SEMIGROUPS WITH IDENTITY ON PEANO CONTINUA

WILLIAM W. WILLIAMS
SEMIGROUPS WITH IDENTITY ON PEANO CONTINUA

W. WILEY WILLIAMS

A continuum is cell-cyclic if every cyclic element is a finite dimensional cell. We show that any finite dimensional cell-cyclic Peano continuum $X$ admits a commutative semigroup with zero and identity, and apply this to show that if $X$ is also homogeneous it is a point.

In [12] we showed that each cell-cyclic Peano continuum (locally connected metric continuum every cyclic element of which is a finite dimensional cell) $X$ admits a semilattice (commutative idempotent topological semigroup). We now extend this result to show that $X$ admits a commutative semigroup with identity and zero, and then apply this to homogeneous continua. Our extension is a partial answer to a question first raised by R. J. Koch in [6].

A semilattice is also a partially ordered Hausdorff topological space in which every two elements have a greatest lower bound and the function $(x, y) \rightarrow \text{glb}(x, y)$ is continuous. For $A \subseteq S$, let $L(A) = \{ z : z \leq x \text{ for some } x \in A \}$ and $M(A) = \{ y : x \leq y \text{ for some } x \in A \}$. A set $A$ is increasing if $M(A) = A$. An arc chain is a totally ordered subset of a semilattice whose underlying space is an arc. We reserve $I$ for the unit interval under min multiplication, and $T$ for the quotient semilattice obtained by identifying $(0, 0)$ and $(1, 0)$ in $\{0, 1\} \times I$. Note that $I^n$ and $T^n$, under coordinatewise multiplication, are semilattices with identity on the $n$-cell, with zero in the boundary and interior respectively.

Let $X$ be a cell-cyclic Peano continuum. We use the cyclic element notation and results of Whyburn [10] and Kuratowski and Whyburn [8], slightly modified in the following way. In $X$ we say a set $A$ separates $a$ and $b$ if each arc from $a$ to $b$ meets $A$. $C(p, q)$ denotes the cyclic chain from $p$ to $q$ and is $\{ x \in X | \text{some arc from } p \text{ to } q \text{ contains } x \}$. An subcontinuum $A$ of $X$ is an $A$-set if each arc in $X$ having end points in $A$ is contained in $A$. Cyclic elements and cyclic chains are $A$-sets. Given a point $x$ and an $A$-set $A$, if $x \notin A$ there is a unique element $y \in A$ such that $y$ separates each element of $A$ from $x$. Denote this $y$ by $P(A, x)$. If $x \in A$ set $P(A, x) = x$. Then for a fixed $A$-set $A$ the function $x \rightarrow P(A, x)$ is a monotone retraction of $X$ onto $A$ mapping $X\setminus A$ into $Fr(A) = \{ x \in A | x \notin D^0 \text{ for any cyclic element } D \text{ of } A \} \cup \{ \text{cut points of } A \}$. A set $M$ is nodal in $X$ if $M \cap (X\setminus M)^*$ contains at most one point. A point is an end point of $X$ if it has a basis of neighborhoods having one point.
boundary. A node of \(X\) is either (i) a true cyclic element which is a nodal set or (ii) an endpoint. By \(\text{Com}(x, A)\) we mean the component of \(x\) in \(A\). The interior of \(A\) is denoted by \(A^\circ\).

I. Preliminary results.

**Theorem 1.1.** [12]. Any cell-cyclic Peano continuum admits the structure of a semilattice.

We note that in the proof of 1.1 given in [12], \(I^n\) and \(T^n\), as defined above, could have been used for the semilattice structures on the individual cyclic elements. Thus the structure may be so constructed that each cyclic element is a semilattice with identity; also the zero may be chosen to be any predetermined point.

The following is an unpublished result due to Phyrne Bacon. We include a proof for completeness.

**Theorem 1.2.** Let \(X\) be a compact semilattice and \(C\) an arc chain containing 0. If \(\Pi_c\) is defined by \(\Pi_c(x) = \sup\{a \in C | x \in M(a)\}\), then

(i) \(\Pi_c\) is a homomorphism from \(X\) onto \(C\)

(ii) \(\Pi_c\) is continuous iff whenever \(x, y \in C\) and \(x < y\) then \(y \in M(x)^\circ\).

**Proof.** \(X\) compact implies \(\Pi_c\) is well-defined. For (i), first note that \(\Pi_c\) is order preserving. Let \(x, y \in X\) and suppose \(\Pi_c(x) \leq \Pi_c(y)\). Since \(\Pi_c\) is order preserving we have \(\Pi_c(xy) \leq \Pi_c(x)\). If \(\Pi_c(xy) < \Pi_c(x)\), then there exists \(z \in C\) such that \(\Pi_c(xy) < z < \Pi_c(x)\). Thus \(x \in M(z)\) and \(xy \in M(z)\). But \(\Pi_c(x) < \Pi_c(y)\) and \(x \in M(z)\) implies \(y \in M(z)\). We conclude \(xy \in M(z)\), a contradiction. Thus

\[
\Pi_c(xy) = \Pi_c(x) = \Pi_c(x)\Pi_c(y).
\]

By symmetry, if \(\Pi_c(y) \leq \Pi_c(x)\) then the same conclusion is reached, and \(\Pi_c\) is a homomorphism.

