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A practical method is given for finding the classes and centralizers for arbitrary p'-elements in the automorphism group of a Chevalley type group over a field of characteristic p.

1. Introduction. In the study of finite simple groups it is important to know their conjugacy classes and the structure of the corresponding centralizer subgroups. For the alternating groups the results are well known; for the sporadic groups the calculations are special to each group. In this article the authors will study the semi-simple classes in Chevalley type groups. Different methods are required for their unipotent classes.

Our approach is to work, as far as possible, in the algebraic group G corresponding to the given finite Chevalley type group. If t is a semi-simple element in G then, in general, $C_G(t)$ is not connected and its component of the identity is not semi-simple but only reductive. Since certain applications **[3]** require the structure of centralizers of pairs of commuting semi-simple elements we are led to study the rather general situation described in §2, 3, and 4. The underlying theory for these sections is quite simple and is based on essentially two results; (i) the algorithm which leads to the fundamental domain $\mathcal{F}_d$ of 2.4, for this see **[2]**, **[14]** or Appendix 2, and (ii) a general result about algebraic groups, see **[14**, §7**]**, which allows one to reduce questions about semi-simple elements in G to linear algebra problems in certain lattices. In a given case, once the situation in G is clear, the step down to the finite group is easily done by application of Lang's theorem, see **[11]**, **[13]** and §5 below.

In two unpublished notes **[4]**, **[5]** this approach was used to calculate (i) the classes of involutions at odd characteristic and (ii) the 3-elements at even characteristic in $\mathrm{Aut}(L)$ for all finite Chevalley type groups L. We also described the layer of $C_L(t)$. Rather than reproduce these results, we include, in Appendix 1, the structure of certain centralizer subgroups that are of interest for current work on simple groups of component type, see for example **[7]**, **[9]**.

We are indebted to the fundamental paper of Steinberg **[14]** for the basic theory. Earlier work on these question occurs in Abe **[1]**, Ree **[12]** and Iwahori **[10]**. In fact the starting point for our work was the attempt to put the ideas of **[10]** in a form which would give rapid and explicit answers to the sort of questions which arise from finite group theory.

Our notation for finite groups follows **[8]** and for algebraic groups **[6]** and **[14]**. In particular, if C is an algebraic group then C^0 denotes its connected component of the identity. All homomorphisms in §2 are algebraic.

2. Reductive groups.

2.1. *Some standard notation.* G is a connected reductive algebraic group over the algebraically closed field k. T is a maximal torus of G. Put $N = N_G(T)$ and $W = N/T$. If $n \in N$ let $w(n)$ denote its image in W.

Let $X = X(T) = \mathrm{Hom}(T, k^*)$ and $\Gamma = \Gamma(T) = \mathrm{Hom}(k^*, T)$, where k^* is the multiplicative group of k. They are Z-lattices of rank $r =$ dimension of T. Identify Γ with the Z-dual of X by putting $\chi(\eta(\zeta)) = \zeta^{\eta(x)}$, where $\chi \in X$, $\eta \in \Gamma$, $\zeta \in k^*$.

The root system of G relative to T is denoted by Σ and consists of all $\alpha \in X$ for which there exists an isomorphism x_α of the additive group k_+ into G satisfying $tx_\alpha(\zeta)t^{-1} = x_\alpha(\zeta\alpha(t))$ for $t \in T$, $\zeta \in k$.

If $\alpha \in \Sigma$ the subgroup $U_\alpha = x_\alpha(k_+)$ is uniquely determined. Let $T_\alpha = T \cap \langle U_\alpha, U_{-\alpha}\rangle$, a one-dimensional torus, and define the co-root $\hat{\alpha} \in \Gamma$ by $\hat{\alpha}(k^*) = T_\alpha$ and $\hat{\alpha}(\alpha) = 2$. If $N_\alpha = N \cap \langle U_\alpha, U_{-\alpha}\rangle$ there exists $n_\alpha \in N$ such that $N_\alpha = \langle T_\alpha, n_\alpha\rangle$. Let $w_\alpha = w(n_\alpha)$ then $W = \langle w_\alpha \colon \alpha \in \Sigma\rangle$ and w_α permutes Σ as follows: $w_\alpha\beta = \beta - \hat{\alpha}(\beta)\alpha$.

If B is a Borel subgroup of G which contains T let Π denote the resulting set of simple roots in Σ and Σ^+ the corresponding positive roots.

Let $E = E(G)$ be the unique maximal connected semi-simple subgroup of G and $F = F(G)$ the unique torus in G satisfying $EF = G$, $[E, F] = 1$. Note that $F \subseteq T$.

2.2. *Lattices and tori.* Let X^{ad} be the sub-lattice of X spanned by all $\alpha \in \Sigma$. Let X^e be the unique smallest sub-lattice containing X^{ad} and such that X/X^e is torsion free. Put $X^f = \{\chi \in X \colon \hat{\alpha}(\chi) = 0 \text{ all } \alpha \in \Sigma\}$; then X/X^f is torsion free, $X^e \cap X^f = 0$ and $X/X^e + X^f$ is finite.

Let Γ^{sc} be the sub-lattice of Γ spanned by all $\hat{\alpha}$ where $\alpha \in \Sigma$. Let Γ^e be the smallest sub-lattice containing Γ^{sc} with Γ/Γ^e torsion free. Put $\Gamma^f = \{\eta \in \Gamma \colon \eta(\alpha) = 0 \text{ all } \alpha \in \Sigma\}$. The pairs X^e, Γ^f and X^f, Γ^e are orthogonal complements, i.e. $X^e = \{\chi \in X \colon \eta(\chi) = 0 \text{ all } \eta \in \Gamma^f\}$, etc. This gives a useful way to compute X^e.

If Y is any sub-lattice of X define $\mathrm{Ann}\, Y = \{t \in T \colon \chi(t) = 1 \text{ all } \chi \in Y\}$. Then $\mathrm{Ann}\, Y$ is always a closed subgroup of T and is a torus if X/Y is torsion free. For example, $T \cap E = \mathrm{Ann}\, X^f$ and $F = \mathrm{Ann}\, X^e$. Hence, we have the natural isomorphisms, $X(T \cap E) \simeq X/X^f$, $\Gamma(T \cap E) \simeq \Gamma^e$ and $X(T/F) \simeq X^e$, $\Gamma(T/F) \simeq \Gamma/\Gamma^f$.

Note that $E \cap F = 1$ if and only if $X^e + X^f = X$.

2.3. *Morphisms.* If f is a morphism of G onto itself which stabilizes T then f induces endomorphisms f_* on X and f on Γ as follows: if $\chi \in X$, $\eta \in \Gamma$, $t \in T$, $\zeta \in k^*$ then $f_*\chi(t) = \chi(ft)$ and $f\eta(\zeta) = f(\eta(\zeta))$. Thus $(ff')_* = f'_*f_*$ and $f\eta(\chi) = \eta(f_*\chi)$.

An example is $f = i_n$ where $n \in N$ and $i_n g = ngn^{-1}$ for $g \in G$. If $w = w(n)$ we put $i_n\eta = w\eta$ for $\eta \in \Gamma$. This gives a representation of W on Γ. If $\alpha \in \Sigma$ one has $w_\alpha\eta = \eta - \eta(\alpha)\hat{\alpha}$. We adopt a common convention by letting w also denote the action of W on X induced by the permutation of Σ given in 2.1, together with trivial action on X^f; thus $w_\alpha\chi = \chi - \hat{\alpha}(\chi)\alpha$ for $\chi \in X$. Then $(i_n)_* = w^{-1}$ and so $w\eta(w\chi) = \eta(\chi)$.

2.4. *Action of W on Γ.* Let d be a fixed, positive integer and put $d\Gamma = \{d\eta : \eta \in \Gamma\}$. Consider the group $\mathscr{A}_d = \mathscr{A}_d(W, \Gamma) = \{a(w, \lambda): w \in W, \lambda \in d\Gamma\}$ of all affine motions $a(w, \lambda)\mu = w\mu + \lambda$ where $\mu \in \Gamma$. We wish to describe the orbits of $\mathscr{A}_d$ on Γ.

Let $\mathscr{A}_d^{sc} = \mathscr{A}_d(W, \Gamma^{sc})$ and note that $\mathscr{A}_d^{sc} \trianglelefteq \mathscr{A}_d$. A standard result, **[2]**, [**14**, §1], or Appendix 2, states that a fundamental domain for $\mathscr{A}_d^{sc}$ on Γ is given by the set

$$\mathscr{F}_d = \mathscr{F}_d(\Gamma) = \{\mu \in \Gamma : 0 \leqq \mu(\alpha) \leqq d \ \text{ all } \ \alpha \in \Sigma^+\}.$$

Using this result we next describe a fundamental domain for $\mathscr{A}_d$.

