy condition (3.3) of Theorem 1, with  $r \ge m + 1$ . Then

$$(4.3) S_R^k = o(R^{r+1/2}),$$

as  $R \rightarrow \infty$ , for  $k = 0, 1, \dots, m + 1$ .

*Proof.* We first note that for  $n_1 + n_2 \neq 0$ ,

$$\sum_{n_1+n_2
eq 0} (n_1+n_2)^{-2} |n|^{-2r+3+arepsilon} |c_n|^2 \ \ge rac{1}{4} \sum_{n_1+n_2
eq 0} |n|^{-2} |n|^{-2r+3+arepsilon} |c_n|^2 \ = rac{1}{4} \sum_{n_1+n_2
eq 0} |n|^{-2r+1+arepsilon} |c_n|^2 \;.$$

Thus, from (3.3),

$$\sum_{n_1+n_2
eq 0} |n|^{-2r+1+arepsilon} |c_n|^2 < \infty$$
 ,

and therefore

$$\sum_{n \in \mathbb{Z}_2} |n|^{-2r+1+\varepsilon} |c_n|^2 < \infty$$
 .

Using Schwartz's inequality,

(4.4)  

$$\sum_{|n|

$$\leq (\sum_{n \in \mathbb{Z}_2} |n|^{-2r+1+\varepsilon} |c_n|^2)^{1/2} (\sum_{|n|

$$= C \cdot (R^{2r+1-\varepsilon})^{1/2}$$

$$= o(R^{r+1/2})$$$$$$

as  $R \rightarrow \infty$ .

Now fix an integer j.

$$\sum_{|i| < R} c_i (R - |i| + j)^{m+1} = \sum_{|i| < R+j} c_i (R - |i| + j)^{m+1} \ - \sum_{R \le |i| < R+j} c_i (R - |i| + j)^{m+1} \ .$$

Since  $\sum c_n e^{in \cdot x}$  is Bochner-Riesz-(m + 1) summable to 0 at 0,

$$\sum_{|i| < R+j} c_i (R - |i| + j)^{m+1} = o(R^{m+1})$$

as  $R \rightarrow \infty$ .

$$\sum\limits_{R \leq |i| < R+j} c_i (R - |i| + j)^{m+1} = o(R^{r+1/2})$$
 ,

because of (4.4). Thus,

(4.5) 
$$\sum_{|i| < R} c_i (R - |i| + j)^{m+1} = o(R^{m+1}) + o(R^{r+1/2}) = o(R^{r+1/2}),$$

as  $R \rightarrow \infty$ .

We next use the fact, see [7], that there are number  $C_{jk}$ , for  $j = 1, \dots, m+2, k = 0, \dots, m+1$  such that for all complex numbers z,

$$\sum_{j=1}^{m+2} C_{jk}(z\,+\,j)^{m+1} = z^k$$
 .

Thus, for  $0 \leq k \leq m + 1$ ,

$$S_{R}^{k} = rac{1}{k!} \sum_{|i| < R} c_{i}(R - |i|)^{k} \ = rac{1}{k!} \sum_{|i| < R} c_{i} \sum_{j=1}^{m+2} C_{jk}(R - |i| + j)^{m+1}$$

$$egin{aligned} &=\sum_{j=1}^{m+2}rac{1}{k!}C_{jk}\sum_{|i|< R}c_i(R-|i|+j)^{m+1}\ &=\sum_{j=1}^{m+2}rac{1}{k!}C_{jk}o(R^{r+1/2})\ &=o(R^{r+1/2})\ , \end{aligned}$$

by (4.5). This proves Lemma 1.

LEMMA 2. Let  $x = (x_1, x_2) = te^{i\theta} \in E_2$  and  $n = (n_1, n_2) \in Z_2$ , with  $|n| \neq 0$ . Define

(4.6) 
$$g_n(x) = \begin{cases} \frac{1}{2}(x_1 + x_2)e^{in \cdot x} & \text{if } n_1 + n_2 = 0\\ \frac{-ie^{in \cdot x}}{n_1 + n_2} & \text{if } n_1 + n_2 \neq 0 \end{cases}$$

Then

$$rac{1}{2\pi} \int_{0}^{2\pi} g_n(te^{i heta})(\cos heta+\sin heta)d heta=rac{J_1(|n|t)}{|n|}$$
 ,

where  $J_1(z)$  is the Bessel function of the first kind of order 1.