For (ii), suppose whenever \(x, y \in C\) and \(x < y\), then \(y \in M(x)^\circ\). For each \(x \in C\) define \(V(x) = X \setminus M(x)\). Then each \(V(x)\) is open, and we claim that \(x < y\) implies \(V(x)^* \subset V(y)\). First note that \(M(M(x)^\circ)\) is open by the continuity of multiplication, contains \(M(x)^\circ\), and is contained in \(M(x)\). Thus \(M(M(x)^\circ) = M(x)^\circ\), and \(M(x)^\circ\) is increasing. So if \(x < y\), then \(y \in M(x)^\circ\), and \(M(y) \supseteq M(M(x)^\circ) = M(x)^\circ\). Thus \(V(y) = X \setminus M(y)\) contains \(X \setminus M(x)^\circ = [X \setminus M(x)]^* = V(x)^*\). Since \(C\) is an arc chain, \(\inf\{a \in C | x \in V(a)\} = \sup\{a \in C | x \in M(a)\} = \Pi_c(x)\). Thus a proof like that for Urysohn's lemma [3] shows \(\Pi_c\) is continuous. This completes the proof.
It is implicit in results of Lawson [9] that if $X$ is a semilattice on a finite dimensional Peano continuum, then (i) each point of $X$ lies on an arc chain $C$ containing 0, and (ii) if $x < y$ in $C$, then $y \in M(x)$. We conclude

**Corollary 1.3.** Each point of a finite dimensional Peano continuum $X$ lies on an arc chain $C$ containing 0 and there is a homomorphic retraction of $X$ onto $C$.

**Theorem 1.4.** Any finite dimensional cell-cyclic chain $C(p, q)$ admits a semilattice with identity. Moreover, if $q \in Fr(C(p, q))$ then $q$ can be chosen to be the identity.

**Proof.** Note that the true cyclic elements of $C(p, q)$ form a countable collection $\{D_i\}$. We consider two cases:

**Case 1.** Some true cyclic element $D_0$ of $C(p, q)$ contains $q$. Then $D_0$ admits a semilattice structure with zero $a = P(D, p) \neq q$ and identity $e$. Moreover if $q \in Fr(C(p, q))$ then $q \in Fr(D_0)$, so we may choose $e = q$. By 1.1, $C(p, a)$ admits a semilattice in which each cyclic element $D_i$ is a semilattice with identity $e_i$ and zero $P(D_i, p)$. In each $D_i$ there is an arc chain $T_i$ from $e_i$ to $b_i = P(D_i, q)$ and also an arc chain $T_0$ in $D_0$ from $e$ to $a$ and a homomorphism $h: D_0 \to T_0$ which is a retraction. Let $f_i: T_0 \to T_i$ be an onto homomorphism for each $i$. Now define a semilattice structure * on $C(p, q)$ to agree with those on $C(p, a)$ and $D_0$ and such that if $x \in C(p, a)$ and $y \in D$ then

$$x * y = y * x = \begin{cases} x & \text{if } x \text{ is a cut point of } C(p, a) \\ x \cdot f_i(h(y)) & \text{if } x \in D_i \end{cases}$$

This obviously idempotent and commutative. Associativity and continuity follow since $h$ and $f_i$ are homomorphisms and continuous. Note that $e$ is an identity for *.

**Case 2.** $q$ is not in any true cyclic element of $C(p, q)$. Then there is a sequence $\{c_i\}$ of distinct cut points of $C(p, q)$ such that $\{c_i\} \to q$ and $c_{i+1}$ separates $c$ from $q$. This implies

$$C(p, q) \setminus \bigcup_{i=1}^{\infty} C(c_i, c_{i+1}) = \{q\}.$$  

Endow each $C(c_i, c_{i+1})$ with a semilattice structure as in 1.1 so that $c_i$ is the zero of $C(c_i, c_{i+1})$ and each cyclic element $D_j$ is a semilattice with zero $P(D_j, p)$ and identity $e_j$, and let $T_j$ be a (possibly degenerate)
are chain in $D_j$ from $e_j$ to $P(D_j, q)$. Let $S_i$ be an arc chain in $C(c_i, c_{i+1})$ from $c_i$ to $c_{i+1}$ and let $h_i: C(c_i, c_{i+1}) \to S_i$ be a homomorphism and retraction. For each $i, j \in \mathbb{Z}^+$, let $f_{i,j}: S_i \to T_j$ be an onto homomorphism. Now define an operation $*$ on $C(p, q)$ to agree with that on each $C(c_i, c_{i+1})$ and such that if $x \in C(c_m, c_{m+1})$ and $y \in C(c_n, c_{n+1})$ then

$$
y \ast x = x \ast y = \begin{cases} x & \text{if } x \text{ is a cut point and } n = m + 1 \\
x & \text{if } n > m + 1 \\
x f_{n+1}(h_n(y)) & \text{if } x \text{ is not a cut point} \\
\text{(i.e., } x \in D_j \text{ for some } j \in \mathbb{Z}^+) \text{ and } n = m + 1 \\
xy & \text{if } n = m
\end{cases}
$$

Define $q$ to be an identity for $C(p, q)$. This is obviously idempotent and commutative. The proof of associativity is similar to that in Case 1 except in the following case: Suppose $x \in C(c_n, c_{n+1})$, $y \in C(c_{n+1}, c_{n+2})$ and $z \in C(c_{n+2}, c_{n+3})$. If $x$ is a cut point, then $x \ast y \ast z = x$ in any order, and if $y$ is a cut point then $x \ast y \ast z = x \ast y$ in any order. If neither is a cut point then $x \in D_z$ and $y \in D_k$ for some true cyclic elements $D_j$ and $D_k$. So