Let Π^* denote the extended system of simple roots obtained by adjoining to Π all lowest roots in Σ; one such root for each connected piece of the Dynkin diagram. Let V denote the stabilizer in W of Π^*; its elements are completely characterized by their action on the subset Π^*-Π. Put, for $d = 1$, $\mathscr{F} = \mathscr{F}_1$ and to each $\theta \in \mathscr{F}$ associate a unique $v_\theta \in V$ by: $v_\theta\alpha_* = \alpha_*$ if $\theta(\alpha_*) = 0$ for $\alpha_* \in \Pi^* - \Pi$, otherwise $v_\theta\alpha_* = \alpha$ where α is the unique element in the orbit of α_* under V which satisfies $\theta(\alpha) = 1$.

$\mathscr{F}$ is a set of coset representatives for Γ^{sc} in Γ. If $\eta \in \Gamma$ let $\theta_\eta \in \mathscr{F}$ satisfy $\theta_\eta - \eta \in \Gamma^{sc}$. We define a map δ_d of Γ into $\mathscr{A}_d$ by $\delta_d\eta = a(v_\theta, d\theta)$ where $\theta = \theta_\eta$. Since W centralizes Γ/Γ^{sc} we see that δ_d is a homomorphism with kernel Γ^{sc} (e.g. $\delta_d(\eta + \eta') = \delta_d(\eta)\delta_d(\eta')$). It is easy to verify that the group $\mathscr{C}_d = \mathscr{C}_d(\Gamma) = \delta_d\Gamma$ stabilizes $\mathscr{F}_d$ and is a complement to $\mathscr{A}_d^{sc}$ in $\mathscr{A}_d$. Thus,

PROPOSITION A. *Any fundamental domain for $\mathscr{C}_d$ on $\mathscr{F}_d$ is also a fundamental domain for $\mathscr{A}_d$ on Γ.*

Now consider a fixed $\mu \in \mathscr{F}_d$ and define the subgroups $W(\mu) = \{w \in W: (1 - w)\mu \in d\Gamma\}$, $W(\mu)^{sc} = \{w \in W: (1 - w)\mu \in d\Gamma^{sc}\}$ and $V(\mu) = \{v_\theta \in V: (1 - v_\theta)\mu = d\theta\}$. Using Proposition A and the connection between $\mathscr{F}$ and Γ and, for (ii), a result in [**14**, §1], we have

PROPOSITION B. (i) $W(\mu)^{sc} \trianglelefteq W(\mu)$ *and* $V(\mu)$ *is a complement.* (ii) $W(\mu)^{sc} = \langle w_\alpha : \alpha \in \Sigma, \mu(\alpha) \in dZ \rangle$.

Define the homomorphism $\bar{\delta}$ of Γ into V by $\bar{\delta}\eta = v_\theta$ where $\theta = \theta_\eta$. Since $\Gamma^f \subseteq \mathcal{F}$, and $v_\theta = v_\phi$ if and only if $\theta - \phi \in \Gamma^f$, we have $\operatorname{Ker} \bar{\delta} = \Gamma^{sc} + \Gamma^f$. Let $\Gamma(\mu) = \{\eta \in \Gamma : (\delta_d \eta)\mu = \mu\}$, i.e. such that $v_\theta \mu + d\theta = \mu$ where $\theta = \theta_\eta$. Then $\bar{\delta}\Gamma(\mu) = V(\mu)$.

Since $\mathcal{C}_d$ is abelain $V(\mu)$ and $\Gamma(\mu)$ depend only on the $\mathcal{C}_d$-orbit of μ in $\mathcal{F}_d$. Note that $V(\mu) = 1$ in three particular cases, (i) $d = 1$ with arbitrary Γ, (ii) $\Gamma^e = \Gamma^{sc}$ with arbitrary d, and (iii) $\mu = 0$.

3. Semi-simple classes.

3.1. *Automorphisms.* Let ψ be an algebraic automorphism of G and assume ψ has finite order, equal to a. We assume that a is relatively prime to the characteristic of k. By [**14**, §7] we may suppose that ψ stabilizes a maximal torus T and a Borel subgroup B containing T. Thus ψ induces a permutation $\beta \to \psi(\beta)$ on both Σ and Π, defined by $\psi U_{\psi(\beta)} = U_\beta$ for $\beta \in \Sigma$.

If ψ is multiplied by an inner automorphism, defined by a suitable element of T, we may suppose that $\psi(x_{\psi(\alpha)}(\zeta)) = x_\alpha(\zeta)$ for all $\zeta \in k^*$, $\pm\alpha \in \Pi$. From now on, we assume that ψ is always in this "standard form" relative to T and B.

The possible actions of ψ on G are easily described: if $\{E_1, \cdots, E_s\}$ is a ψ-orbit of simple components of $E = E(G)$ then $\psi^s E_1 = E_1$ and $\psi^s \neq 1$ on E_1 only if E_1 is of type A, D, or E_6. The restriction of ψ to $F = F(G)$ defines an element of $GL(f, Z) \simeq \operatorname{Aut} F$ where $f =$ dimension of F.

Let $\langle G, \psi \rangle$ be the semi-direct product defined by $\psi g \psi^{-1} = \psi(g)$ and $\psi^a = 1$. We wish to describe the classes of semi-simple elements of finite order in $\langle G, \psi \rangle$. It suffices to describe those classes in the coset $G\psi$ under conjugation by G.

In practice, $\psi = 1$ is the most important example. Many of the following calculations simplify considerably in this case.

3.2. *The root system* Σ_ψ. ψ induces endomorphisms on X and Γ (as in 2.3 but, for convenience, we use ψ instead of ψ_*). The action on X is consistent with the permutation of Σ. Put $C_\psi = \psi - 1$ and $S_\psi = 1 + \psi + \cdots + \psi^{a-1}$. Define $X_c = \{\chi \in X : C_\psi \chi = 0\}$, $X_s = \{\chi \in X : S_\psi \chi = 0\}$ and Γ_c, Γ_s similarly. Note that X_c and $\bar{\Gamma} = \Gamma/\Gamma_s$ are natural Z-duals via $\bar{\eta}(\chi) \equiv (\eta + \Gamma_s)(\chi) = \eta(\chi)$ for $\chi \in X_c$.

Let $T_0 = \{t^{-1}\psi(t) : t \in T\}$, then $T_0 = \operatorname{Ann} X_c$ is a sub-torus of T. Put $\bar{T} = T/T_0$ then we have the natural isomorphisms $X(\bar{T}) \simeq X_c$, $\Gamma(\bar{T}) \simeq \bar{\Gamma}$.

If $n \in N$ define $\psi(w(n)) = \psi(n)T$ and let $W_\psi =$

$\{w \in W: \psi(w) = w\}$. W_ψ acts as a reflection group on $\bar{T}$ and hence on X_c, $\bar{\Gamma}$ (see [**14**, §1] and below).

For $\alpha \in \Sigma$ let $0_\alpha = \{\alpha, \psi\alpha, \cdots\}$ denote the orbit due to the action of ψ. Let 0, $0'$ be two such orbits in Σ, if $\alpha \in 0$, $\alpha' \in 0'$ and $\alpha + \alpha' \in \Sigma$ define $0 + 0' = 0_{\alpha+\alpha'}$. This sum is well-defined; i.e., it is independent of the choice of the representatives α, α'. The orbit 0 is called 'good' if there does not exist an orbit $0'$ such that $0' + 0' = 0$ ('bad' orbits can only occur if $E(G)$ contains components of type A of even rank).

Let 0_α be a good orbit, put

$$A_\alpha = \begin{cases} 2(\alpha + \psi\alpha + \cdots) & \text{if } 0_\alpha + 0_\alpha \text{ exists} \\ (\alpha + \psi\alpha + \cdots) & \text{otherwise,} \end{cases}$$

and then define $\Sigma_\psi = \{A_\alpha: \alpha \in \Sigma,\ 0_\alpha \text{ a good orbit}\}$. Σ_ψ is a root-system in X_c with $\hat{\alpha} + \Gamma_s \in \bar{\Gamma}$ the co-root corresponding to A_α.

If $0_\alpha + 0_\alpha$ exists, there is a unique $\alpha' = \psi^s\alpha \in 0_\alpha$, where $|0_\alpha| = 2s$, such that $\alpha + \alpha' \in \Sigma$. Define

$$W_{A_\alpha} = \begin{cases} w_{\alpha+\alpha'} w_{\psi(\alpha+\alpha')} \cdots & \text{if } 0_\alpha + 0_\alpha \text{ exists} \\ w_\alpha w_{\psi\alpha} \cdots & \text{otherwise.} \end{cases}$$

The products depend only on 0_α and not on the order. From [**14**, §1] we have $W_\psi = \langle w_A: A \in \Sigma_\psi \rangle$. Note that if $A = A_\alpha$ and $\chi \in X_c$ then $w_A\chi = \chi - \hat{\alpha}(\chi)$. Let $\Pi_\psi = \{A_\alpha \in \Sigma_\psi: \alpha \in \Pi\}$ and $\Sigma_\psi^+ = \{A_\alpha \in \Sigma_\psi: \alpha \in \Sigma^+\}$.