Proof. This is the lemma from [5].

### 5. Proof of Theorem 1. Let

$$T_{R}(x) = \sum_{|x_{1}| < R \atop n_{1} + n_{2} = 0} \frac{1}{2} (x_{1} + x_{2}) \frac{c_{n}}{|n|^{2s}} e^{in \cdot x} + \sum_{|x_{1}| < R \atop n_{1} + n_{2} \neq 0} \frac{-ic_{n}}{(n_{1} + n_{2})|n|^{2s}} e^{in \cdot x}$$

The hypothesis (3.3) insures that

$$L(x) = \lim_{R\to\infty} T_R(x)$$

exists a.e. on each circle |x| = t, see [3], Theorem 1. Also, by Theorem 2 of [3],

so, using Lebesgue's Dominated Convergence Theorem,

$$\begin{split} \frac{1}{2\pi} \int_{0}^{2\pi} L(te^{i\theta}) (\cos\theta + \sin\theta) d\theta \\ &= \lim_{R \to \infty} \frac{1}{2\pi} \int_{0}^{2\pi} T_{R}(te^{i\theta}) (\cos\theta + \sin\theta) d\theta \\ &= \lim_{R \to \infty} \sum_{|n| < R} \frac{c_{n}}{|n|^{2s}} \frac{1}{2\pi} \int_{0}^{2\pi} g_{n}(te^{i\theta}) (\cos\theta + \sin\theta) d\theta \end{split}$$

where  $g_n(x)$  is defined by (4.6). Using Lemma 2 we get

(5.1)  

$$\frac{1}{2\pi} \int_{0}^{2\pi} L(te^{i\theta})(\cos\theta + \sin\theta)d\theta$$

$$= \lim_{R \to \infty} \sum_{|n| < R} \frac{c_n}{|n|^{2s}} \frac{J_i(|n|t)}{|n|}$$

$$= \lim_{R \to \infty} \sum_{|n| < R} c_n \frac{J_i(|n|t)}{|n|^r}$$

$$= t^r \lim_{R \to \infty} \sum_{|n| < R} c_n \gamma(|n|t) ,$$

where  $\gamma(t) = z^{-r}J_1(z)$ .

We express the last sum as an integral and integrate by parts m+1 times.

$$\sum_{|n| < R} c_n \gamma(|n|t) = S_R \gamma(Rt) - \int_0^R S_u rac{d}{du} \gamma(ut) du$$
  
=  $S_R \gamma(Rt) - S_R^1 rac{d}{dR} \gamma(Rt) + \int_0^R S_u^1 rac{d^2}{du^2} \gamma(ut) du$   
:

(5.2)

$$egin{aligned} &=S_{\scriptscriptstyle R}\gamma(Rt)-S_{\scriptscriptstyle R}^{\scriptscriptstyle 1}rac{d}{dR}\gamma(Rt)+\cdots+(-1)^mS_{\scriptscriptstyle R}^mrac{d^m}{dR^m}\gamma(Rt)\ &+(-1)^{m+1}\int_{\scriptscriptstyle 0}^{\scriptscriptstyle R}S_{\scriptscriptstyle u}^mrac{d^{m+1}}{du^{m+1}}\gamma(ut)du \;. \end{aligned}$$

From Lemma 1,

 $S^{k}_{R} = o(R^{r+1/2}) \ \ ext{for} \ \ k = 0, \ \cdots, \ m \ .$ 

Repeatedly using the relations from [1],

(5.3) 
$$\frac{d}{dz}(z^{-n}J_n(z)) = z^{-n}J_{n+1}(z)$$

and

$$J_{\nu}(z) = o(z^{-1/2})$$
 ,

as  $z \rightarrow \infty$ , we get

(5.4) 
$$\frac{d^k}{dz_k}\gamma(z) = o(z^{-r-1/2})$$

as  $z \to \infty$ . So, for  $k = 0, \dots, m$ 

(5.5) 
$$S_{R}^{k} \frac{d^{k}}{dR^{k}} \gamma(Rt) = o(R^{r+1/2})o(R^{-r-1/2})$$
$$= o(1) ,$$