$$(x \ast y) \ast z = x \ast y = x f_{n+1,j}(h_{n+1}(y)).$$

Now $x \ast (y \ast z) = x \ast (y f_{n+2,k}(h_{n+2}(z))) = x f_{n+1,j}(h_{n+1}(y f_{n+2,k}(h_{n+2}(z)))) = h_{n+1}(y) h_{n+1}(f_{n+2,k}(h_{n+2}(z)))$ since $h_{n+1}$ is a homomorphism. Also $h_{n+1}(y) \leq P(D_k, q) = h_{n+1}(f_{n+2,k}(h_{n+2}(z)))$ since $S_{n+1} \cap D_k$ is an arc chain with maximum element $P(D_k, q)$ and $T_k$ is an arc chain with minimum element $P(D_k, q)$. It follows that

$$x \ast (y \ast z) = x f_{n+1,j}(h_{n+k}(y)) = x \ast y = (x \ast y) \ast z.$$

Suppose $x_n \to x$ and $y_n \to y$. If $x \neq q \neq y$, then one can prove $x_n \ast y_n \to xy$ using the continuity of the functions $h_i$ and $f_{n,j}$ and the fact that the cyclic chains $C(c_i, c_{i+1})$ meet only at cut points. If $x = q \neq y$ and $y \in C(c_i, c_{i+1})$ then eventually $c_{i-1} \leq y_n \leq c_{i+2}$ and $c_{i+4} \leq x_n$ so that $x_n \ast y_n = y_n \to y = xy$. If $x = q = y$ and if $W(x_n, y_n)$ denotes the smaller of $i$ and $j$ where $x_n \in C(c_i, c_{i+1})$ and $y_n \in C(c_j, c_{j+1})$ then $W(x_n, y_n) \to \infty$ as $n \to \infty$. Since $x_n \ast y_n \in C(c_{W(x_n, y_n)}, c_{W(x_n, y_n)+1})$ and since $C(p, q)$ is locally connected we conclude that $x_n \ast y_n \to q = xy$. This completes the proof.

We note that in Case 2, if $c_{n+1}$ separates $x$ from $p$ and $c_n$ separates $y$ from $q$ then $x \ast y = y$.

II. Ruled continua.

DEFINITION 2.1. Suppose $X$ is a topological space and $E \subseteq X$,
0 \in X$. Let $A = \{[0, e] : e \in E\}$ be a collection of arcs in $X$ satisfying:

(i) $X = \bigcup\{[0, e] : e \in E\}$.

(ii) $[0, e] \cap [0, f]$ is a proper subarc of each when $e$ and $f$ are distinct elements of $E$.

(iii) For each $e \in E$, there is a unique $[0, e] \in A$.

(iv) If $x_a \to x$ then $[0, x_a] \to [0, x]$ in the sense of lim sup-lim inf convergence.

Then $A$ is said to be a ruling of $X$ and $X$ is said to be a ruled space with zero 0. The concept of a ruled space was introduced by Eberhart in his dissertation [4]. Spaces admitting a stronger type of ruling have been studied by Koch and McAuley [7]. We note that if $X$ is ruled then for each $x \in X$ there is a unique arc $[0, x]$ which is contained in every $[0, e]$ containing $x$.

**DEFINITION 2.2.** A metric $d$ is radially convex with respect to a partial order $\leq$ on $X$ if $x \leq y$, $y \leq z$ and $y \neq z$ imply $d(x, y) < d(x, z)$.

**LEMMA 2.3.** Let $X$ be a compact metric ruled space. Define $x \leq y$ iff $x \in [0, y]$. Then $\leq$ is a closed partial order on $X$. Moreover $X$ admits a metric radially convex with respect to this order, so that if $r \leq d(0, e)$ there is a unique $x(r) \in [0, e]$ such that $d(0, x(r)) = r$.

*Proof.* This is clearly a partial order; that it is closed follows from property iv) of ruled spaces. By a result of Carruth [2], $X$ admits a metric radially convex with respect to this order. The lemma now follows.

**THEOREM 2.4.** Any cell-cyclic Peano continuum $X$ admits a ruling, and 0 may be chosen to be any point of $X$.

*Proof.* By 1.1, $X$ admits a semilattice with zero 0 chosen arbitrarily. As in the proof of 1.1 given in [12], for each true cyclic element $D$ of $X$ let $h_D$ denote the homeomorphism from $I^*$ or $T^*$ to $D$ used to define this semilattice. Set $E = \text{Fr}(X)\setminus\{\text{cut points of } X\} \cup \{0\}$. For each $e \in E$ and each true cyclic element $D$ of $C(0, e)$ define $T(D, e)$ to be the image under $h_D$ of the straight line segment $[h_D^{-1}(P(D, 0)), h_D^{-1}(P(D, e))]$ in $I^*$ or $T^*$. Then define $[0, e] = (\cup\{T(D, e) : D \in C(0, e)\}) \cup \{\text{cut points of } C(0, e)\}$. Then $[0, e]$ is a metric, compact (since $C(0, e)[0, e]$ is open in $C(0, e)$) order dense chain in the semilattice $X$ and hence an arc. We now show the four conditions are satisfied.
(i) \( X = U\{0, e\}: e \in E \}. \) If \( x \in X \setminus E \) then \( x \) is either an interior point of some cyclic element \( D \) of \( X \) or a cut point of \( X \). If \( x \) is an interior point of \( D \) then \( x \in h_{\beta}(P(D, 0), h_{\beta}(e)) \) for some \( e \in Fr(D) \). If \( e \in E \) then \( x \in T(D, e) \subseteq [0, e] \). If \( e \in E \) then choosing an end element \( e' \) of a component of \( X \setminus \{e\} \) other than the one containing \( 0, x \in [0, e'] \)

(ii) and (iii) are clear

(iv) If \( e_{a} \to e \), then \( [0, e_{a}] \to [0, e] \). This follows from the fact that \( [0, e_{a}] \subseteq L(e_{a}) \) and from techniques like those in [12]. We omit the details.

