SEMIGROUPS WITH IDENTITY ON PEANO CONTINUA

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A continuum is cell-cyclic if every cyclic element is a finite dimensional cell. We show that any finite dimensional cell-cyclic Peano continuum $X$ admits a commutative semigroup with zero and identity, and apply this to show that if $X$ is also homogeneous it is a point.

In [12] we showed that each cell-cyclic Peano continuum (locally connected metric continuum every cyclic element of which is a finite dimensional cell) $X$ admits a semilattice (commutative idempotent topological semigroup). We now extend this result to show that $X$ admits a commutative semigroup with identity and zero, and then apply this to homogeneous continua. Our extension is a partial answer to a question first raised by R. J. Koch in [6].

A semilattice is also a partially ordered Hausdorff topological space in which every two elements have a greatest lower bound and the function $(x, y) \rightarrow \text{glb}(x, y)$ is continuous. For $A \subseteq S$, let $L(A) = \{z : z \leq x \text{ for some } x \in A\}$ and $M(A) = \{y : x \leq y \text{ for some } x \in A\}$. A set $A$ is increasing if $M(A) = A$. An arc chain is a totally ordered subset of a semilattice whose underlying space is an arc. We reserve $I$ for the unit interval under min multiplication, and $T$ for the quotient semilattice obtained by identifying $(0, 0)$ and $(1, 0)$ in $[0, 1] \times I$. Note that $I^n$ and $T^n$, under coordinatewise multiplication, are semilattices with identity on the $n$-cell, with zero in the boundary and interior respectively.

Let $X$ be a cell-cyclic Peano continuum. We use the cyclic element notation and results of Whyburn [10] and Kuratowski and Whyburn [8], slightly modified in the following way. In $X$ we say a set $A$ separates $a$ and $b$ if each arc from $a$ to $b$ meets $A$. $C(p, q)$ denotes the cyclic chain from $p$ to $q$ and is $\{x \in X \mid \text{some arc from } p \text{ to } q \text{ contains } x\}$. An subcontinuum $A$ of $X$ is an $A$-set if each arc in $X$ having end points in $A$ is contained in $A$. Cyclic elements and cyclic chains are $A$-sets. Given a point $x$ and an $A$-set $A$, if $x \in A$ there is a unique element $y \in A$ such that $y$ separates each element of $A$ from $x$. Denote this $y$ by $P(A, x)$. If $x \in A$ set $P(A, x) = x$. Then for a fixed $A$-set $A$ the function $x \rightarrow P(A, x)$ is a monotone retraction of $X$ onto $A$ mapping $X \setminus A$ into $F_{\gamma}(A) = \{x \in A \mid x \in D^c \}$ for any cyclic element $D$ of $A$ \cup \{cut points of $A\}$. A set $M$ is nodal in $X$ if $M \cap (X \setminus M)^*$ contains at most one point. A point is an end point of $X$ if it has a basis of neighborhoods having one point.
boundary. A node of \(X\) is either (i) a true cyclic element which is a nodal set or (ii) an endpoint. By \(\text{Com}(x, A)\) we mean the component of \(x\) in \(A\). The interior of \(A\) is denoted by \(A^o\).

I. Preliminary results.

**Theorem 1.1.** [12]. Any cell-cyclic Peano continuum admits the structure of a semilattice.

We note that in the proof of 1.1 given in [12], \(I^n\) and \(T^n\), as defined above, could have been used for the semilattice structures on the individual cyclic elements. Thus the structure may be so constructed that each cyclic element is a semilattice with identity; also the zero may be chosen to by any predetermined point.

The following is an unpublished result due to Phyrne Bacon. We include a proof for completeness.

**Theorem 1.2.** Let \(X\) be a compact semilattice and \(C\) an arc chain containing 0. If \(\Pi_c\) is defined by \(\Pi_c(x) = \sup\{a \in C \mid x \in M(a)\}\), then

1. \(\Pi_c\) is a homomorphism from \(X\) onto \(C\)
2. \(\Pi_c\) is continuous iff whenever \(x, y \in C\) and \(x < y\) then \(y \in M(x)^o\).

**Proof.** \(X\) compact implies \(\Pi_c\) is well-defined. For (i), first note that \(\Pi_c\) is order preserving. Let \(x, y \in X\) and suppose \(\Pi_c(x) \leq \Pi_c(y)\). Since \(\Pi_c\) is order preserving we have \(\Pi_c(xy) \leq \Pi_c(x)\). If \(\Pi_c(xy) < \Pi_c(x)\), then there exists \(z \in C\) such that \(\Pi_c(xy) < z < \Pi_c(x)\). Thus \(x \in M(z)\) and \(xy \in M(z)\). But \(\Pi_c(x) < \Pi_c(y)\) and \(x \in M(z)\) implies \(y \in M(z)\). We conclude \(xy \in M(z)\), a contradiction. Thus

\[
\Pi_c(xy) = \Pi_c(x) = \Pi_c(x)\Pi_c(y).
\]

By symmetry, if \(\Pi_c(y) \leq \Pi_c(x)\) then the same conclusion is reached, and \(\Pi_c\) is a homomorphism.