When G is simple and $\psi \neq 1$ the type of Σ_ψ is:

Σ	A_{2r}	A_{2r-1}	D_r	E_6	D_4
Σ_ψ	C_r	B_r	C_{r-1}	F_4	G_2
a	2	2	2	2	3

If $G = E(G)$ and $\Gamma = \Gamma^{sc}$ then $\bar{\Gamma}$ is spanned by the elements $\hat{\alpha} + \Gamma_s$ and hence is the 'simply-connected' lattice for $\bar{T}$ and Σ_ψ.

3.3. *Reduction to* $\bar{T}$. If $t \in T$ let $\bar{t} = tT_0 \in \bar{T}$.

PROPOSITION C. (i) *Any semi-simple class in* $G\psi$ *intersects* $T\psi$.

(ii) *If* $t, t' \in T$ *then* $t\psi$ *is conjugate to* $t'\psi$ *under* G *if and only if* $w\bar{t} = \bar{t}'$ *for some* $w \in W_\psi$.

Proof. (i) Let $g \in G$ and suppose $g\psi$ is semi-simple. By [**14**, §7] $g\psi$ fixes T^x and B^x for some $x \in G$. Since ψ stabilizes T and B and $\psi(x) = \psi x \psi^{-1}$ we have $xg\psi(x^{-1}) \in B \cap N$. Hence $xg\psi(x^{-1}) = t \in T$ and $g\psi = (t\psi)^x$.

(ii) We first show that if $w \in W_\psi$ then we can choose $n \in N$ so that $w = w(n)$ and $\psi n = n$. It suffices to do this for $w = w_A$ where $A \in \Pi_\psi$. For any $\alpha \in \Pi$ define $n_\alpha = x_\alpha(1)x_{-\alpha}(-1)x_\alpha(1)$. We may assume that x_α, $x_{-\alpha}$ are chosen such that $n_\alpha \in N$. Then $w(n_\alpha) = w_\alpha$ and, since ψ is in standard form, $\psi(n_{\psi\alpha}) = n_\alpha$. If $w_A = w_\alpha w_{\psi\alpha} \cdots$ put $n_A = n_\alpha n_{\psi\alpha} \cdots$. In this case, if α, β are distinct elements in 0_α then $[U_\alpha, U_\beta] = 1$ and hence $[n_\alpha, n_\beta] = 1$. Thus $\psi n_A = n_A$. If $w_A = w_{\alpha+\alpha'} w_{\psi(\alpha+\alpha')} \cdots$ (the case where $0_\alpha + 0_\alpha$ exists) let $n_A = (n_\alpha n_{\alpha'} n_\alpha) \cdots$. Now a direct calculation shows that $n_\alpha n_{\alpha'} n_\alpha = n_{\alpha'} n_\alpha n_{\alpha'}$ and since, as above, distinct terms in n_A commute we again have $\psi n_A = n_A$.

To prove (ii) use the Bruhat normal form for the conjugating element. Thus, in an obvious notation, $(usnv)t\psi = t'\psi(usnv)$. This yields $w(n) \in W_\psi$ and by the previous paragraph we may now suppose that $\psi n = n$. Thus, if $w = w(n)$, $wt = t's^{-1}\psi(s)$ and hence $w\bar{t} = \bar{t}'$.

3.4. *Computation of the classes.* Let $\mathcal{K}$ be a class of semi-simple elements of order d in $G\psi$. Thus d is a multiple of a ($=$ order of ψ) and is also relatively prime to the characteristic of k. These are the only restrictions on d. Choose $\xi_d \in k^*$ a fixed primitive dth root of unity.

Let $T_d = \{t \in T: (t\psi)^d = 1\}$ and put $\bar{T}_d$ for its image in $\bar{T}$. Let $\bar{T}_{(d)} = \{\bar{t} \in \bar{T}: \bar{t}^d = 1\}$ and note that $\bar{T}_d \subseteq \bar{T}_{(d)}$.

For $\bar{\mu} \in \bar{\Gamma}$ define $\bar{\mu}(\xi_d) \in \bar{T}$ by $\bar{\mu}(\xi_d) = \overline{\mu(\xi_d)}$ $(\equiv \mu(\xi_d)T_0)$ where $\mu \in \Gamma$ is any inverse image of $\bar{\mu}$. The map $\bar{\mu} \to \bar{\mu}(\xi_d)$ is a homomorphism of $\bar{\Gamma}$ onto $\bar{T}_{(d)}$ with kernel $d\bar{\Gamma}$. Let $\bar{\Gamma}_c$ denote the image of Γ_c in $\bar{\Gamma}$.

PROPOSITION D. *The homomorphism $\bar{\mu} \to \bar{\mu}(\xi_d)$ maps $\bar{\Gamma}_c$ onto $\bar{T}_d$.*

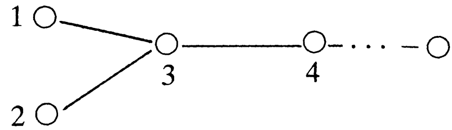
Proof. Let $\mu \in \Gamma_c$ and $e = d/a$ then $(\mu(\xi_d)\psi)^d = eS_\psi\mu(\xi_d) = d\mu(\xi_d) = 1$ and hence $\bar{\mu}(\xi_d) \in \bar{T}_d$. Conversely, suppose $\mu(\xi_d) \in T_d$ then, as above, $eS_\psi\mu \in d\Gamma$. Thus $eS_\psi\mu = d\lambda$ some $\lambda \in \Gamma$. Since $\psi S_\psi = S_\psi\psi$ we have $\lambda \in \Gamma_c$ and hence $d\lambda = eS_\psi\lambda$. Therefore $\mu - \lambda \in \Gamma_s$, as required.

Thus the semi-simple classes, of order dividing d, in $G\psi$ are in 1:1 correspondence with the orbits of $\mathcal{A}_d(W_\psi, \bar{\Gamma})$ on $\bar{\Gamma}_c$. Using proposition A these orbits are easily found.

We have $\mathcal{F}_d(\bar{\Gamma}) = \{\bar{\mu} \in \bar{\Gamma}: 0 \leqq \bar{\mu}(A_\alpha) \leqq d \text{ all } A_\alpha \in \Sigma_\psi^+\}$. If $\{\bar{\mu}, \cdots\}$ is a $\mathcal{C}_d(\bar{\Gamma})$-orbit in $\mathcal{F}_d(\bar{\Gamma}) \cap \bar{\Gamma}_c$ let $\mu \in \Gamma_c$ be an inverse image of $\bar{\mu}$. A representative for the class corresponding to $\{\bar{\mu}, \cdots\}$ is $\mu(\xi_d)\psi$. Since $\Gamma_s \cap \Gamma_c = 0$ μ is uniquely determined by $\bar{\mu}$; also $[\mu(\xi_d), \psi] = 1$.

3.5. *Examples.* Let G be simple of type D_r, with $r \geqq 4$ and even, and suppose $X = X^{ad}$. Let ψ be of order $a = 2$.

Index the simple roots $\Pi = \{\alpha_1, \cdots, \alpha_r\}$ as follows,



Let $\{\eta_j\} \in \Gamma$ be a dual basis to Π, i.e. $\eta_j(\alpha_i) = \delta_{ij}$. We have $\psi\alpha_1 = \alpha_2$, $\psi\alpha_2 = \alpha_1$, $\psi\alpha_i = \alpha_i$ for $i \geqq 3$ and similarly for $\psi\eta_j$. Thus $\Gamma_s = \langle \eta_1 - \eta_2 \rangle$, $\Gamma_c = \langle \eta_1 + \eta_2, \eta_3, \cdots, \eta_r \rangle$ and so $\bar{\Gamma} = \langle \bar{\eta}_1, \bar{\eta}_3, \cdots, \bar{\eta}_r \rangle$ and $\bar{\Gamma}_c = \langle 2\bar{\eta}_1, \bar{\eta}_3, \cdots, \bar{\eta}_r \rangle$. $\Pi_\psi = \{\alpha_1 + \alpha_2, \alpha_3, \cdots, \alpha_r\}$ and $A_* = -(\alpha_1 + \alpha_2 + 2\alpha_3 + \cdots + 2\alpha_r)$ is the corresponding low root in Σ_ψ. Thus $\mathcal{F}(\bar{\Gamma}) = \{0, \bar{\eta}_1\}$ and so $\mathcal{C}_d(\bar{\Gamma}) = \{1, a(v, d\bar{\eta}_1)\}$ where $v \in V_\psi$ (see 4.2), is defined by $v\bar{\eta}_1 = -\bar{\eta}_1$, $v\bar{\eta}_j = -2\bar{\eta}_1 + \bar{\eta}_{r+3-j}$ for $3 \leqq j \leqq r$.

