

# Pacific Journal of Mathematics

**INNER-OUTER FACTORIZATION OF FUNCTIONS WHOSE  
FOURIER SERIES VANISH OFF A SEMIGROUP**

HOWARD LEWIS PENN

# INNER-OUTER FACTORIZATION OF FUNCTIONS WHOSE FOURIER SERIES VANISH OFF A SEMIGROUP

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Let  $G$  be a compact, connected, Abelian group. Its dual,  $\Gamma$ , is discrete and can be ordered. Let  $\Gamma_1$  be a semigroup which is a subset of the positive elements for some ordering, but which contains the origin of  $\Gamma$ . Let  $H^p(\Gamma_1)$  be the subspace of  $L^p(G)$  consisting of functions which have vanishing off  $\Gamma_1$ . The question that this paper is concerned with is what conditions on a function in  $H^p(\Gamma_1)$  assure an inner-outer factorization.

An inner function is a function  $f \in H^\infty(\Gamma_1)$  such that  $|f|=1$  a.e. ( $dx$ ) on  $G$ . A function  $f \in H^p(\Gamma_1)$  is said to be outer if

$$\int_G \log |f(x)| = \log \left| \int_G f(x) dx \right| > -\infty.$$

A function  $f \in H^1(\Gamma_1)$  is said to be in the class  $LRP(\Gamma_1)$  if  $\log |f| \in \Gamma_1(G)$  and  $\log |f|$  has Fourier coefficients equal to zero off  $\Gamma_1 \cup -\Gamma_1$ . The main result of the paper is that if  $\Gamma_1$  is the intersection of half planes and  $f \in H^1(\Gamma_1)$  with  $\int_G \log |f(x)| dx > -\infty$  then  $f$  has an inner-outer factorization if and only if  $\log |f|$  is in  $LRP(\Gamma_1)$ .

A semigroup,  $P$ , in  $\Gamma_1$  is called a half plane if  $P \cup -P = \Gamma$  and  $P \cap -P = \{0\}$ . Helson and Lowdenslager [2] proved that if  $\Gamma_1$  is a half plane then every function  $f \in H^p(\Gamma_1)$  with  $\int \log |f| dx > -\infty$  has a factorization as a product of an outer function,  $h \in H^p(\Gamma_1)$  and an inner function,  $g$ , and this factorization is unique up to multiplication by constants of magnitude 1. From now on we shall assume  $\int \log |f| dx > -\infty$ .

Helson and Lowdenslager also showed [3] that if  $u$  is a real function such that  $u$  and  $e^u$  are summable, and  $v$  is the conjugate function of  $u$  with respect to the half plane,  $\Gamma_1$ , then  $e^{u+iv}$  is an outer function in  $H^1(\Gamma_1)$ . Conversely, if a summable outer function has the representation  $e^{u+iv}$  with  $u$  and  $v$  real then  $u$  is summable and  $v$  is equal to its conjugate modulo  $2\pi$  except for an additive constant.

Let  $P$  be a half plane which contains  $\Gamma_1$ . Then, for  $u \in L^1_R(G)$  there exists a conjugate function,  $v$ , which is unique if we assume  $v(0) = 0$ , such that  $u + iv$  has its Fourier series supported on  $P$ . The function,  $v$ , is in  $L^p$ ,  $p < 1$ . If  $u$  has its Fourier coefficients supported only on  $\Gamma_1 \cup -\Gamma_1$  then  $u + iv$  has its Fourier coefficients supported only on  $\Gamma_1$  [4, Chap. 8, §7]. Therefore,  $f \in LRP(\Gamma_1)$  if

and only if  $\log |f| \in L^1$  and  $\log |f|$  is the real part of a function whose Fourier coefficients vanish off  $\Gamma_1$ .

**THEOREM.** *Assuming that  $f \in H^1(\Gamma_1)$  and  $\int \log |f| dx > -\infty$ , and that  $\Gamma_i = \bigcap_{i \in I} P_i$ , then  $f$  has an inner-outer factorization if and only if  $\log |f|$  has its Fourier coefficients vanish off  $\Gamma_1 \cup -\Gamma_1$ .*

*Proof.* Assume  $f \in LRP(\Gamma_1)$ . Let  $u = \log |f| \in L^1(G)$ . Take any  $i \in I$  and consider  $P_i$ .  $\Gamma_1 \subset P_i$  and  $f$  has an inner-outer factorization with respect to  $P_i$ . The outer factor is given by  $e^{u+iv_i}$  where  $v_i$  is the conjugate function to  $u$  with respect to  $P_i$ . Since  $u$  has its Fourier coefficients supported on  $\Gamma_1 \cup -\Gamma_1$ , it follows that  $v_i$  also has its Fourier coefficients supported there. Therefore,  $v_i$  is the same as the conjugate function of  $u$  with respect to any of the other half planes  $P_j$ ,  $j \in I$ . Therefore, the outer factor of  $f$  in  $H^1(P_i)$  is given by  $e^{u+iv_i}$ . Also, if  $P_j$  is any of the other half planes whose intersection gives  $\Gamma_1$ , then the outer factor of  $f$  in  $H^1(P_j)$  is  $e^{u+iv_j}$ . Therefore,  $e^{u+iv_i} \in \bigcap_{i \in I} H^1(P_i)$ , which is just equal to  $H^1(\Gamma_1)$ . For each half plane  $P_i$ ,  $i \in I$ , we have that the inner factor is given by  $f e^{-(u+iv_i)}$ . Therefore, the inner factor is also in  $H^1(\Gamma_1)$ .