as  $R \rightarrow \infty$ . Thus, returning to (5.2),

$$\lim_{R o\infty}\sum_{\|n\|< R} c_n \gamma(\|n|t) = (-1)^{m+1} \int_0^\infty S_u^m \, rac{d^{m+1}}{du^{m+1}} \gamma(ut) du$$
 ,

and (5.1) becomes,

(5.6) 
$$\frac{1}{2\pi} \int_0^{2\pi} L(te^{i\theta}) (\cos \theta + \sin \theta) d\theta$$
$$= t^r \lim_{R \to \infty} \sum_{|n| \le R} c_n \gamma(|n|t)$$
$$= t^r (-1)^{m+1} \int_0^\infty S_u^m \frac{d^{m+1}}{du^{m+1}} \gamma(ut) du .$$

Now we make use of the series expansion for  $J_i(z)$ , [1], p. 4.

(5.7) 
$$J_{1}(z) = \sum_{k=0}^{\infty} \frac{(-1)^{k} (\frac{1}{2} z)^{1+2k}}{k! (k+1)!}$$

$$= a_1 z + a_3 z^3 + \cdots$$

Then,

$$\begin{array}{l} \gamma(z) \,=\, z^{-r} J_{\scriptscriptstyle 1}(z) \ =\, z^{-r} (a_{\scriptscriptstyle 1} z \,+\, a_{\scriptscriptstyle 3} z^{\scriptscriptstyle 3} \,+\, \cdots \,+\, a_{r-2} z^{r-2} \,+\, a_r z^r \,+\, \cdots) \;. \end{array}$$

We define a polynomial P(z) as follows. If r = 1, let  $P(z) \equiv 0$ . Otherwise, let

$$P(z) = a_1 z + a_3 z^3 + \cdots + a_{r-2} z^{r-2}$$

where the  $a_i$ 's are given by (5.7). Now we let

(5.8) 
$$\lambda(z) = \gamma(z) - z^{-r}P(z) .$$

Then  $\lambda(z)$  is an entire function in the plane and

$$\gamma(z) = z^{-r}P(z) + \lambda(z)$$
.

Returning to (5.6),

.

$$\begin{aligned} \frac{1}{2\pi} \int_{0}^{2\pi} L(te^{i\theta})(\cos\theta + \sin\theta)d\theta \\ &= t^{r}(-1)^{m+1} \int_{0}^{\infty} S_{u}^{m} \frac{d^{m+1}}{du^{m+1}} \gamma(ut)du \\ &= t^{r}(-1)^{m+1} \int_{0}^{\infty} S_{u}^{m} \frac{d^{m+1}}{du^{m+1}} \{(ut)^{-r}P(ut) + \lambda(ut)\}du \\ &= t^{r}(-1)^{m+1} \int_{0}^{\infty} S_{u}^{m} \frac{d^{m+1}}{du^{m+1}} \{(ut)^{-r}P(ut)\}du \\ &+ t^{r}(-1)^{m+1} \int_{0}^{\infty} S_{u}^{m} \frac{d^{m+1}}{du^{m+1}} \lambda(ut)du \\ &= A + t^{r}B(t) .\end{aligned}$$

Since  $c_o = 0$ , therefore  $S_u^m = 0$  for  $0 \le u < 1$ . Thus we may replace the interval of integration of the integral involving A by the interval  $(1/2, \infty)$ .

$$\begin{split} A &= t^r (-1)^{m+1} \int_{1/2}^{\infty} S_u^m \frac{d^{m+1}}{du^{m+1}} \{ (ut)^{-r} P(ut) \} du \\ &= t^r (-1)^{m+1} \int_{1/2}^{\infty} S_u^m \frac{d^{m+1}}{du^{m+1}} (\sum_{k=1}^{r-2} a_k (ut)^{k-r}) du \\ &= \sum_{\substack{k=1\\k \text{ odd}}}^{r-2} t^{r+k-r} a_k (-1)^{m+1} \int_{1/2}^{\infty} S_u^m \frac{d^{m+1}}{du^{m+1}} u^{k-r} du \\ &= \sum_{\substack{k=1\\k \text{ odd}}}^{r-2} t^k a_k (-1)^{m+1} \int_{1/2}^{\infty} o(u^m) O(u^{k-r-m-1}) du \\ &= \sum_{\substack{k=1\\k \text{ odd}}}^{r-2} t^k a_k (-1)^{m+1} \int_{1/2}^{\infty} o(u^{k-r-1}) du \\ &= \sum_{\substack{k=1\\k \text{ odd}}}^{r-2} b_k t^k . \end{split}$$