Consider the case $d = 2$. Proposition A gives the following $\mathcal{C}_2(\bar{\Gamma})$ orbits in $\mathcal{F}_2(\bar{\Gamma})$: $\{0, 2\bar{\eta}_1\}$, $\{\bar{\eta}_1\}$, $\{\bar{\eta}_3, \bar{\eta}_r\}$, $\{\bar{\eta}_4, \bar{\eta}_{r-1}\}, \cdots, \{\bar{\eta}_p, \bar{\eta}_{p+1}\}$ where $p = (r+2)/2$. Except for $\{\bar{\eta}_1\}$, all lie in $\bar{\Gamma}_c$. Thus there are $r/2$ classes of involutions in $G\psi$. Representative elements are ψ, $\eta_3(-1)\psi$, $\eta_4(-1)\psi, \cdots, \eta_p(-1)\psi$.

If $r = 4$ and ψ is of order 3 a similar calculation with $d = 3$ gives two classes, with representatives ψ, $\eta_3(\xi_3)\psi$.

Calculations for other G, ψ are entirely similar. The only requirement is a practical description of Π, Π^*, X, Γ and V.

4. Centralizer subgroups.

4.1. *The component of the identity.* We continue with the notation introduced in §3. Let $\mu \in \Gamma_c$ and $\bar{\mu} \in \mathcal{F}_d(\bar{\Gamma})$ then $\varphi = \mu(\xi_d)\psi$ is a typical semi-simple element in $G\psi$ with order dividing d. We put $C = C_G(\varphi)$ and will begin by describing the structure of C^0. From [**14**, §8] we know it is reductive (see also our proof of proposition E).

Put $T_\psi = \{t \in T: \psi t = t\}$ then T_ψ^0 is a maximal torus of C^0. Since $T_\psi^0 = \operatorname{Ann} X_s$ we have the isomorphisms $X(T_\psi^0) \simeq X/X_s = \bar{X}$ and $\Gamma(T_\psi^0) \simeq \Gamma_c$. If $\alpha \in \Sigma$ let $\bar{\alpha}$ denote its image in $\bar{X}$ and put $\Sigma(\varphi) = \{\bar{\alpha}: \mu(A_\alpha) \in dZ, A_\alpha \in \Sigma_\psi\}$.

PROPOSITION E. *$\Sigma(\varphi)$ is a root system for C^0 in $\bar{X}$.*

Proof. Let $\alpha \in \Sigma$ and 0_α a good orbit (see 3.2) and suppose first that $0_\alpha + 0_\alpha$ does not exist. Choose x_α and then define $x_{\psi\alpha}, x_{\psi^2\alpha}, \cdots$ by $\psi(x_{\psi^i\alpha}) = x_{\psi^{i-1}\alpha}$ for $i = 1, 2, \cdots, s-1$ where $s = |0_\alpha|$. A simple argument, using induction on the height of α and the fact that ψ is in standard form, shows that $\psi x_\alpha = x_{\psi^{s-1}\alpha}$. This is the key-point. Since

$(\alpha + \psi\alpha + \cdots)\mu(\xi_d) = \xi_d^{\mu(A_\alpha)}$ we now find that

$$C \cap (U_\alpha U_{\psi\alpha} \cdots) = \begin{cases} 1 & \text{if} \quad \mu(A_\alpha) \notin dZ \\ U_{\bar{\alpha}} & \text{if} \quad \mu(A_\alpha) \in dZ \end{cases}$$

where $U_{\bar{\alpha}} = x_{\bar{\alpha}}(k_+)$ and $x_{\bar{\alpha}}(\zeta) = x_\alpha(\zeta)x_{\psi\alpha}(\epsilon_1\zeta)\cdots x_{\psi^{s-1}\alpha}(\epsilon_{s-1}\zeta)$ for $\zeta \in k$ and $\epsilon_i = \xi_d^{-e_i}$ with $e_i = \mu(\alpha + \psi\alpha + \cdots + \psi^{i-1}\alpha)$. If $t \in T_\psi^0$ then $\alpha(t) = \psi\alpha(t) = \cdots = \bar{\alpha}(t)$, hence $tx_{\bar{\alpha}}(\zeta)t^{-1} = x_{\bar{\alpha}}(\zeta\bar{\alpha}(t))$ and so $\bar{\alpha}$ is a root.

If $0_\alpha + 0_\alpha$ exists the argument is completely similar but slightly more involved. The case where G is simple of type A_2 and $\psi \neq 1$ is quite typical. In this case suppose $\Pi = \{\alpha, \beta\}$ and $\psi \neq 1$. Using the commutation rules in U and the fact that ψ is in standard form we get

$$\psi(x_\alpha(\zeta_\alpha)x_\beta(\zeta_\beta)x_{\alpha+\beta}(\zeta)) = x_\alpha(\zeta_\beta)x_\beta(\zeta_\alpha)x_{\alpha+\beta}(\epsilon\zeta_\alpha\zeta_\beta - \zeta)$$

where $\epsilon = \pm 1$.

Since $\mu(A_\alpha) = \mu(2(\alpha + \beta))$ we have $\xi_d^{\mu(\alpha+\beta)} = \pm 1$ for $\mu(A_\alpha) \in dZ$. Thus

$$C \cap U = \begin{cases} 1 & \text{if} \quad \mu(A_\alpha) \notin dZ \\ U_{\bar{\alpha}} & \text{if} \quad \mu(A_\alpha) \in dZ \end{cases}$$

where the form of $U_{\bar{\alpha}} = x_{\bar{\alpha}}(k_+)$ depends on whether or not $\mu(A_\alpha) \in 2dZ$. We find $x_{\bar{\alpha}}(\zeta) = x_{\alpha+\beta}(\zeta)$ if $\mu(A_\alpha) \notin 2dZ$ and $x_{\bar{\alpha}}(\zeta) = x_\alpha(\zeta)x_\beta(\xi_d^{\mu(\alpha)}\zeta)x_{\alpha+\beta}(\epsilon\xi_d^{\mu(\alpha)}\zeta^2/2)$ if $\mu(A_\alpha) \in 2dZ$.

The co-root in Γ_c corresponding to $\bar{\alpha} \in \Sigma(\varphi)$ is denoted by $\hat{A}_\alpha$ and is given by replacing the α's by $\hat{\alpha}$'s in the definition of A_α.

Using the Bruhat normal form in G we have $C = \langle C \cap U, C \cap N \rangle$ where U is the unipotent radical of B. Thus the above proof shows that $C^0 = \langle U_{\bar{\alpha}}, T_\psi^0 \colon \bar{\alpha} \in \Sigma(\varphi)\rangle$. Hence its Weyl group is $(C^0 \cap N)/T_\psi^0 \simeq \langle w_A : \mu(A) \in dZ, A \in \Sigma_\psi \rangle$.

The structure of C^0 may be described by the methods of 2.1 and 2.2 using $\bar{X}, \Gamma_c, \Sigma(\varphi)$ in place of X, Γ, Σ. The fact that $\{\bar{\alpha} \in \Sigma(\varphi) \colon A_\alpha \in \Pi_\psi^*\}$ contains a set of simple roots for $\Sigma(\varphi)$ is very useful in calculations.

4.2. *Structure of C/C^0.* We will use proposition B but with W, Γ replaced by $W_\psi, \bar{\Gamma}$. Thus V_ψ will denote the stabilizer in W_ψ of the extended root system Π_ψ^* corresponding to Π_ψ in Σ_ψ.

Observe that $C = C^0(C \cap N)$ and if $n \in C \cap N$ then $w(n) \in W_\psi$. Let ϵ denote the homomorphism of $C \cap N$ into W_ψ given by $\epsilon(n) = w(n)$. A simple calculation, using the proof of part (ii) of proposition C, yields that $\operatorname{Ker}\epsilon = T_\psi$ and $\operatorname{Im}\epsilon = W_\psi(\bar{\mu})$. Now the results of 4.1 together with proposition B imply that $\epsilon(C^0 \cap N) = W_\psi(\bar{\mu})^{sc}$ and hence $(C \cap N)/T_\psi(C^0 \cap N) \simeq V_\psi(\bar{\mu})$.