**Theorem 2.5.** Any cell-cyclic Peano continuum with a nodal cyclic element admits a commutative semigroup with identity and zero.

**Proof.** Let \( X \) be a cell cyclic Peano continuum and suppose \( X = C \cup D \), where \( C \cap D = \{0\} \) and \( D \) is a true cyclic element. Then \( C \) is a cell-cyclic Peano continuum and hence admits a ruling \( A = \{[0, e]: e \in E \} \) with zero \( 0 \) and a radially convex metric. Let \( h \) be a homeomorphism from \( I^n \) or \( \Gamma^m \) to \( D > \), depending on whether \( 0 \) is in the boundary or interior of \( D \), and define a semilattice with identity \( e \) on \( D \) using \( h \). Then there is in \( D \) an arc chain \( S \) from \( 0 \) to \( e \) and a retraction \( f: D \to S \) which is a homomorphism. Moreover we may assume that \( S \) is radially convex so that for \( x, y \in S \), \( d(0, xy) = \min \{d(0, x), d(0, y)\} \). Without loss of generality we may assume \( d(0, e) \) is maximal among \( \{d(0, x) \mid x \in X\} \). Now define a semigroup on \( X \) by

\[
y \ast x = x \ast y = \begin{cases} 0 & \text{if } x, y \in C \\ xy & \text{if } x, y \in D \\ \text{The point in } [0, x] \text{ of distance } r = \min \{d(0, x), d(0, f(y))\} \text{ from } 0 & \text{if } x \in C, y \in D \end{cases}
\]

Associativity is obvious in all cases except the following: Suppose \( x \in C \) and \( y, z \in D \). Then \( (x \ast y) \ast z \) is the point in \( [0, x] \) of distance \( \min \{d(0, x), d(0, f(y)), d(0, f(z))\} \) from \( 0 \), whereas \( x \ast (y \ast z) \) is the point in \( [0, x] \) of distance \( \min \{d(0, x), d(0, f(y), z))\} \) from \( 0 \). But \( d(0, f(yz)) = d(0, f(y)f(z)) = \min \{d(0, f(y)), d(0, f(z))\} \) so \( x \ast (y \ast z) = x \ast (y \ast z) \). Continuity follows from the properties of ruled spaces and the fact that \( f \) is continuous. It is clear that \( e \) is an identity and \( 0 \) a zero. This completes the proof.

We conjecture that any \( X \) as in 2.5 admits a semilattice with identity. In fact, if \( X \) can be embedded in a plane then \( X \) can be embedded in a two-cell \( N \) and ruled in such a way that \( X \cap Fr(N) \) is one of the arcs ruling \( X \). One can now apply a theorem from...
Eberhart's dissertation to show that $X$ admits a semilattice with identity.

III. Cell-cyclic Peano continua without a nodal cyclic element. The goal of this section is a result like 2.5 for finite dimensional cell-cyclic Peano continua without a nodal cyclic element.

**Lemma 3.1.** Let $X$ be a cell-cyclic Peano continuum. Then there exist two sequences $\{p_i\}$ and $\{q_i\}$ in $Fr(X)$, with $p_i$ and $q_i$ chosen arbitrarily, such that

(i) If we set $H_n = \bigcup_{i=1}^n C(p_i, q_i)$, then for each $n > 1$, $\{p_n\} = C(p_n, q_n) \cap H_{n-1}$

(ii) If we set $H = \bigcup_{n=1}^{\infty} H_n$, then each point of $X \setminus H$ is an end point of $X$, and so $H^* = X$.

(iii) The diameter of the components of $S \setminus H_n$ tends to 0 uniformly with $1/n$.

Proof. This was proved by Whyburn ([10], p. 73) without the condition that $\{p_i\}$ and $\{q_i\}$ are in $Fr(X)$. We show this condition can also be assumed. Whyburn's proof considers a dense sequence $\{r_j\}$ and sets $p_\lambda = r_1$, $q_\gamma = r_2$. Clearly these may be chosen arbitrarily in $Fr(X)$. In Whyburn's proof, for $j > 1$ $q_j$ is the $r_i$ of smallest index such that $r_i \notin H_{j-1}$ and $p_j = P(H_{j-1}, q_j)$. Thus $p_j \in Fr(X)$. If $q_j \notin Fr(X)$, then $q_j$ is an interior point of some true cyclic element $D$. Let $q'_j$ be any point in $Fr(D)$ other than $P(D, p_j)$. Then $C(p_j, q'_j)$ is a semilattice with zero $p_j$ and identity $g^*$ by 1.3. We now define inductively an algorithm for defining a semilattice with identity on $H$.

Now let $X$ be a finite dimensional cell-cyclic Peano continuum without a nodal cyclic element. Then $X$ has at least 2 end points ([10], p. 77); let 0 and 1 denote end points of $X$. Let $\{p_i\}$, $\{q_i\}$, $\{H_n\}$, and $H$ be as described in 3.1, with $p_1 = 0$, $q_1 = 1$. Each $C(p_i, q_i)$ admits a semilattice with zero $p_i$ and identity $q_i$ by 1.3. We now define inductively an algorithm for defining a semilattice with identity on $H$.