Returning to (5.9),

$$egin{aligned} &rac{1}{2\pi}\int_{0}^{2\pi}L(te^{i heta})(\cos heta\,+\,\sin heta)d heta\ &=A+t^{r}B(t)\ &=b_{1}t+b_{3}t^{3}+\cdots+b_{r-2}t^{r-2}+0{\cdot}t^{r}+t^{r}B(t)\;. \end{aligned}$$

The proof of Theorem 1 will be complete when we establish  $B(t) \rightarrow 0$  as  $t \rightarrow 0$ .

(5.10)  
$$B(t) = (-1)^{m+1} \int_0^\infty S_u^m \frac{d^{m+1}}{du^{m+1}} \lambda(ut) du$$
$$= \int_0^{1/t} + \int_{1/t}^\infty$$
$$= B_1(t) + B_2(t) .$$

To estimate  $B_1(t)$  we use the fact that  $\lambda(z)$  is entire, so for  $|z| \leq 1$ ,

$$\left|rac{d^k}{dz^k}\,\lambda(z)
ight| < K$$
 .

Since  $|ut| \leq 1$  in the interval of integration involving  $B_1(t)$ ,

$$\left|rac{d^{m+1}}{du^{m+1}}\,\lambda(ut)
ight| \leq t^{m+1}K$$

in this interval.

$$egin{aligned} B_1(t) &= (-1)^{m+1} \int_0^{1/t} o(u^m) t^{m+1} K du \ &= o(t^{m+1}) \int_0^{1/t} u^m du \ &= o(t^{m+1}) \left(rac{1}{t}
ight)^{m+1} \ &= o(1) \end{aligned}$$

as  $t \rightarrow 0$ .

For the estimate of  $B_2(t)$  we use the decomposition

$$\lambda(z) = \gamma(z) - z^{-r}P(z)$$
.

Clearly, as  $z \to \infty$ 

$$rac{d^{m+1}}{dz^{m+1}}z^{-r}P(z)=O(z^{-m-3})$$
 ,

and by (5.4),

$$rac{d^{m+1}}{dz^{m+1}} \gamma(z) = O(z^{-r-1/2}) \;.$$

Thus, for  $z \rightarrow \infty$ 

(5.11) 
$$\frac{d^{m+1}}{dz^{m+1}}\lambda(z) = O(z^{-r-1/2}),$$

and

$$egin{aligned} B_2(t) &= (-1)^{m+1} \int_{1/t}^\infty S_u^m \, rac{d^{m+1}}{du^{m+1}} \lambda(ut) du \ &= (-1)^{m+1} \int_{1/t}^\infty o(u^m) t^{m+1} O(ut)^{-r-1/2} du \ &= o(t^{m+1-r-1/2}) \int_{1/t}^\infty o(u)^{m-r-1/2} du \ &= o(t^{m-r+1/2}) \, o\left(rac{1}{t}
ight)^{m-r+1/2} \ &= o(1) \ . \end{aligned}$$

(Note we needed m - r - 1/2 < -1 to perform the last integration.) Thus  $B_2(t) \rightarrow 0$  as  $t \rightarrow 0$ , and returning to (5.10), the proof of Theorem 1 is complete.

6. Proof of Theorem 2. We may assume that the fractional part of  $\beta$  is not zero. Otherwise Theorem 2 reduces to Theorem 1. Write  $\beta = m + \alpha$ , where m is an integer and  $0 < \alpha < 1/2$ .

We again assume  $c_o = 0$ ,  $x_o = 0$ ,  $s_o = 0$ . We proceed as in the beginning of the proof of Theorem 1.

$$egin{aligned} &rac{1}{2\pi} \int_{0}^{2\pi} L(t e^{i heta})(\cos heta\,+\,\sin heta) d heta \ &= t^r \lim_{R o\infty} \sum_{|n|< R} c_n \gamma(|n|t) \;, \end{aligned}$$

with  $\gamma(z) = z^{-r}J_1(z)$ .