Since $T_\psi = \operatorname{Ann} C_\psi X$ and $Z(G) = \operatorname{Ann} X^{ad}$ we have $T_\psi \cap Z(G) = \operatorname{Ann}(C_\psi X + X^{ad})$ and thus $T_\psi^0(T_\psi \cap Z(G)) = \operatorname{Ann}(X_s \cap (C_\psi X + X^{ad}))$. Since ψ permutes the simple roots Π and $X^{ad} = Z\Pi$ we have $C_\psi X^{ad} = X^{ad} \cap X_s$ and hence find that $T_\psi^0(T_\psi \cap Z(G)) = T_\psi$. Thus,

PROPOSITION F. (i) *$C = C^1(C \cap N)$ where $C^1 = C^0(T_\psi \cap Z(G))$.*

(ii) *$C/C^1 \simeq V_\psi(\bar{\mu})$ and $C^1/C^0 \simeq X_s/C_\psi X$.*

In particular $|C/C^0|$ is finite and relatively prime to the characteristic of k. If $\psi = 1$ then $C^1 = C^0$ and $C/C^0 \simeq V(\mu) \simeq \Gamma(\mu)/\Gamma^{sc} + \Gamma^f$, as in 2.4.

The action of C/C^0 on C^0 is found from the permutations induced on $\Sigma(\varphi)$ by $V_\psi(\bar{\mu})$. In general C will not split over C^0, however in certain cases it does: an example is G simple, $X = X^{ad}$, $\psi = 1$ and $d = 2$. We will not prove this result but the example below should indicate the line of argument.

4.3. *Examples.* Suppose G is simple of type D_4, $X = X^{ad}$ and $\psi = 1$. We use the notation in 3.5 and consider the involution $\eta_3(-1)$ in G. We wish to describe the structure of $C = C_G(\eta_3(-1))$.

Since $d = 2$ and $\alpha_* = -(\alpha_1 + \alpha_2 + \alpha_4 + 2\alpha_3)$ we find $\Pi^* \cap \Sigma(\eta_3(-1)) = \{\alpha_1, \alpha_2, \alpha_4, \alpha_*\}$ as a set of simple roots for C^0. Since $-\alpha_3 = (\alpha_1 + \alpha_2 + \alpha_4 + \alpha_*)/2$ we see that C^0 is a central product of four $SL_2(k)$'s (and $Z(C^0) = \langle \eta_3(-1) \rangle$).

In D_4; $\mathcal{F} = \{0, \eta_1, \eta_2, \eta_4\}$ and if $v_i = v_{\eta_i}$ we have $v_i\eta_3 = -2\eta_i + \eta_3$ for $i = 1, 2, 4$. Thus $V(\eta_3) = V$ and hence $C/C^0 \simeq Z_2 \times Z_2$ and acts as a regular permutation group on the four components of C^0.

If n_1 lies in the coset $v_1 \in W$, it induces an algebraic (outer) automorphism of C^0 and we may choose n_1 so that it is in standard form (see 3.1) relative to the simple roots $\{\alpha_1, \alpha_2, \alpha_4, \alpha_*\}$. Thus n_1^2 will centralize C^0. Now $n_1 x_3(\zeta) n_1^{-1} = x_3(\pm\zeta)$ and hence n_1^2 centralizes U_{α_3}. Since $\langle C^0, U_{\alpha_3} \rangle = G$ we conclude that $n_1^2 = 1$. Choose n_2 similarly, so that $n_2^2 = 1$, and put $n_4 = n_1 n_2$. Since n_1, n_2 are in standard form so also is n_4 and thus $n_4^2 = 1$. This implies $n_1 n_2 = n_2 n_1$ and so $\langle n_1, n_2 \rangle$ is a complement to C^0 in C and the extension splits.

As a further example we describe the structure of C when G is simple, $X = X^{ad}$, $\psi \neq 1$ and where φ runs over the classes of order $d = a$ in the coset $G\psi$. A typical calculation of these classes was given in 3.5. We find that C^0 is always semi-simple; the following table describes the various cases.

G	a	C^0
A_{2r}	2	B_r
A_{2r-1}, $r \geqq 2$	2	C_r, D_r
D_r $r \geqq 4$	2	B_{r-1}, $B_i \times B_{r-1-i}$ for $i = 1, 2, \cdots, \left[\frac{r-1}{2}\right]$
D_4	3	G_2, A_2
E_6	2	F_4, C_4

The lattice for C^0 is always adjoint and $C = C^0$ with two exceptions: (i) $G = A_{2r-1}$, $C^0 = D_r$ when $C = \langle C^0, n\rangle$ where n is the automorphism interchanging α_1 and α_2 and $n^2 = 1$, (ii) $G = D_r$ $r = \text{odd}$, $C^0 = B_{(r-1)/2} \times B_{(r-1)/2}$ when $C = \langle C^0, n\rangle$ where n interchanges the two components and $n^2 = 1$.

5. The finite groups.

5.1. *Endomorphisms of finite type.* Let σ be a morphism of G onto itself with $\ker \sigma = 1$ and such that $G(\sigma) = \{g \in G : \sigma g = g\}$ is finite. If $E(G) \neq 1$ the existence of such a σ implies that the characteristics of k is $p \neq 0$ and that σ^{-1} is not a morphism (although σ is an automorphism of G considered as an abstract group). The possibilities for σ are well known, see [**14**, §11].

We may suppose that σ stabilizes T and hence that σ induces a permutation on Σ. This permutation is consistent with the action of σ_* on X as defined in 2.3. If $n \in N$ and $\rho = i_n\sigma$ then $\rho_* = \sigma_* w^{-1}$ where $w = w(n)$ (and $\rho = w\sigma$ on Γ). Using a theorem of Lang [**11**] we may find $h \in G$ such that $h\sigma(h^{-1}) = n$ and hence $g \to i_h g$ gives an isomorphism of $G(\sigma)$ onto $G(\rho)$.

5.2. *Structure of* $G(\sigma)$. Put $L = 0^{p'}(G(\sigma))$ where p is the characteristic of k. $0^{p'}(X)$ denotes the normal subgroup of X generated by all elements of order p. Thus L is that subgroup of $G(\sigma)$ generated by all the unipotent elements. Put $J = C_{G(\sigma)}(L)$, $Z = J \cap L$ and $D = G(\sigma)/JL$. Then $J = Z(G(\sigma))$ and $Z = Z(L)$ and D induces 'diagonal type' automorphisms of L. If we make allowance for the usual 8 exceptions ($A_1(2)$, ${}^2A_2(2)$, $C_2(2)$, ${}^2C_2(2)$, $G_2(2)$, ${}^2F_4(2)$ and $A_1(3)$, ${}^2G_2(3)$) then L is the layer of $G(\sigma)$, i.e., L/Z is a direct product of simple components.

The components in L/Z can be immediately described once the action of σ on Σ (or on Π) is given. To describe J we introduce $\tau_* = \sigma_* - 1$ and note that it is always nonsingular on X. Then $T(\sigma) = \{t \in T\colon \sigma t = t\} = \operatorname{Ann} \tau_* X$. Since $C_G(L) = C_G(E(G))$ we have $J = \operatorname{Ann}(\tau_* X + X^{ad})$ and hence $J \simeq X/\tau_* X + X^{ad}$.

We now describe Z and D. Let E^{sc} be a simply-connected covering group of $E = E(G)$ and let π be the natural isogeny of E^{sc} onto E. Choose T^{sc} a maximal torus of E^{sc} so that $\pi T^{sc} = T \cap E$. Let π_* denote the natural homomorphism of X into $X^{sc} = X(T^{sc})$ induced by π. We have $\operatorname{Ker} \pi_* = X^f$ and $\operatorname{Ker} \pi = \operatorname{Ann}(\operatorname{Im} \pi_*)$. Define $\Delta^{sc} = \Delta^{sc}(G) = X^{sc}/\pi_* X^{ad}$; this is a finite group, in fact just the 'weight lattice/root lattice' for the root system Σ.

We may extend σ to E^{sc} so that $\sigma\pi = \pi\sigma$ (see [**14**, §9]). Hence τ_* is defined on X^{sc} and so on Δ^{sc}. We use the notation $\operatorname{Ker} \tau_*$, $\operatorname{Im} \tau_*$ to denote the kernel and image for the action of τ_* on Δ^{sc}. Let Δ, Δ^e denote the images of X, X^e respectively in Δ^{sc}.