For (ii), suppose whenever \(x, y \in X\) and \(x < y\), then \(y \in M(x)^o\). For each \(x \in C\) define \(V(x) = X \setminus M(x)\). Then each \(V(x)\) is open, and we claim that \(x < y\) implies \(V(x)^o \subset V(y)\). First note that \(M(M(x)^o)\) is open by the continuity of multiplication, contains \(M(x)^o\), and is contained in \(M(x)\). Thus \(M(M(x)^o) = M(x)^o\), and \(M(x)^o\) is increasing. So if \(x < y\), then \(y \in M(x)^o\), and \(M(y) \subseteq M(M(x)^o) = M(x)^o\). Thus \(V(y) = X \setminus M(y)\) contains \(X \setminus M(x)^o = [X \setminus M(x)]^o = V(x)^o\). Since \(C\) is an arc chain, \(\inf\{a \in C \mid x \in V(a)\} = \sup\{a \in C \mid x \in M(a)\} = \Pi_c(x)\). Thus a proof like that for Urysohn’s lemma [3] shows \(\Pi_c\) is continuous. This completes the proof.
It is implicit in results of Lawson [9] that if $X$ is a semilattice on a finite dimensional Peano continuum, then (i) each point of $X$ lies on an arc chain $C$ containing 0, and (ii) if $x < y$ in $C$, then $y \in M(x)^e$. We conclude

**Corollary 1.3.** Each point of a finite dimensional Peano continuum $X$ lies on an arc chain $C$ containing 0 and there is a homomorphic retraction of $X$ onto $C$.

**Theorem 1.4.** Any finite dimensional cell-cyclic chain $C(p, q)$ admits a semilattice with identity. Moreover, if $q \in Fr(C(p, q))$ then $q$ can be chosen to be the identity.

**Proof.** Note that the true cyclic elements of $C(p, q)$ form a countable collection $\{D_i\}$. We consider two cases:

**Case 1.** Some true cyclic element $D_0$ of $C(p, q)$ contains $q$. Then $D_0$ admits a semilattice structure with zero $a = P(D, p) \neq q$ and identity $e$. Moreover if $q \in Fr(C(p, q))$ then $q \in Fr(D_0)$, and so we may choose $e = q$. By 1.1, $C(p, a)$ admits a semilattice in which each cyclic element $D_i$ is a semilattice with identity $e_i$ and zero $P(D_i, p)$. In each $D_i$ there is an arc chain $T_i$ from $e_i$ to $b_i = P(D_i, q)$ and also an arc chain $T_0$ in $D_0$ from $e$ to $a$ and a homomorphism $h: D_0 \rightarrow T_0$ which is a retraction. Let $f_i: T_0 \rightarrow T_i$ be an onto homomorphism for each $i$. Now define a semilattice structure $*$ on $C(p, q)$ to agree with those on $C(p, a)$ and $D_0$ and such that if $x \in C(p, a)$ and $y \in D$ then

$$x*y = y*x = \begin{cases} x & \text{if } x \text{ is a cut point of } C(p, a) \\ x*f_i(h(y)) & \text{if } x \in D_i \end{cases}$$

This obviously idempotent and commutative. Associativity and continuity follow since $h$ and $f_i$ are homomorphisms and continuous. Note that $e$ is an identity for $*$.

**Case 2.** $q$ is not in any true cyclic element of $C(p, q)$. Then there is a sequence $\{c_i\}$ of distinct cut points of $C(p, q)$ such that $\{c_i\} \rightarrow q$ and $c_{i+1}$ separates $c$ from $q$. This implies

$$C(p, q) \bigcup_{i=1}^{\infty} C(c_i, c_{i+1}) = \{q\}.$$ 

Endow each $C(c_i, c_{i+1})$ with a semilattice structure as in 1.1 so that $c_i$ is the zero of $C(c_i, c_{i+1})$ and each cyclic element $D_j$ is a semilattice with zero $P(D_j, p)$ and identity $e_j$, and let $T_j$ be a (possibly degenerate)
are chain in $D_j$ from $e_j$ to $P(D_j, q)$. Let $S_i$ be an arc chain in $C(c_i, c_{i+1})$ from $c_i$ to $c_{i+1}$ and let $h_i: C(c_i, c_{i+1}) \rightarrow S_i$ be a homomorphism and retraction. For each $i, j \in \mathbb{Z}^+$, let $f_{i,j}: S_i \rightarrow T_j$ be an onto homomorphism. Now define an operation $*$ on $C(p, q)$ to agree with that on each $C(c_i, c_{i+1})$ and such that if $x \in C(c_m, c_{m+1})$ and $y \in C(c_n, c_{n+1})$ then

$$
 y \ast x = x \ast y = \begin{cases} 
 x \text{ if } x \text{ is a cut point and } n = m + 1 \\
 x \text{ if } n > m + 1 \\
 x f_{n,i}(h_n(y)) \text{ if } x \text{ is not a cut point} \\
 \text{(i.e., } x \in D_j \text{ for some } j \in \mathbb{Z}^+)\text{ and} \\
 n = m + 1 \\
 xy \text{ if } n = m 
\end{cases}
$$

Define $q$ to be an identity for $C(p, q)$.