Let $\{c_j\}$ be the sequence of cut points of $C(0, 1)$ converging to 1 such that $c_{j+1}$ separates $c_j$ from 1 used in 1.3 to define the semilattice on $C(0, 1)$. Let $n_i$ be one more than the smallest $i$ such that $c_i$ separates $p_2$ from 1 in $X$. Set $Q_i = C(p_i, q_i)$, $P_i = [Com(1, C(0, 1)\setminus \{c_{n_i}\})]^*$, and $R_i = [Com(0, C(0, 1)\setminus \{c_{n_i}\})]^*$. Let $T_i$ be an arc chain from $p_2$ to $q_2$ in $Q_i$, and $S_i$ be an arc chain from $c_{n_i}$ to 1 in $C(0, 1)$. Let $f_i: S_i \rightarrow T_i$ be a continuous onto homomorphism such that $f_i^{-1}(q_2) = M(c_{n_i+1}) \cap S_i$, and let $h_i: P_i \rightarrow T_i$ be the continuous onto homomorphism obtained by composing $f_i$ and a homomorphic retraction $r_i$ of $C(c_{n_i}, 1)$ onto $S_i$. We now define a semilattice $\ast$ on $H_i = C(p_i, q_i) \cup C(p_2, q_2) = H_i \cup Q_i$ by
\[ x * y = y * x = \begin{cases} 
  xy & \text{if } x, y \in C(0, 1) = P_1 \cup R_1 \text{ or } x, y \in Q_1 \\
  xp_z & \text{if } x \in R_1, \ y \in Q_1 \\
  h_z(x)y & \text{if } x \in P_1, \ y \in Q_1 
\end{cases} \]

where juxtaposition means whichever of the previously defined operations on \( H_1 \) or \( Q_1 \) fits the context.

Associativity is clear in all cases except when \( r \in R_1, \ p \in P_1, \ q \in Q_1 \). In this case \( r^*(p*q) = r^*(h_z(p)q) = rp_z \), whereas

\[ (r*p)*q = (rp)p_z = r(pp_z) = rp_z \]

by the note at the end of Section I. Continuity is easily checked since \( P_1, \ Q_1 \) and \( R_1 \) meet only at cut points of \( X \). Note that any point in \( C(c_{n_2+1}, 1) \) acts as an identity for any point in

\[ [\text{Com}(0, H_2\{c_{n_1}\})]^* \]

and 1 acts as an identity for all of \( H_2 \).

Suppose that a semilattice structure with zero 0 and identity 1 has been defined on \( H_{k-1} \) so that the structure agrees with those on \( C(P_i, q_i) \) for each \( i \leq k \). Also suppose \( c_{n_{k-1}} \in \{c_i\} \) has been chosen so that any element of \( [\text{Com}(1, H_{k-1}\{c_{n_{k-1}+1}\})]^* \) acts as an identity for any element \( [\text{Com}(0, H_{k-1}\{c_{n_{k-1}}\})]^* \).

Let \( n_k \) be one more than the smallest integer greater than \( n_{k-1} \) such that \( c_{n_k} \) separates \( p_{k+1} \) from 1. Set \( Q_k = C(p_{k+1}, q_{k+1}), \ P_k = C(c_{n_k}, 1) = [\text{Com}(1, H_k\{c_{n_k}\})]^* \), and \( R_k = [\text{Com}(0, H_k\{c_{n_k}\})]^* \). Let \( T_k \) be an arc chain from \( P_{k+1} \) to \( q_{k+1} \) in \( Q_k \) and \( S_k = S_1 \cap P_k \). Let \( f_k: S_k \rightarrow T_k \) be a continuous onto homomorphism such that

\[ f_k^{-1}(q_{k+1}) = M(c_{n_k+1}) \cap S_k \]

in \( P_k \), and let \( h_k: P_k \rightarrow T_k \) be a continuous onto homomorphism obtained by composing \( f_k \) and the homomorphic retraction \( r_k = r|P_k \) of \( C(c_{n_k}, 1) = P_k \) onto \( S_k \). We now define a semilattice * with identity 1 on \( H_k \) by

\[ x * y = y * x = \begin{cases} 
  xy & \text{if } x, y \in H_{k-1} \text{ or } x, y \in Q_k \\
  xp_k & \text{if } x \in R_k, \ y \in Q_k \\
  h_z(x)y & \text{if } x \in P_k, \ y \in Q_k 
\end{cases} \]

where juxtaposition means whichever of the previously defined operations on \( H_k \) or \( Q_k \) fits the context.

Again associativity is clear in all cases except when \( r \in R_k, \ p \in P_k, \ q \in Q_k \). In this case \( r^*(p*q) = r^*(h_z(p)q) = rp_z \), whereas \( (r*p)*q = (rp)p_z = r(pp_z) \) since the operation on \( H_{k-1} \) is associative. But \( p \in [\text{Com}(1, H_{k-1}\{c_{n_{k-1}+1}\})]^* \) and \( p_z \in [\text{Com}(0, H_{k-1}\{c_{n_{k-1}}\})]^* \) so by hypothesis \( pp_z = p_z \), and \( r^*(p*q) = (r*p)*q \). Continuity is again
easily checked. Again any point in \([\text{Com}(1, H_k\{c_{n+1}\})]^*\) acts as an identity for any element \([\text{Com}(0, H_k\{c_n\})]^*\). By induction we have proved the following:

**Lemma 3.2.** Each \(H_n\) admits a semilattice with zero 0 and identity 1 so that the operations agree whenever possible.