PROPOSITION G. *$Z \simeq \Delta/\Delta \cap \operatorname{Im} \tau_*$ and $D \simeq \operatorname{Ker} \tau_*/\Delta^e \cap \operatorname{Ker} \tau_*$.*

Proof. $T \cap L = \pi(T^{sc}(\sigma))$ and so $T \cap L = \operatorname{Ann} Y$ where $Y = \pi_*^{-1}(\tau_* X^{sc} \cap \pi_* X)$. Thus, using $\tau_* X \subseteq Y$, we find $Z = J \cap (T \cap L) = \operatorname{Ann}(Y + X^{ad})$ and $J(T \cap L) = \operatorname{Ann}(\tau_* X + X^{ad} \cap Y)$. Hence

$$Z \simeq X/(Y + X^{ad})$$

and

$$D \simeq (\tau_* X + X^{ad} \cap Y)/\tau_* X \simeq (X^{ad} \cap Y)/(X^{ad} \cap \tau_* X).$$

Now $\operatorname{Im} \tau_* = (\tau_* X^{sc} + \pi_* X^{ad})/\pi_* X^{ad}$ and, since τ_* is nonsingular on X^{sc}, $\operatorname{Ker} \tau_* = (X^{sc} \cap \tau_*^{-1} \pi_* X^{ad})/\pi_* X^{ad}$. Using the above expressions for Z and D and the fact that $X \cap \tau_*^{-1} X^{ad} = X^e \cap \tau_*^{-1} X^{ad}$ we immediately obtain the required formulas.

In practice the action of τ_* on Δ^{sc} is easily found and the above expression for J, Z, D give a most useful and easy method for describing $G(\sigma)$; see for example [**5**]. To find the explicit extension of $G(\sigma)$ over JL use the description of $J(T \cap L)$ given in the proof. Note that Δ^{sc} may be replaced by $0_{p'}(\Delta^{sc})$.

The above results extend to the disconnected case as follows: suppose $G^+ = \langle G, \psi_1, \psi_2, \cdots \rangle$ where the ψ_i are automorphisms of G. If σ acts on G^+ it permutes the cosets $G\psi_i$. Using [**11**] we see that $G^+(\sigma) \cap G\psi \neq \phi$ if and only if σ fixes $G\psi$.

5.3. *Classes and centralizers in* $G(\sigma)$. Let ψ be as in 3.1 and put $G^+ = \langle G, \psi\rangle$. We suppose that σ is as described above and acts on G^+. We also assume that $\sigma(\psi) = \psi$. This is a minor restriction which is justified by the applications (by putting ψ and σ in standard form we may always assume $\sigma(\psi) = z\psi$, where $z \in Z(G)$; it is not difficult to extend the following arguments to this case). Using the results of §3 and §4 together with Lang's theorem see [**13**, §2, 3], we will now describe the semi-simple classes in $G^+(\sigma)$, under conjugation by $G(\sigma)$, and the structure of their corresponding centralizer subgroups in $G(\sigma)$.

Let $\mathcal{K}$ be as in 3.4 and suppose $\mu(\xi_d)\psi \in \mathcal{K}$ where $\mu \in \Gamma_c$ and $\bar{\mu} \in \mathcal{F}_d(\bar{\Gamma})$. For convenience, we put $\mathcal{K} = [\mu]$, with the understanding that ψ, d, ξ_d are fixed.

Now $[\mu] \cap G^+(\sigma) \neq \phi$ if and only if $\sigma[\mu] = [\mu]$, which is equivalent to $\sigma\bar{\mu}$ and $\bar{\mu}$ lying in the same $\mathcal{A}_d(W_\psi, \bar{\Gamma})$ orbit. The proof that $\mathcal{F}_d$ is a fundamental domain for $\mathcal{A}_d^{sc}$ yields a practical algorithm for finding the representative in $\mathcal{F}_d$ of any $\mathcal{A}_d^{sc}$ orbit (see Appendix 2). Thus proposition A will determine whether $[\mu]$ intersects $G^+(\sigma)$. Suppose now that $G^+(\sigma) \cap [\mu] \neq \phi$, then the algorithm yields a $w \in W_\psi$ such that $(w\sigma - 1)\bar{\mu} \in d\bar{\Gamma}$. Let $w = nT$ and put $\rho = i_n\sigma$. Thus $(\rho - 1)\bar{\mu} \in d\bar{\Gamma}$ which, since $d\Gamma \cap \Gamma_c = d\Gamma_c$, is equivalent to $\mu(\xi_d) \in G(\rho)$ and hence $\mu(\xi_d)\psi \in G^+(\rho)$.

Assume $\mu(\xi_d) \in G(\rho)$. The set $[\mu] \cap G^+(\rho)$ will, in general, split into several classes under conjugation by $G(\rho)$. Let $C = C_G(\mu(\xi_d)\psi)$ and put $[C, \rho] = \{c^{-1}\rho(c) \colon c \in C\}$. By [**11**] $[C, \rho] \supseteq C^0$ and hence $[C, \rho]$ is a group and the quotient $\tilde{C} = C/[C, \rho]$ is finite and abelian. Choose $\tilde{c} \in \tilde{C}$, then by 4.2 we may find $m \in \tilde{c} \cap N$ and may choose $h \in G$ such that $h^{-1}\rho(h) = m$. Then, again using [**11**], the $G(\rho)$ classes in $[\mu] \cap G^+(\rho)$ are in 1:1 correspondence with the elements of $\tilde{C}$: $\tilde{c}$ corresponds to the class containing $h\mu(\xi_d)\psi h^{-1}$. It is easily checked that this correspondence is independent of the choices of m, h. Note that $h\mu(\xi_d)\psi h^{-1} \in G^+(\rho)$ since $\rho(h) = hm$ and $m \in C$.

In practice it is not necessary to compute m, h. Instead, define $\rho_m = i_m\rho$ then, since $i_h(G(\rho_m)) = G(\rho)$ we may replace $h\mu(\xi_d)\psi h^{-1} \in G(\rho)$ by $\mu(\xi_d)\psi \in G(\rho_m)$. Thus we describe the $G(\rho)$ classes in $[\mu] \cap G^+(\rho)$ by the pairs $[\mu, \rho]$, $[\mu, \rho_m], \cdots$ where m runs over coset representatives for $[C, \rho]$ in C.

If $[\mu, \rho_m]$ is such a class the centralizer in $G(\rho)$ of an element of this class is isomorphic to $C(\rho_m)$ (where $C = C_G(\mu(\xi_d)\psi)$). The structure of $C(\rho_m)$ is determined by the methods of 5.2. Note that we only need $w(m)$ for these calculations.

5.4. *The semi-simple case.* Throughout this subsection we assume that G is semi-simple. Thus $X^e = X$, $X^f = 0$, etc. We introduce the symbol $G(\mu)^u$ in place of $0^{p'}(G(\sigma)) = L$ of 5.2. $G(\sigma)^u$ is generated by the unipotent elements ($=p$-elements) in $G(\sigma)$. If $X = X^{ad}$ and G is simple then $G(\sigma)^u$ is, with the usual 8 exceptions, a non-abelian simple group.

The expression for J in 5.2 may be simplified to $J \simeq \Delta/\tau_*\Delta$. Together with $\Delta^e = \Delta$ and the formula for D this yields $|G(\sigma)| = |G(\sigma)^u/Z| \cdot |\mathrm{Ker}\,\tau_*|$. In particular, $|G(\sigma)|$ depends only on Σ and σ and not on X.

We now consider two questions: first, if $[\mu]$ is a semi-simple class in G and $G(\sigma) \cap [\mu] \neq \phi$ when is $G(\sigma)^u \cap [\mu] \neq \phi$? and second, if $C = C_G(\mu(\xi_d))$ what is the structure of $C \cap G(\sigma)^u$? We have phrased these questions for the case $\psi = 1$ but their extension to $\psi \neq 1$ is quite obvious.

To answer the first question we may suppose $\mu \in \mathscr{F}_d$ and $(\sigma - 1)\mu \in d\Gamma$, and hence $\mu(\xi_d) \in G(\sigma)$. Then,

Proposition H. *$\mu(\xi_d) \in G(\sigma)^u$ if and only if $(1/d)(\sigma - 1)\mu \in (\sigma - 1)\Gamma + \Gamma^{sc}$.*

Proof. Let $\mathscr{S} = \{\mu - d\theta \colon \theta \in \mathscr{F}\}$. By proposition A the set $\mathscr{S}$ contains a set of representatives for those $\mathscr{A}_d^{sc}$ orbits which lie in the $\mathscr{A}_d$ orbit containing μ. There is a positive integer e such that $e\eta \in \Gamma^{sc}$ for all $\eta \in \mathscr{S}$. Choose $\zeta \in k^*$ a primitive (ed)th root of unity such that $\zeta^e = \xi_d$. Let G^{sc} be the simply connected covering group of G and π the isogeny of G^{sc} onto G. We may identify Γ^{sc} with the Γ-lattice for G^{sc}. Then since $\ker \pi = \{ed\theta(\zeta) \colon \theta \in \mathscr{F}(\Gamma)\}$ we have $\pi^{-1}(\mu(\xi_d)) = \{e\eta(\zeta) \colon \eta \in \mathscr{S}\} \subseteq G^{sc}$.

Since $\pi(G^{sc}(\sigma)) = G(\sigma)^u$ we see that $\mu(\xi_d) \in G(\sigma)^u$ if and only if $e\eta(\zeta) \in G^{sc}(\sigma)$ for some $\eta \in \mathscr{S}$, i.e., if and only if $(\sigma - 1)e(\mu - d\theta) \in ed\Gamma^{sc}$ for some $\theta \in \mathscr{F}$. This gives the required result.