As in the proof of Theorem 1 we integrate the last sum by parts. We now integrate by parts m + 2 times.

$$\sum_{|n| < R} c_n \gamma(|n|t) = S_R \gamma(Rt) - \int_0^R S_u \frac{d}{du} \gamma(ut) du$$
  
:  
(6.1)  

$$= S_R \gamma(Rt) - S_R^1 \frac{d}{dR} \gamma(Rt) + \dots + (-1)^{m+1} S_R^{m+1} \frac{d^{m+1}}{dR^{m+1}} \gamma(Rt)$$
  

$$+ (-1)^{m+2} \int_0^R S_u^{m+1} \frac{d^{m+2}}{du^{m+2}} \gamma(ut) du$$

We are now assuming the series (3.1) is summable Bochner-Riesz- $\beta$  to 0 at  $x_o = 0$ , so it is also summable Bochner-Riesz-(m + 1) to 0 at  $x_o = 0$ . Therefore we may again apply Lemma 1. For  $0 \le k \le m + 1$ ,

$$S^k_R rac{d^k}{dR^k} \gamma(Rt) = o(R^{r+1/2}) O(R^{-r-1/2}) 
onumber \ = o(1)$$
 ,

as  $R \to \infty$ , so  $\frac{1}{2\pi} \int_{0}^{2\pi} L(te^{i\theta}) (\cos \theta + \sin \theta) d\theta$ (6.2)  $= t^{r} \lim_{R \to \infty} \sum_{|n| < R} c_{n} \gamma(|n|t)$   $(r(-1)m \int_{0}^{\infty} C^{m+1} d^{m+2} \gamma(nt) dx$ 

$$=t^{r}(-1)^{m}\int_{0}^{\infty}S_{u}^{m+1}rac{d^{m+2}}{du^{m+2}}\gamma(ut)du$$
 .

We define P(z) and  $\lambda(z)$  as in the proof of Theorem 1:

$$P(z) = egin{cases} 0 & ext{if} & r = 1 \ a_1 z + a_3 z^3 + \cdots + a_{r-2} z^{r-2} & ext{if} & r 
eq 1 \end{cases}$$

and

$$\lambda(z) = \gamma(z) - z^{-r}P(z)$$
 .

Then (6.2) becomes,

$$egin{aligned} &rac{1}{2\pi} \int_{\scriptscriptstyle 0}^{\scriptscriptstyle 2\pi} L(te^{i heta})(\cos heta\,+\,\sin heta)d heta\ &=t^r(-1)^m \int_{\scriptscriptstyle 0}^{\infty}\!S_u^{\,m+1} rac{d^{m+2}}{du^{m+2}} [(ut)^{-r}P(ut)\,+\,\lambda(ut)]du \end{aligned}$$

$$= t^{r}(-1)^{m} \int_{0}^{\infty} S_{u}^{m+1} \frac{d^{m+2}}{du^{m+2}} [(ut)^{-r} P(ut)] du$$
  
+  $t^{r}(-1)^{m} \int_{0}^{\infty} S_{u}^{m+1} \frac{d^{m+2}}{du^{m+2}} \lambda(ut) du$   
=  $A(t) + t^{r} B(t)$ .  
$$A = t^{r}(-1)^{m} \int_{1/2}^{\infty} S_{u}^{m+1} \frac{d^{m+2}}{du^{m+2}} (\sum_{\substack{k=1\\k \text{ odd}}}^{r-2} a_{k}(ut)^{k-r}) du$$
  
=  $\sum_{\substack{k=1\\k \text{ odd}}}^{r-2} t^{r+k-r} a_{k}(-1)^{m} \int_{1/2}^{\infty} o(u)^{m+1} \frac{d^{m+2}}{du^{m+2}} u^{k-r} du$   
=  $\sum_{\substack{k=1\\k \text{ odd}}}^{r-2} b_{k} t^{k}$ .