**Lemma 3.3.** The function \(P(H_n, \cdot): H \to H_n\) is a retraction and a homomorphism for each \(n\).

*Proof.** It has been previously noted that each \(P(H_n, \cdot)\) is a retraction. To show that each is a homomorphism it suffices to show that the restriction of \(P(H_n, \cdot)\) to \(H_{n+1}\) is a homomorphism, since \(P(H_n, \cdot)\) is the composition of this restriction and \(P(H_{n+1}, \cdot)\). Let \(x, y \in H_{n+1} = H_n \cup Q_n\). If \(x, y \in Q_n\) then

\[
P(H_n, x) \ast P(H_n, y) = p_n \ast p_n = p_n = P(H_n, x \ast y)
\]

since \(x \ast y \in Q_n\). If \(x \in Q_n, y \in H_n\) then there are two cases. If \(y \in P_n\) then \(P(H_n, x) \ast P(H_n, y) = p_n \ast y = p_n\) since \(p_n \in R_{n-1}\) by definition and any element of \(P_n\) acts as an identity for any element of \(R_{n-1}\). However \(P(H_n, x \ast y) = P(H_n, x \ast h_n(y)) = p_n\) since \(x \ast h_n(y) \in Q_n\). If \(y \in R_n\) then \(P(H_n, x) \ast P(H_n, y) = p_n \ast y = x \ast y = P(H_n, x \ast y)\). This completes the proof of the lemma.

**Lemma 3.4.** Let \(X\) be as above and let \(x, y \in X\), and suppose \(\{x_n\}, \{y_n\}\) are sequences in \(H\) such that \(x_n \to x, y_n \to y\). Then there exists \(z \in X\) such that \(x_n \ast y_n \to z\), where \(\ast\) denotes the operation on any \(H_n\) containing \(x_n\) and \(y_n\), and \(z\) is independent of the choice of the sequences.

*Proof.** We distinguish four cases.

**Case I.** \(x = y = 1\). From the definition of multiplication on \(H\), if \(a, b \in P_k = [\text{Com}(1, H_k\{c_k\})]^*\) then \(a \ast b \in P_k\). Now \(\{P_n\}\) forms a neighborhood basis at the end point 1. Since both \(\{x_n\}\) and \(\{y_n\}\) are eventually in each \(P_k\), \(\{x_n \ast y_n\}\) is eventually in each \(P_k\) and hence \(\{x_n \ast y_n\} \to 1\).

**Case II.** \(x, y,\) and \(1\) all distinct. Let \(N\) be an integer so large that \(P(H_n, x)\) and \(P(H_n, y)\) are in \(\text{Com}(0, H_n\{c_n\})\) and that the diameter of any component of \(X \setminus H_N < d(x, y)/2\). This implies \(\text{Com}(x, X \setminus H_N)\) and \(\text{Com}(y, X \setminus H_N)\) are disjoint open sets, and we may assume \(x_n \in \text{Com}(x, X \setminus H_N)\) and \(y_n \in \text{Com}(y, X \setminus H_N)\) for all \(n\). Also we may assume \(d(x_n, y_n) > d(x, y)/2\) for all \(n\). We now show
\[ x_n \ast y_n = P(H_N, y_n) \ast P(H_N, y_n) \] for all \( n \). The statement is obvious if \( x_n, y_n \in H_N \). Suppose it is true whenever \( x_n, y_n \in H_m \) for some \( m \leq N \), and let \( x_n, y_n \in H_{m+1} = H_m \cup Q_m \). If \( x_n \in Q_m \) and \( y_n \in H_m \) then \( x_n \ast y_n = p_m \ast y_n \). By hypothesis, since \( p_m, y_n \in H_m \) then
\[ p_m \ast y_n = P(H_N, p_m) \ast P(H_N, y_n). \]
But \( P(H_N, p_m) = P(H_N, x_n) \) since \( Q_m \subset \text{Com} \{x_n, X \backslash H_N \} \). Thus \( x_n \ast y_n = P(H_N, x_n) \ast P(H_N, y_n) \). By symmetry the statement is true if \( x_n \in H_m \) and \( y_n \in Q_m \). The statement is obvious if both \( x_n, y_n \in H_m \), whereas the case \( x_n, y_n \in Q_m \) is impossible for it implies \( d(x_n, y_n) < d(x, y)/2 < d(x_n, y_n) \).

We know \( H_N \) is a semilattice and hence
\[ x_n \ast y_n = P(H_N, x_n) \ast P(H_N, y_n) \rightarrow P(H_N, x) \ast P(H_N, y) \]

since \( P(H_N, \cdot) \) is continuous.

**Case III.** \( x = y \neq 1 \)

(a) \( x = y \in H \). Then \( x = y \) is an end point of \( X \) and \( \{U_i\} = \{\text{Com} \{x, X \backslash H_1\}\} \) is a neighborhood basis at \( x = y \). We show that if \( U_i \) is fixed and if \( x_n, y_n \in U_i \cap H_N \) then \( x_n \ast y_n \in U_i \cap H_N \), for any \( N \). Note the statement is true for \( N \leq i \). Suppose it is true whenever \( x_n, y_n \in U_i \cap H_m \) for some \( m \geq i \), and let
\[ x_n, y_n \in U_i \cap H_{m+1} = U_i \cap (H_m \cup Q_m). \]
If \( x_n \in Q_m \) and \( y_n \in H_m \) then \( x_n \ast y_n = p_m \ast y_n \in U_i \cap H_m \subset U_i \cap H_{m+1} \) by the induction hypothesis. By symmetry the statement is true if \( x_n \in H_m \) and \( y_n \in Q_m \). If \( x_n, y_n \in Q_m \) then \( x_n \ast y_n \in Q_m \subset U_i \cap H_{m+1} \), and if \( x_n, y_n \in H_m \) the statement follows from the induction hypothesis.