This is obviously idempotent and commutative. The proof of associativity is similar to that in Case 1 except in the following case: Suppose $x \in C(c_i, c_{i+1}), y \in C(c_{n+1}, c_{n+2})$ and $z \in C(c_{n+2}, c_{n+3})$. If $x$ is a cut point, then $x \ast y \ast z = x$ in any order, and if $y$ is a cut point then $x \ast y \ast z = x \ast y$ in any order. If neither is a cut point then $x \in D_x$ and $y \in D_y$ for some true cyclic elements $D_x$ and $D_y$. So

$$(x \ast y) \ast z = x \ast y = x f_{n+1,i}(h_{n+1}(y)).$$

Now $x \ast (y \ast z) = x \ast (y f_{n+2,k}(h_{n+2}(z))) = x f_{n+1,i}(h_{n+1}(y f_{n+2,k}(h_{n+2}(z))))$. But $h_{n+1}(y f_{n+2,k}(h_{n+2}(z))) = h_{n+1}(y) h_{n+1}(f_{n+2,k}(h_{n+2}(z)))$ since $h_{n+1}$ is a homomorphism. Also $h_{n+1}(y) \leq P(D_h, q) = h_{n+1}(f_{n+2,k}(h_{n+2}(z)))$ since $S_{n+1} \cap D_h$ is an arc chain with maximum element $P(D_h, q)$ and $T_h$ is an arc chain with minimum element $P(D_h, q)$. It follows that

$$x \ast (y \ast z) = x f_{n+1,i}(h_{n+1}(y)) = x \ast y = (x \ast y) \ast z.$$  

Suppose $x_n \rightarrow x$ and $y_n \rightarrow y$. If $x \neq q \neq y$, then one can prove $x_n \ast y_n \rightarrow xy$ using the continuity of the functions $h_i$ and $f_{n,j}$ and the fact that the cyclic chains $C(c_i, c_{i+1})$ meet only at cut points. If $x = q \neq y$ and $y \in C(c_i, c_{i+1})$ then eventually $c_{i-1} \leq y_n \leq c_{i+2}$ and $c_{i+4} \leq x_n$ so that $x_n \ast y_n = y_n \rightarrow y = xy$. If $x = q = y$ and if $W(x_n, y_n)$ denotes the smaller of $i$ and $j$ where $x_n \in C(c_i, c_{i+1})$ and $y_n \in C(c_j, c_{j+1})$ then $W(x_n, y_n) \rightarrow \infty$ as $n \rightarrow \infty$. Since $x_n \ast y_n \in C(c_{W(x_n, y_n)}, c_{W(x_n, y_n)+1})$ and since $C(p, q)$ is locally connected we conclude that $x_n \ast y_n \rightarrow q = xy$. This completes the proof.

We note that in Case 2, if $c_{n+1}$ separates $x$ from $p$ and $c_n$ separates $y$ from $q$ then $x \ast y = y$.

II. Ruled continua.

**Definition 2.1.** Suppose $X$ is a topological space and $E \subseteq X$,
Let \( A = \{[0, e]: e \in E\} \) be a collection of arcs in \( X \) satisfying:

(i) \( X = U\{[0, e]: e \in E\} \).

(ii) \([0, e] \cap [0, f]\) is a proper subarc of each when \( e \) and \( f \) are distinct elements of \( E \).

(iii) For each \( e \in E \), there is a unique \([0, e] \in A\).

(iv) If \( x_n \to x \) then \([0, x_n] \to [0, x]\) in the sense of lim sup-lim inf convergence.

Then \( A \) is said to be a ruling of \( X \) and \( X \) is said to be a ruled space with zero 0. The concept of a ruled space was introduced by Eberhart in his dissertation [4]. Spaces admitting a stronger type of ruling have been studied by Koch and McAuley [7]. We note that if \( X \) is ruled then for each \( x \in X \) there is a unique arc \([0, x]\) which is contained in every \([0, e]\) containing \( x \).

**Definition 2.2.** A metric \( d \) is radially convex with respect to a partial order \( \leq \) on \( X \) if \( x \leq y, y \leq z \) and \( y \neq z \) imply \( d(x, y) < d(x, z) \).

**Lemma 2.3.** Let \( X \) be a compact metric ruled space. Define \( x \leq y \) iff \( x \in [0, y] \). Then \( \leq \) is a closed partial order on \( X \). Moreover \( X \) admits a metric radially convex with respect to this order, so that if \( r \leq d(0, e) \) there is a unique \( x(r) \in [0, e] \) such that \( d(0, x(r)) = r \).

**Proof.** This is clearly a partial order; that it is closed follows from property iv) of ruled spaces. By a result of Carruth [2], \( X \) admits a metric radially convex with respect to this order. The lemma now follows.

**Theorem 2.4.** Any cell-cyclic Peano continuum \( X \) admits a ruling, and 0 may be chosen to be any point of \( X \).