Hence,

(6.3) 
$$\frac{1}{2\pi}\int_0^{2\pi} L(te^{i\theta})(\cos\theta + \sin\theta)d\theta = \sum_{\substack{k=1\\k \text{ odd}}}^{r-2} b_k t^k + t^r B(t)$$

where

(6.4) 
$$B(t) = (-1)^m \int_0^\infty S_u^{m+1} \frac{d^{m+2}}{du^{m+2}} \lambda(ut) du .$$

To complete the proof of Theorem 2 we must show  $B(t) \rightarrow 0$  as  $t \rightarrow 0$ .

If f(u) is a function defined for u > 0 and  $\eta$  is a positive real number, denote by

$$I^\eta f(z) = rac{1}{\Gamma(\eta)} \int_0^z (z-u)^{\eta-1} f(u) du \; ,$$

the fractional integral of order  $\eta$ , see [6]. Now if we set

$$f(u)=S_u=\sum\limits_{|n|< u}c_n$$
 ,

then by (4.1),

$$S_u^{\scriptscriptstyle\eta}=I^{\scriptscriptstyle\eta}S_u$$
 ,

 $\mathbf{SO}$ 

$$egin{array}{lll} S^{m+1}_u &= I^{m+1}S_u \ &= I^{1-lpha}I^{m+lpha}S_u \ &= I^{1-lpha}S^{m+lpha}_u \,. \end{array}$$

Thus,

$$S_{u}^{m+1} = \frac{1}{\Gamma(1-\alpha)} \int_{0}^{u} (u-z)^{1-\alpha-1} S_{z}^{m+\alpha} dz$$
$$= \frac{1}{\Gamma(1-\alpha)} \int_{0}^{u} (u-z)^{-\alpha} S_{z}^{m+\alpha} dz .$$

Returning to (6.4)

$$\begin{split} B(t) &= (-1)^m \int_0^\infty S_u^{m+1} \frac{d^{m+2}}{du^{m+2}} \lambda(ut) du \\ &= \lim_{R \to \infty} (-1)^m \int_0^R \frac{1}{\Gamma(1-\alpha)} \int_0^u (u-z)^{-\alpha} S_z^{m+\alpha} dz \, \frac{d^{m+2}}{du^{m+2}} \lambda(ut) du \\ &= \lim_{R \to \infty} \frac{(-1)^m}{\Gamma(1-\alpha)} \int_0^R S_z^{m+\alpha} \int_z^R (u-z)^{-\alpha} \frac{d^{m+2}}{du^{m+2}} \lambda(ut) du dz \\ &= \lim_{R \to \infty} \frac{(-1)^m}{\Gamma(1-\alpha)} \int_0^R S_z^{m+\alpha} H(z, t, R) dz , \end{split}$$

where

$$egin{aligned} H(z,\,t,\,R) &= \int_{z}^{R} (u-z)^{-lpha} rac{d^{m+2}}{du^{m+2}} \lambda(ut) du \; . \ B(t) &= \lim_{R o \infty} rac{(-1)^m}{\Gamma(1-lpha)} \int_{0}^{1/t} S_z^{m+lpha} H(z,\,t,\,R) dz \ &+ \lim_{R o \infty} rac{(-1)^m}{\Gamma(1-lpha)} \int_{1/t}^{R} S_z^{m+lpha} H(z,\,t,\,R) dz \ &= B_1(t) + B_2(t) \; . \end{aligned}$$

We will make separate estimates of H(z, t, R) for  $B_1(t)$  and for  $B_2(t)$ . First, in the interval of integration involving  $B_1(t)$ ,  $0 \le z \le 1/t$ .

(6.5)  
$$H(z, t, R) = \int_{z}^{R} (u - z)^{-\alpha} \frac{d^{m+2}}{du^{m+2}} \lambda(ut) du$$
$$= \int_{z}^{1/t} + \int_{1/t}^{R}$$
$$= H_{1} + H_{2}.$$

Using the fact that  $\lambda$  is entire,

$$egin{aligned} |H_1| &\leq \int_z^{1/t} (z-u)^{-lpha} t^{m+2} \cdot K du \ &\leq K t^{m+2} \int_z^{1/t} (z-u)^{-lpha} du \ &= O(t^{m+2}) \Big(rac{1}{t} - z\Big)^{1-lpha} \,. \end{aligned}$$