To answer the second question we first note that $G(\sigma) = G(\sigma)^u T(\sigma)$ and $T \subseteq C^0$. This implies that $C(\sigma)/C^0(\sigma) \simeq C \cap G(\sigma)^u/C^0 \cap G(\sigma)^u$ and thus we need only describe $C^0 \cap G(\sigma)^u$.

Let $X^{ad}(\mu), X^e(\mu), \cdots$ denote the lattices in X associated with C^0. Thus $X^{ad}(\mu)$ is spanned by $\Sigma(\mu) = \{\alpha \in \Sigma \colon \mu(\alpha) \in dZ\}$ (this is the $\Sigma(\varphi)$ of 4.1) and $X(\mu) = X$. Let $\Delta^{sc} = \Delta^{sc}(C^0)$ be defined as in 5.2 (but for C^0) and let Δ, Δ^e denote the images of $X(\mu)$ and $X^e(\mu)$ respectively. Both Δ and Δ^e depend on X as it varies between X^{ad} and X^{sc} (the root and weight lattices *of* G). Let $\Delta(sc)$, $\Delta^e(sc)$ denote their values when $X = X^{sc}$.

Put $L(\mu)$ and $J(\mu)$ for the subgroups associated with C^0 as in 5.2. Since $L(\mu) \subseteq G(\sigma)^u$ the structure of $C^0 \cap G(\sigma)^u$ is determined once $J(\mu)^u = J(\mu) \cap G(\sigma)^u$ and $D(\mu)^u = C^0 \cap G(\sigma)^u/L(\mu)J(\mu)^u$ are given. Now $J(\mu)^u = \mathrm{Ann}(\tau_* X^{sc} \cap X + X^{ad}(\mu))$ and hence $J(\mu)^u \simeq$

$X/(\tau_* X^{sc} \cap X + X^{ad}(\mu))$. If C^0 is semi-simple this simplifies to $J(\mu)^u \simeq \Delta/\Delta \cap \tau_* \Delta(sc)$. A slight extension of the arguments in the proof of proposition H gives $D(\mu)^u \simeq \mathrm{Ker}\, \tau_*/\Delta^e(sc) \cap \mathrm{Ker}\, \tau_*$.

5.5. *Field automorphisms.* Let G be semi-simple. Suppose ρ is a morphism of G onto itself such that $\rho^f = \sigma$ for some integer $f \geqq 2$. Then ρ induces an automorphism of order f on $G(\sigma)$ whose centralizer is clearly $G(\rho) \subseteq G(\sigma)$. We call the restriction of ρ to $G(\sigma)$ a 'field automorphism'. If $X = X^{ad}$ the field automorphisms together with those induced by $G(\sigma)$ (inner and diagonal) and by any $\psi \neq 1$ (graph) give a complete set of representatives for $\mathrm{Aut}(G(\sigma))$. If $X \neq X^{ad}$ the only difference is that the diagonal automorphisms occur in $N_G(G(\sigma))$.

An example may clarify the implications of our definition of a field automorphism. Let G be of type E_6 (and $X = X^{ad}$) and let ψ denote its graph automorphism of order 2. Put σ_q for the Frobenius morphism of G whose fixed points on k is the Galois field $\mathbf{F}(q)$, with $q = p^m$. We may suppose that ψ and σ_q are in standard form relative to a fixed Π and hence $\psi\sigma_q = \sigma_q\psi$. Put ${}^2\sigma_q = \psi\sigma_q$ then $G(\sigma)^u \simeq E_6(q)$ or ${}^2E_6(q)$ depending on $\sigma = \sigma_q$ or ${}^2\sigma_q$. Suppose m is even, then $G(\sigma_q)$ has two field automorphisms of order 2, namely $\sigma_{\sqrt{q}}$ and ${}^2\sigma_{\sqrt{q}}$; however $G({}^2\sigma_q)$ has none. Of course $G(\sigma_q)$ and $G({}^2\sigma_q)$ both have two classes of graph automorphisms of order 2 (see 4.3, the centralizers are isomorphic to $F_4(q)$ or $C_4(q)$).

Consider ρ as restricted to $G(\sigma)$, then $\langle G(\sigma), \rho\rangle$ is a semi-direct product (with $\rho^f = 1$). If $g \in G(\sigma)$ and $(g\rho)^f = 1$ an application of Lang's theorem shows that $g\rho$ is conjugate to ρ under the action of $G(\sigma)$.

In practice it is useful to know the structure of $G(\rho) \cap G(\sigma)^u$. If $X = X^{ad}$ one need only determine the diagonal part $D(\rho)^u \equiv G(\rho) \cap G(\sigma)^u / G(\rho)^u$. The usual lattice calculations give $D(\rho)^u \simeq \mathrm{Ker}(\rho_* - 1)/\mathrm{Ker}(\rho_* - 1) \cap \mathrm{Im}(1 + \rho_* + \cdots + \rho_*^{f-1})$, computed in $\Delta^{sc}(G)$.

5.6. *Examples.* We consider the D_4 example discussed in 4.5. Let σ be in standard form relative to Π and suppose $\sigma_* \alpha_1 = q\alpha_2$, $\sigma_* \alpha_2 = q\alpha_1$, $\sigma_* \alpha_3 = q\alpha_3$, $\sigma_* \alpha_4 = q\alpha_4$ where $q = p^m$, p = characteristic of k. Thus $G(\sigma)^u \simeq {}^2D_4(q)$. We assume q is odd and consider the involution $\eta_3(-1)$; since $(\sigma - 1)\eta_3 = (q-1)\eta_3 \in 2\Gamma$ we have $\eta_3(-1) \in G(\sigma)$. Since $V(\eta_3) = V \simeq Z_2 \times Z_2$, $\tilde{C} \simeq V/(1-\sigma)V$ giving coset representatives 1, n_1, where $v_1 = w(n_1)$. Hence $[\eta_3]$ splits into the two classes $[\eta_3, \sigma]$ and $[\eta_3, \sigma_1]$ where $\sigma_1 = i_{n_1}\sigma$.

Since $\frac{1}{2}(\sigma - 1)\eta_3 \in \Gamma^{sc}$ prop. H gives $\eta_3(-1) \in G(\sigma)^u$, however $\frac{1}{2}(\sigma_1 - 1)\eta_3 = -q\eta_1 + ((q-1)/2)\eta_3 \equiv \eta_1 (\mathrm{mod}\, \Gamma^{sc})$ and, since $\eta_1 \notin$

$(\sigma_1 - 1)\Gamma$, $\eta_3(-1) \notin G(\sigma_1)^u$. Thus the class $[\eta_3, \sigma]$ is "inner" (in fact central) while $[\eta_3, \sigma_1]$ is outer-diagonal.

We now describe the structure of $C(\sigma)$ and $C(\sigma_1)$ where C is given in 4.5. We use the methods of 5.2. From the permutations induced on $\{\alpha_1, \alpha_2, \alpha_4, \alpha_*\}$ by σ and σ_1 we have $L/Z \simeq A_1(q^2) \times A_1(q) \times A_1(q)$ for σ and $L/Z \simeq A_1(q^4)$ for σ_1. Next let $\Delta^{sc} = \langle \epsilon_1, \epsilon_2, \epsilon_4, \epsilon_* \rangle \simeq Z_2 \times Z_2 \times Z_2 \times Z_2$ where ϵ_i is the image of the fundamental weight $\alpha_i/2$. Thus $\Delta = \Delta^e = \langle \epsilon_1 + \epsilon_2 + \epsilon_4 + \epsilon_* \rangle$ and (see 5.4) $\Delta(sc) = \Delta^e(sc) = \langle \epsilon_1 + \epsilon_2, \epsilon_2 + \epsilon_4, \epsilon_4 + \epsilon_* \rangle$. We now calculate in Δ^{sc};

	$\operatorname{Ker} \tau_*$	$\operatorname{Im} \tau_*$	$\tau_* \Delta$	$\Delta \cap \tau_* \Delta(sc)$
σ	$\langle \epsilon_1 + \epsilon_2, \epsilon_4 + \epsilon_* \rangle$	$\langle \epsilon_1 + \epsilon_2 \rangle$	0	0
σ_1	Δ	$\Delta(sc)$	0	Δ

Using proposition G and 5.4 we thus have:

	Z	J	D	J^u	D^u
σ	Z_2	Z_2	$Z_2 \times Z_2$	Z_2	Z_2
σ_1	1	Z_2	1	1	1

So, for example, $C^0(\sigma_1) \simeq A_1(q^4) \times Z_2$ while $C^0 \cap G(\sigma_1)^u \simeq A_1(q^4)$.