Conversely, assume that  $f$  has an inner-outer factorization,  $gh$ , in  $H^1(\Gamma_1)$ . Choose  $P_j$ ,  $j \in I$ , then the outer factor,  $h$ , of  $f$  in  $H^1(\Gamma_1)$ , and hence in  $H^1(P_j)$ , is given by  $e^{u+v_j}$ , where  $v_j$  is the conjugate function of  $u = \log |f|$  with respect to  $P_j$ . Since this is true for all  $P_j$ ,  $j \in I$ , it follows that  $e^{iv_j}$  is the same regardless of which half plane,  $P_j$  is used. Now assume  $P_k$  is another of the half planes whose intersection is  $\Gamma_1$ . Then  $e^{iv_k} = e^{iv_j}$  where  $v_k$  is the conjugate function of  $u$  with respect to  $P_j$ . It follows that  $v_k(x) = v_j(x) + 2n\pi$  where  $n$  might change from point to point. We will now show that  $n = 0$ . Consider the function  $h^{1/2}$  which is outer in  $H^2(\Gamma_1) \subset H^1(\Gamma_1)$ . It follows that  $\log |h^{1/2}| = u/2$ . The conjugate function of  $u/2$  with respect to  $P_j$  is  $v_j/2$  and its conjugate function with respect to  $P_k$  is  $v_k/2$ . By the Helson and Lowdenslager theorem  $h^{1/2} = e^{(u+iv_j)/2}$  and also  $h^{1/2} = e^{(u+iv_k)/2}$ . Therefore,

$$h = h^{1/2} h^{1/2} = e^{u+i(v_j+v_k)/2}.$$

Hence

$$v_k(x) = (v_j(x) + v_k(x))/2 + 2n\pi.$$

So,

$$v_k(x) = v_j(x) + 4n\pi.$$

Now consider  $h^{1/4} = e^{(u+iv_j)/4} = e^{(u+iv_k)/4}$ . Therefore,

$$h = h^{3/4} \bar{h}^{1/4} = e^{3(u+iv_k)/4} e^{(u+iv_j)/4} = e^{u+i(3v_k+v_j)/4}.$$

Hence,

$$v_k(x) = v_j(x) + 8n\pi.$$

By considering the  $2^m$ th roots of  $h$  we can show that the difference between  $v_k$  and  $v_j$  must be  $2^{m+1}n\pi$ . This must hold for all values of  $m$ . The only integer for which this is true is 0. Therefore  $u$  has the same conjugate function with respect to each of the half planes.

We will show that  $u$  has its Fourier coefficients supported on  $\Gamma_1 \cup -\Gamma_1$ . Suppose that  $\hat{u}(\gamma) \neq 0$ , where  $\gamma \notin \Gamma_1 \cup -\Gamma_1$ . Then there exists  $P_j, j \in I$  such that  $\gamma \notin P_j$ . There also exists  $P_k, k \in I$ , such that  $\gamma \notin -P_k$ . Let  $v_j$  be the conjugate functions of  $u$  with respect to the half plane,  $P_j$  and let  $v_k$  be the conjugate function of  $u$  with respect to  $P_k$ . Since  $\gamma \notin P_j$ , we have

$$\hat{v}_j(\gamma) = i\hat{u}(\gamma)$$

[4, Chap. 8, §7]. Likewise, since  $\gamma \notin -P_k$  it follows that  $\gamma \in P_k \setminus 0$  and that

$$\hat{v}_k(\gamma) = -i\hat{u}(\gamma).$$

But since  $\hat{u}(\gamma) \neq 0$ , we have  $\hat{v}_j(\gamma) \neq \hat{v}_k(\gamma)$ , and hence  $v_j$  and  $v_k$  are different functions. But we have just shown that  $u$  has the same conjugate function with respect to each half plane. Therefore  $\hat{u}(\gamma) = 0$  and  $u$  has its Fourier series supported on  $\Gamma_1 \cup -\Gamma_1$ . Therefore  $f \in LRP(\Gamma_1)$ .