Since \( \{x_n\} \) and \( \{y_n\} \) are eventually in each \( U_i \), and since for each \( n \) and each \( i \) we can find \( N(n, i) \) such that \( x_n, y_n \in U_i \cap H_{N(n, i)} \), we conclude that \( \{x_n \ast y_n\} \) is eventually in each \( U_i \). Thus \( \{x_n \ast y_n\} \rightarrow x = y \).

(b) \( x = y \in H_N \), some \( N \). Let \( \varepsilon > 0 \). There exists \( L > N \) so that the diameter of any component of \( X \backslash H_L \) is less that \( \varepsilon/2 \), and so that \( B(x, \varepsilon/2) \cap P_L = \emptyset \). We may assume \( d(x_n, x) < \varepsilon/2 \) and \( d(y_n, y) < \varepsilon/2 \) for each \( n \). Divide \( \{x_n \ast y_n\} \) into two (perhaps finite) sequences: If \( x_n \ast y_n \in H_L \) then
\[ x_n \ast y_n = P(H_L, x_n) \ast P(H_L, y_n) \]
\[ = P(H_L, x_n) \ast P(H_L, y_n) \rightarrow P(H_L, x) \ast P(H_L, x) = xy = x = y, \]
by Lemma 3.3 and the continuity of multiplication on \( H_L \). If \( x_n \ast y_n \notin H_L \), then \( x_n \in H_L \) and \( y_n \in H_L \) because \( B(x, \varepsilon/2) \cap P_L = \emptyset \) and using the definition of multiplication on \( H \). Also, using the definition of
multiplication $x_n * y_n \in \text{Com} (x_n, X \setminus H_L)$ or $x_n * y_n \in \text{Com} (y_n, X \setminus H_L)$. Thus
\[ d(x, x_n * y_n) \leq d(x, x_n) + d(x_n, x_n * y_n) < \varepsilon \]
or
\[ d(y, x_n * y_n) \leq d(y, y_n) + d(y_n, x_n * y_n) < \varepsilon . \]
In either case $d(x, x_n * y) = d(y, x_n * y_n) < \varepsilon$. We conclude that \{x_n * y_n\} → x = y.

Case IV. $y \neq x = 1$. We first establish two facts.

(A) If $a, b \in H$ so that $P(H_0, a) \in \text{Com} (1, H_0 \setminus \{c_n\})$ and $P(H_0, b) \in \text{Com} (0, H_0 \setminus \{c_n\})$ for some $n$, then $a * b = P(H_0, a)^* b$.

The proof is by the induction on the $H_i$ containing $a$. It is clear for $a \in H_0$. Suppose the statement is true for $a \in H_m$, $m \geq 0$, and let $a \in H_{m+1} = H_m \cup Q_{m+1}$. Suppose $a \in Q_{m+1}$, for the induction hypothesis assures the statement is true if $a \in H_m$. Then since $a$ and $b$ are separated by $c_m$, $b \in Q_{m+1}$. Hence $a * b = p_{m+1} * b$. But $p_{m+1} * b = P(H_0, p_{m+1}) * b$ by the induction hypothesis, and
\[ P(H_0, p_{m+1}) = P(H_0, a), \]
so
\[ a * b = P(H_0, a)^* b . \]
Thus (A) is established.

(B) If $a, b \in H$ so that $a \in \text{Com} (1, H_0 \setminus \{c_n\})$ and $b \in \text{Com} (0, H_0 \setminus \{c_n\})$ for some $n$, then either $a * b = a * P(H_n, b)$ or $a * b \in \text{Com} (b, X \setminus H_n)^*$. The proof is by induction on the $H_i$ containing $b$. If $b \in H_n$ then $P(H_n, b) = b$ and the statement is true. Suppose the statement is true when $b \in K_m$ for some $m \geq n$, and let $b \in Q_{m+1}$. If $a \in \text{Com} (1, H_0 \setminus \{c_m\})$ then $a * b \in Q_{m+1} \subset \text{Com} (b, X \setminus H_n)^*$. If $a \in \text{Com} (0, H_0 \setminus \{c_m\})^*$ then $a * b = a * p_m$. But $a * p_m = a * P(H_n, p_m)$ by the induction hypothesis, and $P(H_n, p_m) = P(H_n, b)$. Thus $a * b = a * P(H_n, b)$ and (B) is established.

We now distinguish two subcases of Case IV.

Subcase 1. $y \in H_M$, some $M$. Let $\varepsilon > 0$. Choose $M$ so large that $c_M$ does not separate $y$ from 0 and the diameter of any component of $X \setminus H_M$ is less than $\varepsilon/2$. We may assume that for each $n$, $P(H_0, y_n) \in \text{Com} (0, H_0 \setminus \{c_M\})$ and $P(H_0, x_n) \in \text{Com} (1, H_0 \setminus \{c_M\})$. Then by (A), $x_n * y_n = P(H_0, x_n) * y_n$, and by (B), $P(H_0, x_n) * y_n = P(H_0, x_n) * P(H_M, y_n)$ or $P(H_0, x_n) * y_n \in \text{Com} (b, X \setminus H_n)^*$. If the former then
\[ x_n * y_n = P(H_0, x_n) * P(H_M, y_n) \rightarrow 1 * P(H_M, y) = y \]
by the continuity of the multiplication on $H_M$ and Lemma 3.3. In
the latter case $d(P(H_0, x_n)\ast y_n, y_n) < \varepsilon/2$. We may assume $d(y_n, y) < \varepsilon/2$,
so $d(y, P(H_0, x_n)\ast y_n) < \varepsilon$. Thus we conclude that \{x_n\ast y_n\} \to y.