We estimate  $H_2$  by employing (5.11)

$$egin{aligned} H_2 &= \int_{1/t}^{R} (u-z)^{-lpha} rac{d^{m+2}}{du^{m+2}} \lambda(ut) du \ &= \int_{1/t}^{\infty} (u-z)^{-lpha} t^{m+2} O(ut)^{-r-1/2} du \end{aligned}$$

$$egin{aligned} &= O(t^{m-r+3/2}) \Big( rac{1}{t} - z \Big)^{-lpha} \int_{1/t}^{\infty} u^{-r-1/2} du \ &= O(t^{m-r+3/2}) \Big( rac{1}{t} - z \Big)^{-lpha} \Big( rac{1}{t} \Big)^{-r+1/2} \ &= O(t^{m+1}) \Big( rac{1}{t} - z \Big)^{-lpha} \, . \end{aligned}$$

Returning to (6.5),

$$H(z, t, R) = O(t^{m+2}) \Big( rac{1}{t} - z \Big)^{1-lpha} + O(t^{m+1}) \Big( rac{1}{t} - z \Big)^{-lpha} \, .$$

and

$$egin{aligned} B_1(t) &= rac{(-1)^m}{\Gamma(1-lpha)} \int_0^{1/t} S_z^{m+lpha} H(z,\,t,\,R) dz \ &= \int_0^{1/t} o(z^{m+lpha}) \left\{ O(t^{m+2}) \Big(rac{1}{t} - z\Big)^{1-lpha} + O(t^{m+1}) \Big(rac{1}{t} - z\Big)^{-lpha} 
ight\} dz \ &= o \Big(rac{1}{t}\Big)^{m+lpha} \left\{ O(t^{m+2}) \int_0^{1/t} \Big(rac{1}{t} - z\Big)^{1-lpha} dz + O(t^{m+1}) \int_0^{1/t} \Big(rac{1}{t} - z\Big)^{-lpha} dz 
ight\} \ &= o \Big(rac{1}{t}\Big)^{m+lpha} \left\{ O(t^{m+2}) \Big(rac{1}{t}\Big)^{2-lpha} + O(t^{m+1}) \Big(rac{1}{t}\Big)^{1-lpha} 
ight\} \ &= o(1) \ , \end{aligned}$$

as  $t \rightarrow 0$ .

It remains to be shown that  $B_2(t) \to 0$ . In the interval of integration for  $B_2$ ,  $1/t \leq z \leq R$ , and

$$\begin{split} H(z, t, R) &= \int_{z}^{R} (u - z)^{-\alpha} \frac{d^{m+2}}{du^{m+2}} \lambda(ut) du \\ &= \int_{z}^{R} (u - z)^{-\alpha} \frac{d^{m+2}}{du^{m+2}} \Big( \frac{-P(ut)}{(ut)^{r}} \Big) du \\ &+ \int_{z}^{R} (u - z)^{-\alpha} \frac{d^{m+2}}{du^{m+2}} \gamma(ut) du \\ &= H_{a} + H_{b} \; . \end{split}$$

$$\begin{split} H_{a} &= -\int_{z}^{R} (u - z)^{-\alpha} \frac{d^{m+2}}{du^{m+2}} (\sum_{\substack{k=1\\k \text{ odd}}}^{r-2} a_{k}(ut)^{k-r}) du \\ &= \int_{z}^{R} (u - z)^{-\alpha} t^{m+2} O(ut)^{-m-4} du \\ &= t^{-2} \left\{ \int_{z}^{2z} (u - z)^{-\alpha} O(u)^{-m-4} du + \int_{2z}^{\infty} (u - z)^{-\alpha} O(u)^{-m-4} du \right\} \\ &= t^{-2} \{O(z)^{1-\alpha} z^{-m-4} + O(z^{-\alpha}) z^{-m-3}\} \\ &= t^{-2} O(z^{-m-\alpha-3}) \; . \end{split}$$

We change variables in the interval for  $H_b$  by letting x = ut.

$$egin{aligned} H_b(z,\,t,\,R) &= \int_z^R (u\,-\,z)^{-a}\,rac{d^{m+2}}{du^{m+2}}\gamma(ut)du \ &= \int_{tz}^{tR} &igg(rac{x}{t}\,-\,zigg)^{-a}\,t^{m+2}\,rac{d^{m+2}}{du^{m+2}}\,\gamma(x)\,rac{dx}{t} \ &= t^{m+1+lpha}\!\int_{tz}^{tR} &(x\,-\,tz)^{-lpha}\gamma^{(m+2)}(x)dx \ &= t^{m+1+lpha}\!\left\{\!\int_{tz}^{tz+1}+\int_{tz}^{tR}igg\} \ &= H_b'+H_b'' \ . \end{aligned}$$