The only coset of C/C^0 fixed by σ (or σ_1) is $n_4 C^0$. From the standard form of n_4 we have $[n_4, \sigma] = [n_4, \sigma_1] = 1$ and so the structure of $C(\sigma)$, $C(\sigma_1)$ is easily computed. For example, put $x_+(\zeta) = x_{\alpha_1}(\zeta) x_{\alpha_2}(\zeta^{q^3}) x_{\alpha_4}(\zeta^{q^2}) x_{\alpha_*}(\zeta^{q^4})$ and define x_- similarly, then $C^0(\sigma_1) = \langle x_\pm(\zeta) \colon \zeta \in \mathbf{F}(q^4) \rangle \times \langle \eta_3(-1) \rangle$. Since $n_4 x_\pm(\zeta) n_4^{-1} = x_\pm(\zeta^{q^2})$ we have $C(\sigma_1) \simeq \mathrm{FL}_2(q^4) \times Z_2$ (see the first paragraph of Appendix 1 for notation).

Appendix 1. We list the pairs (L, t) where L is a finite simple Chevalley type group over a field of odd characteristic and t is an involution in $\operatorname{Aut} L$ such that the layer of $C_L(t)$ consists of a single PSL_2-group. We also describe $C_L(t)$. The classes are those occurring *under conjugation by* $G(\sigma)$ where $L = G(\sigma)^u$. The results are given in the table below. The notation is as follows: $q = p^m$, p some odd prime; $L_2(q) = \mathrm{PSL}_2(q)$,

$\mathrm{PGL}_2(q)$ is $L_2(q)$ extended by diagonal automorphism
$\mathrm{FL}_2(q^2)$ is $L_2(q^2)$ extended by field aut. (of order 2)
$\mathrm{PFL}_2(q^2)$ is $L_2(q^2)$ extended by (field) $\times$ (diagonal) aut.
For example, if $q = 3$, $\mathrm{FL}_2(9) \simeq \Sigma_6$ and $\mathrm{PFL}_2(9) \simeq M_{10}$. Z_n and D_n are respectively the cyclic and dihedral group of order n. In the entry for ${}^2D_4(3)$ H is a four-dimensional orthogonal group; it has a subgroup H_0 of index 2 with $H_0 \simeq \mathrm{SL}_2(3) * \mathrm{SL}_2(3)$ (a central product) and s flips these factors.

We sketch the calculations: Let G be a simple algebraic group with $X = X^{ad}$ and σ a morphism so that $L = G(\sigma)^u$. We consider the various involutions t in $\mathrm{Aut}\, L$; (i) t a field automorphism, then G is either of type A_1 or we have the exceptional case G_2 as given in the table,

L	t		$C_L(t)$
$A_1(q^2)$	field	$q \neq 3$	$\mathrm{PGL}_2(q)$
${}^2G_2(q)$	inner	$q = 3^{2m+1},\ m \geqq 1$	$L_2(q) \times Z_2$
$A_2(q)$	graph	$q \neq 3$	$\mathrm{PGL}_2(q)$
${}^2A_2(q)$	graph	$q \neq 3$	$\mathrm{PGL}_2(q)$
$C_2(q)$	inner diagonal diagonal	$\left.\begin{matrix} \\ \\ \end{matrix}\right\}$ $q \equiv \epsilon(4),\ q \neq 3$	$\left.\begin{matrix}\langle L_2(q) \times D_{q-\epsilon}, \tau\rangle \\ \langle L_2(q) \times D_{q+\epsilon}, \tau\rangle\end{matrix}\right\} \begin{matrix}\tau^2 = 1, \langle L_2(q), \tau\rangle \simeq \mathrm{PGL}_2(q) \\ \langle D_{q\mp\epsilon}, \tau\rangle \simeq D_{2(q\mp\epsilon)}\end{matrix}$ $\mathrm{FL}_2(q^2)$
${}^2D_4(q)$	diagonal		$\mathrm{FL}_2(q^4)$
$A_3(q)$	diagonal inner graph 	$q \equiv 1(4)$ $q \equiv -1(4)$ $\left\{\begin{matrix} q \equiv 1(8) \\ q \equiv 3, 7(8) \\ q \equiv 5(8)\end{matrix}\right.$	$\left.\begin{matrix}\langle L_2(q^2) \times Z_{(q+1)/2}, \theta\rangle \\ \langle L_2(q^2) \times Z_{(q+1)}, \theta\rangle\end{matrix}\right\} \begin{matrix}\theta^2 = 1, \langle L_2(q^2), \theta\rangle \simeq \mathrm{FL}_2(q^2) \\ \theta \text{ inverts cyclic factor.}\end{matrix}$ $\mathrm{FL}_2(q^2)$ $\mathrm{PGL}_2(q^2)$ $\mathrm{PFL}_2(q^2)$
${}^2A_3(q)$	inner diagonal graph 	$q \equiv 1(4)$ $q \equiv -1(4)$ $\left\{\begin{matrix} q \equiv 1, 5(8) \\ q \equiv 3(8) \\ q \equiv 7(8)\end{matrix}\right.$	$\left.\begin{matrix}\langle L_2(q^2) \times Z_{q-1}, \theta\rangle \\ \langle L_2(q^2) \times Z_{(q-1)/2}, \theta\rangle\end{matrix}\right\} \theta \text{ as for } A_3(q).$ $\mathrm{PGL}_2(q^2)$ $\mathrm{PFL}_2(q^2)$ $\mathrm{FL}_2(q^2)$
$G_2(3)$ $B_3(3)$ ${}^2D_4(3)$	graph diagonal inner		${}^2G_2(3) \simeq \langle L_2(2^3), f\rangle, f^3 = 1$, field aut. $\mathrm{PGL}_2(3) \times \mathrm{FL}_2(q)$. $\langle H \times L_2(q), s\rangle\ s^2 = 1, \langle L_2(9), s\rangle = \mathrm{FL}_2(9)$ (see text for H and action of s)

(ii) t is a graph automorphism, then Table 4.3 shows that G is of type A_2 or A_3, (iii) t is inner (in G), i.e., the case $\psi = 1$ in §3, 4, then it is easily seen that G has rank $\leqq 4$. At this stage it is necessary to list the various cases, compute $C^0 = C_G(t)^0$ and $O^{P'}(C^0(\sigma))$, and see when the layer is an L_2-group. The calculations given in 5.6 are typical.

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Appendix 2. We use the notation of §2, especially 2.4. For a given $\mu \in \Gamma$ we describe an algorithm which finds the unique $\lambda \in \mathscr{F}_d$ such that μ and λ lie in the same $\mathscr{A}_d^{sc}$-orbit in Γ. This algorithm is an essential tool in practical calculations (especially if $d > 3$).

Without restriction we may suppose that the Dynkin diagram for Σ is connected. Let $\Pi = \{\alpha_1, \cdots, \alpha_r\}$ and let $\{\eta_1, \cdots, \eta_r\}$ be the dual basis for Γ^{ad}, i.e. $\eta_i(\alpha_j) = \delta_{ij}$. There is a natural inclusion $\Gamma \subseteq \Gamma^{ad}$ and if $\Gamma \neq \Gamma^{ad}$ an element $\eta = \Sigma e_i \eta_i$ $(e_i \in Z)$ lies in Γ if and only if certain (easily found) congruences hold. Let $\alpha_* = -(m_1\alpha_1 + \cdots + m_r\alpha_r)$ where the m_i are (known) positive integers then $\mathscr{F}_d(\Gamma) = \{\eta \in \Gamma: 0 \leqq e_i$ all i and $(m_1e_1 + \cdots + m_re_r) \leqq d\}$.

Now suppose $\mu = \Sigma a_i \eta_i \in \Gamma$ $(a_i \in Z)$ the algorithm goes as follows (let $w_i = w_{\alpha_i}$ and $w_* = w_{\alpha_*}$);

(1) if some $a_i < 0$ replace μ by $w_i\mu$,

(2) if all $a_i \geqq 0$ but $m_1a_1 + \cdots + m_ra_r > d$ replace μ by $w_*\mu - d\hat{\alpha}_*$.

By repeating these two steps we eventually obtain an element of $\mathscr{F}_d$. Note that (i) $w_i\eta_j = \eta_j$ if $i \neq j$ while $w_i\eta_i = \eta_i - \hat{\alpha}_i$, (ii) $w_*\mu = \mu + (m_1a_1 + \cdots + m_ra_r)\hat{\alpha}_*$ and (iii) the expressions for $\hat{\alpha}_i$, $\hat{\alpha}_*$ in terms of the η_j's can be read off from the extended Dynkin diagram.

The operation in steps (1), (2) are just reflections on the bounding hyperplanes of $\mathscr{F}_d$. Using result 1.16 of **[14]** it may be shown that we have the best possible algorithm.

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