**Proof.** By 1.1, \( X \) admits a semilattice with zero 0 chosen arbitrarily. As in the proof of 1.1 given in [12], for each true cyclic element \( D \) of \( X \) let \( h_D \) denote the homeomorphism from \( I^a \) or \( T^a \) to \( D \) used to define this semilattice. Set \( E = Fr(X)\setminus\{\text{cut points of } X\}\cup\{0\} \). For each \( e \in X \) and each true cyclic element \( D \) of \( C(0, e) \), define \( T(D, e) \) to be the image under \( h_D \) of the straight line segment \([h_D^{-1}(P(D, 0)), h_D^{-1}(P(D, e))] \) in \( I^a \) or \( T^a \). Then define \([0, e] = (\cup\{T(D, e): D \in C(0, e)\}) \cup \{\text{cut points of } C(0, e)\} \). Then \([0, e]\) is a metric, compact (since \( C(0, e)\cup\{0, e\} \) is open in \( C(0, e) \)) order dense chain in the semilattice \( X \) and hence an arc. We now show the four conditions are satisfied.
(i) $X = U[0, e]: e \in E}$. If $x \in X\setminus E$ then $x$ is either an interior point of some cyclic element $D$ of $X$ or a cut point of $X$. If $x$ is an interior point of $D$ then $x \in h(p_0^{-1}(p(D, 0), h^{-1}(e)))$ for some $e \in Fr(D)$. If $e \in E$ then $x \in T(D, e) \subseteq [0, e]$. If $e \in E$ then choosing an end element $e'$ of a component of $X\setminus \{e\}$ other than the one containing $0$, $x \in [0, e']$

(ii) and (iii) are clear.

(iv) If $e_a \rightarrow e$, then $[0, e_a] \rightarrow [0, e]$. This follows from the fact that $[0, e_a] \subseteq L(e_a)$ and from techniques like those in [12]. We omit the details.

**Theorem 2.5.** Any cell-cyclic Peano continuum with a nodal cyclic element admits a commutative semigroup with identity and zero.

**Proof.** Let $X$ be a cell cyclic Peano continuum and suppose $X = C \cup D$, where $C \cap D = \{0\}$ and $D$ is a true cyclic element. Then $C$ is a cell-cyclic Peano continuum and hence admits a ruling $A = \{[0, e]: e \in E\}$ with zero $0$ and a radially convex metric. Let $h$ be a homeomorphism from $I^n$ or $\Gamma^n$ to $D$, depending on whether $0$ is in the boundary or interior of $D$, and define a semilattice with identity $e$ on $D$ using $h$. Then there is in $D$ an arc chain $S$ from $0$ to $e$ and a retraction $f: D \rightarrow S$ which is a homomorphism. Moreover we may assume that $S$ is radially convex so that for $x, y \in S$, $d(0, xy) = \min \{d(0, x), d(0, y)\}$. Without loss of generality we may assume $d(0, e)$ is maximal among $\{d(0, x): x \in X\}$. Now define a semigroup on $X$ by

$$y * x = x * y = \begin{cases} 0 & \text{if } x, y \in C \\ xy & \text{if } x, y \in D \\ \text{The point in } [0, x] \text{ of distance } r = \min \{d(0, x), d(0, f(y))\} & \text{from } 0 \text{ if } x \in C, y \in D. \end{cases}$$

Associativity is obvious in all cases except the following: Suppose $x \in C$ and $y, z \in D$. Then $(x * y) * z$ is the point in $[0, x]$ of distance $\min \{d(0, x), d(0, f(y)), d(0, f(z))\}$ from $0$, whereas $x * (y * z)$ is the point in $[0, x]$ of distance $\min \{d(0, x), d(0, f(y, z))\}$ from $0$. But $d(0, f(yz)) = d(0, f(y)f(z)) = \min \{d(0, f(y)), d(0, f(z))\}$ so $(x * y) * z = x * (y * z)$. Continuity follows from the properties of ruled spaces and the fact that $f$ is continuous. It is clear that $e$ is an identity and $0$ a zero. This completes the proof.

We conjecture that any $X$ as in 2.5 admits a semilattice with identity. In fact, if $X$ can be embedded in a plane then $X$ can be embedded in a two-cell $N$ and ruled in such a way that $X \cap Fr(N)$ is one of the arcs ruling $X$. One can now apply a theorem from
Eberhart's dissertation to show that $X$ admits a semilattice with identity.

III. Cell-cyclic Peano continua without a nodal cyclic element. The goal of this section is a result like 2.5 for finite dimensional cell-cyclic Peano continua without a nodal cyclic element.

**Lemma 3.1.** Let $X$ be a cell-cyclic Peano continuum. Then there exist two sequences $\{p_i\}$ and $\{q_i\}$ in $Fr(X)$, with $p_i$ and $q_i$ chosen arbitrarily, such that

(i) If we set $H_n = \bigcup_{i=1}^n C(p_i, q_i)$, then for each $n > 1$, $\{p_n\} = C(p_n, q_n) \cap H_{n-1}$

(ii) If we set $H = \bigcup_{n=1}^\infty H_n$, then each point of $X \setminus H$ is an end point of $X$, and so $H^* = X$.

(iii) The diameter of the components of $S \setminus H_n$ tends to 0 uniformly with $1/n$.

**Proof.** This was proved by Whyburn ([10], p. 73) without the condition that $\{p_i\}$ and $\{q_i\}$ are in $Fr(X)$. We show this condition can also be assumed. Whyburn's proof considers a dense sequence $\{r_i\}$ and sets $p_i = r_i$, $q_i = r_{i+1}$. Clearly these may be chosen arbitrarily in $Fr(X)$. In Whyburn's proof, for $j > 1$, $q_j$ is the $r_i$ of smallest index such that $r_i \in H_{j-1}$, and $p_j = P(H_{j-1}, q_j)$. Thus $p_j \in Fr(X)$. If $q_j \notin Fr(X)$, then $q_j$ is an interior point of some true cyclic element $D$. Let $q'_j$ be any point in $Fr(D)$ other than $P(D, p_j)$. Then $C(p_j, q_j) \neq C(p_j, q'_j)$, so we may assume $q_j \in Fr(X)$. The lemma follows.