Recall that  $1/t \leq z$ , so in the interval of integration for  $H_b$ ,  $x > tz \geq 1$ . Thus, by (5.11)

$$|\gamma^{(m+2)}(x)| \leq C x^{-r-1/2}$$
 ,

and

$$egin{aligned} H_b' &= t^{m+1+lpha}\!\!\int_{tz}^{tz+1}\!\!(x-tz)^{-lpha}\gamma^{(m+2)}(x)dx \ &= t^{m+1+lpha}O(tz)^{-r-1/2}\!\int_{tz}^{tz+1}\!\!(x-tz)^{-lpha}dx \ &= t^{m+1+lpha}O(tz)^{-r-1/2} \;. \end{aligned}$$

We estimate  $H_b''$  by integrating by parts.

$$\begin{split} H_b'' &= t^{m+1+\alpha} \int_{tz+1}^{tR} (x-tz)^{-\alpha} \gamma^{(m+2)}(x) dx \\ &= t^{m+1+\alpha} (x-tz)^{-\alpha} \gamma^{(m+1)}(x) \Big|_{tz+1}^{tR} \\ &+ t^{m+1+\alpha} \alpha \int_{tz+1}^{tR} (x-tz)^{-\alpha-1} \gamma^{(m+1)}(x) dx \\ &= t^{m+1+\alpha} (x-tz)^{-\alpha} \gamma^{(m+1)}(x) \Big|_{tz+1}^{tR} \\ &+ t^{m+1+\alpha} O(tz)^{-r-1/2} \int_{tz+1}^{tR} (x-tz)^{-\alpha-1} dx \\ &= t^{m+1+\alpha} (tR-tz)^{-\alpha} \gamma^{(m+1)}(tR) - t^{m+1+\alpha} \gamma^{(m+1)}(tz+1) \\ &+ t^{m+1+\alpha} O(tz)^{-r-1/2} \left(\frac{1}{-\alpha}\right) \{ (tR-tz)^{-\alpha} - 1 \} \\ &= t^{m+1+\alpha} (tR-tz)^{-\alpha} O(tz)^{-r-1/2} + t^{m+1+\alpha} O(tz)^{-r-1/2} \\ &= t^{m+1+\alpha} O(tz)^{-r-1/2} \,. \end{split}$$

Hence, in the interval of integration for  $B_2$ ,

$$egin{aligned} H_b(z,\,t,\,R) &= H_b' + H_b'' \ &= t^{m+1+lpha} O(tz)^{-r-1/2} \ , \end{aligned}$$

and

$$egin{array}{ll} H(z,\,t,\,R) &= H_a \,+\, H_b \ &= t^{-2} O(z^{-m-lpha-3}) \,+\, t^{m+1+lpha} O(tz)^{-r-1/2} \,. \end{array}$$

So,

$$egin{aligned} B_2(t) &= \lim_{R o \infty} rac{(-1)^m}{\Gamma(1-lpha)} \int_{1/t}^R S_z^{m+lpha} H(z,\,t,\,R) dz \ &= \lim_{R o \infty} rac{(-1)^m}{\Gamma(1-lpha)} \int_{1/t}^R o(z)^{m+lpha} \{t^{-2}O(z^{-m-lpha-3}) + t^{m+1+lpha}O(tz)^{-r-1/2}\} dz \ &= t^{-2} \int_{1/t}^\infty o(z^{m+lpha-m-lpha-3}) dz + t^{m+1+lpha-r-1/2} \int_{1/t}^\infty o(z^{m+lpha-r-1/2}) dz \ &= t^{-2}o(z^{-2}) \left|_{1/t}^\infty + t^{m+1/2+lpha-r}o(z^{m+lpha-r+1/2}) \right|_{1/t}^\infty \ &= o(1) \;. \end{aligned}$$

(Note that the hypothesis  $\alpha < 1/2$  is necessary here to insure that the last integral converge.) This completes the proof of Theorem 2.

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