**COROLLARY.** *If  $f \in H^1(\Gamma_1)$  where  $\Gamma_1$  is the intersection of half planes and  $f \in LRP(\Gamma_1)$ , then  $f = p_1 p_2$  where  $p_1, p_2 \in H^2(\Gamma_1)$  and  $|p_1|^2 \equiv |p_2|^2 \equiv |f|$*

**EXAMPLE.** In [1] Ebenstein discusses the  $H^p$  functions on a semigroup which is the intersection of a countable collection of half planes. This semigroup fulfills the hypothesis of the theorem. Let  $T^\omega$  be the compact group which is the Cartesian product of countably many circles. The dual  $\sum_{i=1}^\infty Z$ , is the direct sum of countably many copies of the integers. Define  $A \subset \sum_{i=1}^\infty Z$  by

$$A = \{x: x_i \geq 0 \text{ for all } i\}.$$

We may define  $H^p(T^\omega)$ ,  $p \geq 1$  as the subset of  $L^p(T^\omega)$  consisting of these functions whose Fourier coefficients vanish off  $A$ . The semigroup,  $A$ , is the intersection of half planes  $P_i$  defined as follows:

$$P_i = \{x: x_i \geq 0, \text{ if } x_i = 0 \text{ then } x_1 \geq 0, \\ \text{if } x_1, x_2, \dots, x_j = 0 \text{ then } x_{j+1} = 0\}.$$

Therefore the theorem applies to  $H^p(T^\omega)$ .

REMARK. One might hope that certain theorems which hold for the  $H^p$  spaces of the disk would remain true, at least for the class  $LRP(\Gamma_1)$ . One such theorem is Szego's theorem which states if  $w \in L^1(dx)$  and  $w \geq 0$ , then

$$\inf_{g \in A_0} \int |1 - g|^2 w dx = \exp \int \log(w) dx$$

where  $A_0$  consists of those polynomials supported on  $\Gamma_1$ , with zero-th coefficient equal to zero. This theorem is true if  $\Gamma_1$  is a half plane [4, Chap. 8, §3]. Rudin has an example [5, Theorem 4.4.8] of a function,  $f$ , which is outer, but does not span. This same function can be used to show that Szego's theorem fails even for the class  $LRP$ .

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Carol Alf and Thomas Alfonso O'Connor, <i>Unimodality of the Lévy spectral function</i> .....	285
S. J. Bernau and Howard E. Lacey, <i>Bicontractive projections and reordering of <math>L_p</math>-spaces</i> .....	291
Andrew J. Berner, <i>Products of compact spaces with bi-k and related spaces</i> .....	303
Stephen Richard Bernfeld, <i>The extendability and uniqueness of solutions of ordinary differential equations</i> .....	307
Marilyn Breen, <i>Decompositions for nonclosed planar <math>m</math>-convex sets</i> .....	317
Robert F. Brown, <i>Cohomology of homomorphisms of Lie algebras and Lie groups</i> .....	325
Jack Douglas Bryant and Thomas Francis McCabe, <i>A note on Edelstein's iterative test and spaces of continuous functions</i> .....	333
Victor P. Camillo, <i>Modules whose quotients have finite Goldie dimension</i> .....	337
David Downing and William A. Kirk, <i>A generalization of Caristi's theorem with applications to nonlinear mapping theory</i> .....	339
Daniel Reuven Farkas and Robert L. Snider, <i>Noetherian fixed rings</i> .....	347
Alessandro Figà-Talamanca, <i>Positive definite functions which vanish at infinity</i> .....	355
Josip Globevnik, <i>The range of analytic extensions</i> .....	365
André Goldman, <i>Mesures cylindriques, mesures vectorielles et questions de concentration cylindrique</i> .....	385
Richard Grassl, <i>Multisectioned partitions of integers</i> .....	415
Haruo Kitahara and Shinsuke Yorozu, <i>A formula for the normal part of the Laplace-Beltrami operator on the foliated manifold</i> .....	425
Marvin J. Kohn, <i>Summability <math>R_r</math> for double series</i> .....	433
Charles Philip Lanski, <i>Lie ideals and derivations in rings with involution</i> .....	449
Solomon Leader, <i>A topological characterization of Banach contractions</i> .....	461
Daniel Francis Xavier O'Reilly, <i>Cobordism classes of fiber bundles</i> .....	467
James William Pendergrass, <i>The Schur subgroup of the Brauer group</i> .....	477
Howard Lewis Penn, <i>Inner-outer factorization of functions whose Fourier series vanish off a semigroup</i> .....	501
William T. Reid, <i>Some results on the Floquet theory for disconjugate definite Hamiltonian systems</i> .....	505
Caroll Vernon Riecke, <i>Complementation in the lattice of convergence structures</i> .....	517
Louis Halle Rowen, <i>Classes of rings torsion-free over their centers</i> .....	527
Manda Butchi Suryanarayana, <i>A Sobolev space and a Darboux problem</i> .....	535
Charles Thomas Tucker, II, <i>Riesz homomorphisms and positive linear maps</i> .....	551
William W. Williams, <i>Semigroups with identity on Peano continua</i> .....	557
Yukinobu Yajima, <i>On spaces which have a closure-preserving cover by finite sets</i> .....	571