Now let $X$ be a finite dimensional cell-cyclic Peano continuum without a nodal cyclic element. Then $X$ has at least 2 end points ([10], p. 77); let 0 and 1 denote end points of $X$. Let $\{p_i\}$, $\{q_i\}$, $\{H_n\}$, and $H$ be as described in 3.1, with $p_i = 0$, $q_i = 1$. Each $C(p_i, q_i)$ admits a semilattice with zero $p_i$ and identity $q_i$ by 1.3. We now define inductively an algorithm for defining a semilattice with identity on $H$.

Let $\{c_i\}$ be the sequence of cut points of $C(0,1)$ converging to 1 such that $c_{j+1}$ separates $c_j$ from 1 used in 1.3 to define the semilattice on $C(0,1)$. Let $n_i$ be one more than the smallest $i$ such that $c_i$ separates $p_i$ from 1 in $X$. Set $Q_i = C(p_i, q_i)$, $P_i = [\text{Com}(1, C(0,1)\setminus \{c_{n_i}\})]^*$, and $R_i = [\text{Com}(0, C(0,1)\setminus \{c_{n_i}\})]^*$. Let $T_i$ be an arc chain from $p_i$ to $q_i$ in $Q_i$ and $S_i$ be an arc chain from $c_{n_i}$ to 1 in $C(0,1)$. Let $f_i : S_i \to T_i$ be a continuous onto homomorphism such that $f_i^{-1}(q_i) = M(c_{n_i+1}) \cap S_i$, and let $h_i : P_i \to T_i$ be the continuous onto homomorphism obtained by composing $f_i$ and a homomorphic retraction $r_i$ of $C(c_{n_i},1)$ onto $S_i$. We now define a semilattice $*$ on $H_i = C(p_i, q_i) \cup C(p_n, q_n) = H_i \cup Q_i$ by...
\[ x \ast y = y \ast x = \begin{cases} xy & \text{if } x, y \in C(0, 1) = P \cup R, \text{ or } x, y \in Q_1 \\ xp_2 & \text{if } x \in R_1, y \in Q_1 \\ h_2(x)y & \text{if } x \in P_1, y \in Q_1 \end{cases} \]

where juxtaposition means whichever of the previously defined operations on \( H_1 \) or \( Q_1 \) fits the context.

Associativity is clear in all cases except when \( r \in R_1, p \in P_1, q \in Q_1 \). In this case \( r \ast (p \ast q) = r \ast (h_2(p)q) = r p_2 \), whereas

\[ (r \ast p) \ast q = (r p_2)p_2 = r(p p_2) = r p_2 \]

by the note at the end of Section I. Continuity is easily checked since \( P_1, Q_1 \) and \( R_1 \) meet only at cut points of \( X \). Note that any point in \( C(c_{n+k-1}, 1) \) acts as an identity for any point in

\[ [\Com (0, H_k \setminus \{c_{n_k}\})]^* \]

and 1 acts as an identity for all of \( H_2 \).

Suppose that a semilattice structure with zero 0 and identity 1 has been defined on \( H_{k-1} \) so that the structure agrees with those on \( C(P_i, q_i) \) for each \( i \leq k \). Also suppose \( c_{n_{k-1}} \in \{c_i\} \) has been chosen so that any element of \( [\Com (1, H_{k-1} \setminus \{c_{n_{k-1}-1}\})]^* \) acts as an identity for any element \( [\Com (0, H_{k-1} \setminus \{c_{n_{k-1}}\})]^* \).

Let \( n_k \) be one more than the smallest integer greater than \( n_{k-1} \) such that \( c_{n_k} \) separates \( p_{k+1} \) from 1. Set \( Q_k = C(p_{k+1}, q_{k+1}), P_k = C(c_{n_k}, 1) = [\Com (1, H_{k-1} \setminus \{c_{n_{k-1}}\})]^*, \) and \( R_k = [\Com (0, H_{k-1} \setminus \{c_{n_{k-1}}\})]^* \). Let \( T_k \) be an arc chain from \( P_{k+1} \) to \( q_{k+1} \) in \( Q_k \) and \( S_k = S_1 \cap P_k \). Let \( f_k: S_k \to T_k \) be a continuous onto homomorphism such that

\[ f_k^{-1}(q_{k+1}) = M(c_{n_k+1}) \cap S_k \]

in \( P_k \), and let \( h_k: P_k \to T_k \) be a continuous onto homomorphism obtained by composing \( f_k \) and the homomorphic retraction \( r_k = r_1|P_k \) of \( C(c_{n_k}, 1) = P_k \) onto \( S_k \). We now define a semilattice \( \ast \) with identity 1 on \( H_k \) by

\[ x \ast y = y \ast x = \begin{cases} xy & \text{if } x, y \in H_{k-1} \text{ or } x, y \in Q_k \\ xp_2 & \text{if } x \in R_k, y \in Q_k \\ h_2(x)y & \text{if } x \in P_k, y \in Q_k \end{cases} \]

where juxtaposition means whichever of the previously defined operations on \( H_k \) or \( Q_k \) fits the context.