Subcase 2. $y \in H$. If $V_k = [\text{Com}(y, X\setminus H_k)]^*$ then \{V_k\} is a
neighborhood basis, so we need only show \{x_n\ast y_n\} is eventually in
each $V_k$. Fix a $V_k$. We may assume again that for each $n$,
$P(H_0, y_n) \in \text{Com}(0, H_0\setminus\{c_M\})$, $P(H_0, x_n) \in \text{Com}(1, H_0\setminus\{c_M\})$, and $y_n \in V_k$ for
some $M \geq k$. By (A) and (B), $x_n\ast y_n = P(H_0, x_n)\ast P(H_M, y_n)$ or $x_n\ast y_n \in
\text{Com}(y_n, X\setminus H_M)^* \subset V_k$. However $P(H_M, y_n) \in V_k$, and $P(H_0, x_n) \in H_0$, so
$P(H_0, x_n)\ast P(H_M, y_n) \in V_k$. This completes the proof of the lemma.

**Theorem 3.5.** Let $X$ be a finite dimensional cell-cyclic Peano
continuum without a nodal element. Then $X$ admits a semilattice
with identity.

**Proof.** By the above, the dense set $H$ admits a semilattice with
identity. For each $x, y \in X$ let \{x_n\} \to x, \{y_n\} \to y where \{x_n\}, \{y_n\} are
sequences in $H$. Define $xy = \lim \{x_n\ast y_n\}$. By 3.4 this limit exists
and is independent of the choice of the sequences. It follows that
this operation is a semilattice with identity on $X$. Combining this
with Theorem 2.3 we have

**Corollary 3.6.** Let $X$ be a finite dimensional cell-cyclic Peano
continuum. Then $X$ admits a commutative semigroup with identity
and zero.

**Corollary 3.6.** Any retract of a two-cell admits a commutative
semigroup with identity.

**Proof.** Borsuk [1] has shown that a subset $X$ of a two-cell $A$
is a retract of $A$ if and only if $A$ is a locally connected continuum
which does not separate the plane. Whyburn [11] has shown that
for locally connected continua in the plane, not separating the plane
is equivalent to every cyclic element being a simple closed curve
with interior, i.e., a two-cell. Thus a retract of a two-cell is a
cell-cyclic Peano continuum, and the result follows from Corollary
3.6.

**Definition 3.8.** A space $X$ is homogeneous if for each pair of
points $x$ and $y$ in $X$ there is a homeomorphism of $X$ onto itself
carrying $x$ to $y$.

**Theorem 3.9.** Any finite dimensional homogeneous cell-cyclic
Peano continuum (in particular, any homogeneous retract of a two-cell) is a point.

**Proof.** By a result of Hudson and Mostert [5], any homogeneous compact connected semigroup with identity is a group. Combining this with Corollaries 3.6 and 3.7, unless \( X \) is a point, \( X \) admits the structure of a group with two idempotents, a contradiction.

**References**


Received November 3, 1975. Portions of this paper generalize unpublished parts of a dissertation presented to the Graduate School of Louisiana State University. The author wishes to thank Professor R. J. Koch for encouraging the work on these results.
Carol Alf and Thomas Alfonso O'Connor, *Unimodality of the Lévy spectral function* ................................................................. 285
S. J. Bernau and Howard E. Lacey, *Bicontractive projections and reordering of $L_p$-spaces* .......................................................... 291
Andrew J. Berner, *Products of compact spaces with bi-$k$ and related spaces* .......... 303
Marilyn Breen, *Decompositions for nonclosed planar $m$-convex sets* ............... 317
Robert F. Brown, *Cohomology of homomorphisms of Lie algebras and Lie groups* ......................................................................................... 325
Victor P. Camillo, *Modules whose quotients have finite Goldie dimension* ........ 337
David Downing and William A. Kirk, *A generalization of Caristi’s theorem with applications to nonlinear mapping theory* ............. 339
Daniel Reuven Farkas and Robert L. Snider, *Noetherian fixed rings* ................. 347
Alessandro Figà-Talamanca, *Positive definite functions which vanish at infinity* ....................................................................................... 355
Josip Globevnik, *The range of analytic extensions* ............................................. 365
André Goldman, *Mesures cylindriques, mesures vectorielles et questions de concentration cylindrique* .................................................. 385
Richard Grassl, *Multisectioned partitions of integers* ......................................... 415
Haruo Kitahara and Shinsuke Yorozu, *A formula for the normal part of the Laplace-Beltrami operator on the foliated manifold* ................. 425
Marvin J. Kohn, *Summability $R_r$ for double series* ........................................... 433
Charles Philip Lanski, *Lie ideals and derivations in rings with involution* .......... 449
Solomon Leader, *A topological characterization of Banach contractions* .......... 461
Daniel Francis Xavier O’Reilly, *Cobordism classes of fiber bundles* .................. 467
James William Pendergrass, *The Schur subgroup of the Brauer group* ............... 477
Howard Lewis Penn, *Inner-outer factorization of functions whose Fourier series vanish off a semigroup* .................................................. 501
William T. Reid, *Some results on the Floquet theory for disconjugate definite Hamiltonian systems* .................................................. 505
Caroll Vernon Riecke, *Complementation in the lattice of convergence structures* ..................................................................................... 517
Louis Halle Rowen, *Classes of rings torsion-free over their centers* .................... 527
Manda Butchi Suryanarayana, *A Sobolev space and a Darboux problem* ............. 535
Charles Thomas Tucker, II, *Riesz homomorphisms and positive linear maps* ........ 551
William W. Williams, *Semigroups with identity on Peano continua* ................... 557
Yukinobu Yajima, *On spaces which have a closure-preserving cover by finite sets* ......................................................................................... 571