Again associativity is clear in all cases except when \( r \in R_k, p \in P_k, q \in Q_k \). In this case \( r \ast (p \ast q) = r \ast (h_2(p)q) = r p_2 \), whereas \( (r \ast p) \ast q = (r p_2)p_2 = r(p p_2) \) since the operation on \( H_{k-1} \) is associative. But \( p \in [\Com (1, H_{k-1} \setminus \{c_{n_{k-1}-1}\})]^* \) and \( p_k \in [\Com (0, H_{k-1} \setminus \{c_{n_{k-1}}\})]^* \) so by hypothesis \( p p_k = p_k \), and \( r \ast (p \ast q) = (r \ast p) \ast q \). Continuity is again...
easily checked. Again any point in \([\text{Com}(1, H_k \{c_{n+1}\})]^*\) acts as an identity for any element \([\text{Com}(0, H_k \{c_n\})]^*\). By induction we have proved the following:

**Lemma 3.2.** Each \(H_n\) admits a semilattice with zero 0 and identity 1 so that the operations agree whenever possible.

**Lemma 3.3.** The function \(P(H_n, \cdot): H \rightarrow H_n\) is a retraction and a homomorphism for each \(n\).

**Proof.** It has been previously noted that each \(P(H_n, \cdot)\) is a retraction. To show that each is a homomorphism it suffices to show that the restriction of \(P(H_n, \cdot)\) to \(H_{n+1}\) is a homomorphism, since \(P(H_n, \cdot)\) is the composition of this restriction and \(P(H_{n+1}, \cdot)\). Let \(x, y \in H_{n+1} = H_n \cup Q_n\). If \(x, y \in Q_n\) then

\[
P(H_n, x)^*P(H_n, y) = p_n^*p_n = p_n = P(H_n, x*y)
\]

since \(x*y \in Q_n\). If \(x \in Q_n, y \in H_n\) then there are two cases. If \(y \in P_n\) then \(P(H_n, x)^*P(H_n, y) = p_n^*y = p_n\) since \(p_n \in R_{n-1}\) by definition and any element of \(P_n\) acts as an identity for any element of \(R_{n-1}\). However \(P(H_n, x*y) = P(H_n, x*h_n(y)) = p_n\) since \(x*h_n(y) \in Q_n\). If \(y \in R_n\) then \(P(H_n, x)^*P(H_n, y) = p_n^*y = x*y = P(H_n, x*y)\). This completes the proof of the lemma.

**Lemma 3.4.** Let \(X\) be as above and let \(x, y \in X\), and suppose \([x_n], [y_n]\) are sequences in \(H\) such that \(x_n \rightarrow x, y_n \rightarrow y\). Then there exists \(z \in X\) such that \([x_n*y_n] \rightarrow z\), where \(*\) denotes the operation on any \(H_n\) containing \(x_n\) and \(y_n\), and \(z\) is independent of the choice of the sequences.

**Proof.** We distinguish four cases.

**Case I.** \(x = y = 1\). From the definition of multiplication on \(H\), if \(a, b \in P_k = [\text{Com}(1, H \{c_i\})]^*\) then \(a*b \in P_k\). Now \([P_k]\) forms a neighborhood basis at the end point 1. Since both \([x_n]\) and \([y_n]\) are eventually in each \(P_k\), \([x_n*y_n]\) is eventually in each \(P_k\) and hence \([x_n*y_n] \rightarrow 1\).

**Case II.** \(x, y, 1\) all distinct. Let \(N\) be an integer so large that \(P(H_n, x)\) and \(P(H_n, y)\) are in \(\text{Com}(0, H_0 \{c_N\})\) and that the diameter of any component of \(X \setminus H_N < d(x, y)/2\). This implies \(\text{Com}(x, X \setminus H_N)\) and \(\text{Com}(y, X \setminus H_N)\) are disjoint open sets, and we may assume \(x_n \in \text{Com}(x, X \setminus H_N)\) and \(y_n \in \text{Com}(y, X \setminus H_N)\) for all \(n\). Also we may assume \(d(x_n, y_n) > d(x, y)/2\) for all \(n\). We now show
x_n * y_n = P(H_N, x_n) * P(H_N, y_n) for all n. The statement is obvious if 
\( x_n, y_n \in H_N \). Suppose it is true whenever \( x_n, y_n \in H_m \) for some \( m \geq N \), and let \( x_n, y_n \in H_{m+1} = H_m \cup Q_m \). If \( x_n \in Q_m \) and \( y_n \in H_m \) then \( x_n * y_n = p_m * y_n \). By hypothesis, since \( p_m, y_n \in H_m \) then

\[ P(H_N, x_n) = P(H_N, y_n) \]

for all \( n \). The statement is obvious if \( x_n, y_n \notin H_N \).

Suppose it is true whenever \( x_n, y_n \notin H_m \) for some \( m \geq N \), and let \( x_n, y_n \in H_{m+1} = H_m \cup Q_m \). If \( x_n \in Q_m \) and \( y_n \in H_m \) then \( x_n * y_n = p_m * y_n \). By hypothesis, since \( p_m, y_n \in H_m \) then

\[ P(H_N, x_n) = P(H_N, y_n) \]

But \( P(H_N, x_n) = P(H_N, x_n) \) since \( Q_m \subset \text{Com}(x_n, X \setminus H_N) \). Thus \( x_n * y_n = P(H_N, x_n) * P(H_N, y_n) \). By symmetry the statement is true if \( x_n \in H_m \) and \( y_n \in Q_m \). The statement is obvious if both \( x_n, y_n \in H_m \), whereas the case \( x_n, y_n \in Q_m \) is impossible for it implies \( d(x_n, y_n) < d(x, y)/2 < d(x_n, y_n) \).

We know \( H_N \) is a semilattice and hence

\[ x_n * y_n = P(H_N, x_n) * P(H_N, y_n) \rightarrow P(H_N, x) * P(H_N, y) \]

since \( P(H_N, \cdot) \) is continuous.

Case III. \( x = y \neq 1 \)

(a) \( x = y \in H \). Then \( x = y \) is an end point of \( X \) and \( \{ U_i \} = \{ \text{Com}(x, X \setminus H_i) \} \) is a neighborhood basis at \( x = y \). We show that if \( U_i \) is fixed and if \( x_n, y_n \in U_i \cap H_N \) then \( x_n * y_n \in U_i \cap H_N \), for any \( N \). Note the statement is true for \( N = i \), and let

\[ x_n, y_n \in U_i \cap H_{m+1} = U_i \cap (H_m \cup Q_m) \].

If \( x_n \in Q_m \) and \( y_n \in H_m \) then \( x_n * y_n = p_m * y_n \in U_i \cap H_m \subset U_i \cap H_{m+1} \) by the induction hypothesis. By symmetry the statement is true if \( x_n \in H_m \) and \( y_n \in Q_m \). If \( x_n, y_n \in Q_m \) then \( x_n * y_n \in Q_m \subset U_i \cap H_{m+1} \), and if \( x_n, y_n \in H_m \) the statement follows from the induction hypothesis.

Since \( \{ x_n \} \) and \( \{ y_n \} \) are eventually in each \( U_i \), and since for each \( n \) and each \( i \) we can find \( N(n, i) \) such that \( x_n, y_n \in U_i \cap H_N(n, i) \), we conclude that \( \{ x_n * y_n \} \) is eventually in each \( U_i \). Thus \( \{ x_n * y_n \} \rightarrow x = y \).

(b) \( x = y \in H_N \), some \( N \). Let \( \varepsilon > 0 \). There exists \( L > N \) so that the diameter of any component of \( X \setminus H_L \) is less that \( \varepsilon/2 \), and so that \( B(x, \varepsilon/2) \cap P_L = \emptyset \). We may assume \( d(x_n, x) < \varepsilon/2 \) and \( d(y_n, y) < \varepsilon/2 \) for each \( n \). Divide \( \{ x_n * y_n \} \) into two (perhaps finite) sequences: If \( x_n * y_n \in H_L \) then

\[ x_n * y_n = P(H_L, x_n) * P(H_L, y_n) \rightarrow P(H_L, x) * P(H_L, y) = x y = x = y \]

by Lemma 3.3 and the continuity of multiplication on \( H_L \). If \( x_n * y_n \notin H_L \), then \( x_n \in H_L \) and \( y_n \in H_L \) because \( B(x, \varepsilon/2) \cap P_L = \emptyset \) and using the definition of multiplication on \( H \). Also, using the definition of
multiplication $x_n y_n \in \text{Com}(x_n, X \setminus H_L)$ or $x_n y_n \in \text{Com}(y_n, X \setminus H_L)$. Thus

$$d(x, x_n y_n) \leq d(x, x_n) + d(x_n, x_n y_n) < \varepsilon$$

or

$$d(y, x_n y_n) \leq d(y, y_n) + d(y_n, x_n y_n) < \varepsilon.$$ 

In either case $d(x, x_n y) = d(y, x_n y) < \varepsilon$. We conclude that $\{x_n y_n\} \to x = y$.

**Case IV.** $y \neq x = 1$. We first establish two facts.

(A) If $a, b \in H$ so that $P(H_0, a) \in \text{Com}(1, H_0(c_n))$ and $P(H_0, b) \in \text{Com}(0, H_0(c_n))$ for some $n$, then $a^*b = P(H_0, a)^*b$.

The proof is by the induction on the $H_i$ containing $a$. It is clear for $a \in H_0$. Suppose the statement is true for $a \in H_m$, $m \geq 0$, and let $a \in H_{m+1} = H_m \cup Q_{m+1}$. Suppose $a \in Q_{m+1}$, for the induction hypothesis assures the statement is true if $a \in H_m$. Then since $a$ and $b$ are separated by $c_m, b \in Q_{m+1}$. Hence $a^*b = p_{m+1}^*b$. But $p_{m+1}^*b = P(H_0, p_{m+1})^*b$ by the induction hypothesis, and

$$P(H_0, p_{m+1}) = P(H_0, a),$$

so

$$a^*b = P(H_0, a)^*b.$$

Thus (A) is established.

(B) If $a, b \in H$ so that $a \in \text{Com}(1, H_0(c_n))$ and $b \in \text{Com}(0, H_0(c_n))$ for some $n$, then either $a^*b = a^*P(H_n, b)$ or $a^*b \in \text{Com}(b, X \setminus H_0)^*$. The proof is by induction on the $H_i$ containing $b$. If $b \in H_0$ then $P(H_n, b) = b$ and the statement is true. Suppose the statement is true when $b \in K_m$ for some $m \geq n$, and let $b \in Q_{m+1}$. If $a \in \text{Com}(1, H_0(c_n))$ then $a^*b \in Q_{m+1} \subset \text{Com}(b, X \setminus H_0)^*$. If $a \in [\text{Com}(0, H_0(c_n))]^*$ then $a^*b = a^*p_m$. But $a^*p_m = a^*P(H_n, p_m)$ by the induction hypothesis, and $P(H_n, p_m) = P(H_n, b)$. Thus $a^*b = a^*P(H_n, b)$ and (B) is established.

We now distinguish two subcases of Case IV.

**Subcase 1.** $y \in H_M$, some $M$. Let $\varepsilon > 0$. Choose $M$ so large that $c_M$ does not separate $y$ from $0$ and the diameter of any component of $X \setminus H_M$ is less than $\varepsilon/2$. We may assume that for each $n, P(H_0, x_n) \in \text{Com}(0, H_0(c_M))$ and $P(H_0, x_n) \in \text{Com}(1, H_0(c_M))$. Then by (A), $x_n y_n = P(H_0, x_n)^*y_n$, and by (B), $P(H_0, x_n)^*y_n = P(H_0, x_n)^*P(H_M, y_n)$ or $P(H_0, x_n)^*y_n \in \text{Com}(b, X \setminus H_0)^*$. If the former then

$$x_n y_n = P(H_0, x_n)^*P(H_M, y_n) \to 1^*P(H_M, y) = y$$
by the continuity of the multiplication on $H^\prime$ and Lemma 3.3. In the latter case $d(P(H_0, x_n)*y_n, y_n) < \varepsilon/2$. We may assume $d(y_n, y) < \varepsilon/2$, so $d(y, P(H_0, x_n)*y_n) < \varepsilon$. Thus we conclude that $\{x_n*y_n\} \rightarrow y$.

**Subcase 2.** $y \notin H$. If $V_k = \{\text{Com}(y, X\setminus H_k)\}^*$ then $\{V_k\}$ is a neighborhood basis, so we need only show $\{x_n*y_n\}$ is eventually in each $V_k$. Fix a $V_k$. We may assume again that for each $n$, $P(H_0, y_n) \in \text{Com}(0, H_0 \setminus \{c_M\})$, $P(H_0, x_n) \in \text{Com}(1, H_0 \setminus \{c_M\})$, and $y_n \in V_k$ for some $M \geq k$. By (A) and (B), $x_n*y_n = P(H_0, x_n)*P(H_M, y_n)$ or $x_n*y_n \in \text{Com}(y_n, X\setminus H_k)^* \subseteq V_k$. However $P(H_M, y_n) \in V_k$, and $P(H_0, x_n) \in H_0$, so $P(H_0, x_n)*P(H_M, y_n) \in V_k$. This completes the proof of the lemma.

**Theorem 3.5.** Let $X$ be a finite dimensional cell-cyclic Peano continuum without a nodal element. Then $X$ admits a semilattice with identity.

**Proof.** By the above, the dense set $H$ admits a semilattice with identity. For each $x, y \in X$ let $\{x_n\} \rightarrow x$, $\{y_n\} \rightarrow y$ where $\{x_n\}, \{y_n\}$ are sequences in $H$. Define $xy = \lim \{x_n*y_n\}$. By 3.4 this limit exists and is independent of the choice of the sequences. It follows that this operation is a semilattice with identity on $X$. Combining this with Theorem 2.3 we have

**Corollary 3.6.** Let $X$ be a finite dimensional cell-cyclic Peano continuum. Then $X$ admits a commutative semigroup with identity and zero.

**Corollary 3.6.** Any retract of a two-cell admits a commutative semigroup with identity.

**Proof.** Borsuk [1] has shown that a subset $X$ of a two-cell $A$ is a retract of $A$ if and only if $A$ is a locally connected continuum which does not separate the plane. Whyburn [11] has shown that for locally connected continua in the plane, not separating the plane is equivalent to every cyclic element being a simple closed curve with interior, i.e., a two-cell. Thus a retract of a two-cell is a cell-cyclic Peano continuum, and the result follows from Corollary 3.6.

**Definition 3.8.** A space $X$ is homogeneous if for each pair of points $x$ and $y$ in $X$ there is a homeomorphism of $X$ onto itself carrying $x$ to $y$.

**Theorem 3.9.** Any finite dimensional homogeneous cell-cyclic
Peano continuum (in particular, any homogeneous retract of a two-cell) is a point.

Proof. By a result of Hudson and Mostert [5], any homogeneous compact connected semigroup with identity is a group. Combining this with Corollaries 3.6 and 3.7, unless \( X \) is a point \( X \) admits the structure of a group with two idempotents, a contradiction.

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Received November 3, 1975. Portions of this paper generalize unpublished parts of a dissertation presented to the Graduate School of Louisiana State University. The author wishes to thank Professor R. J. Koch for encouraging the work on these results.

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