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In this note, we characterize maximal topologies of a class of topological properties which include lightly compact spaces and *QHC*-spaces and, when restricted to completely regular spaces, pseudocompact spaces. In addition we prove some results relating maximal lightly compact and maximal pseudocompact spaces.

A. B. Raha **[12]** has shown that maximal lightly compact spaces are submaximal as are maximal pseudocompact spaces, and Douglas E. Cameron **[6]** has characterized maximal QHC-spaces and shown these to be submaximal. In Tychonoff spaces, lightly compact and pseudocompact are equivalent; and in Hausdorff spaces, QHC and H-closed are equivalent. We shall show that the maximal topologies of a class of topologies which include lightly compact and QHC are submaximal and T_1 spaces.

The topological space with topology τ on set X shall be denoted by (X,τ), the closure of a subset A of X with respect to τ is $\mathrm{cl}_\tau A$ and the interior of A with respect to τ is $\mathrm{int}_\tau A$, the complement of A with respect to X is $X - A$, the relative topology of τ on A is $\tau \mid A$, and the product of spaces (X_α, τ_α) for $\alpha \in \mathfrak{A}$ is $(\pi_{\mathfrak{A}} X_\alpha,\ \pi_{\mathfrak{A}} \tau_\alpha)$.

A topological space (X,τ) with property R is *maximal* R if whenever τ' is stronger than $\tau (\tau' \supset \tau)$, then (X,τ') does not have property R. In **[5]** it was shown that for a topological property R, (X,τ) is maximal R if and only if every continuous bijection from a space (Y,τ) with property R to (X,τ) is a homeomorphism. A topological space (X,τ) for which there exists a stronger maximal R topology is said to be *strongly* R. For $A \subseteq X$ the topology $\tau(A)$ with subbase $\tau \cup \{A\}$ is the *simple expansion* of τ by A.

We shall restrict our study to topological properties which satisfy some or all of the following:

P-1: contractive (preserved by continuous surjections)

P-2: regular closed hereditary

P-3: semi-regular (A topological property R is *semi-regular* if (X,τ) has property R if and only if (X,τ_S) has property R where τ_S is the semi-regularization of τ.)

P-4: contagious (A topological property R is *contagious* if whenever a dense subset of a space has property R, the entire space has property R [8]).

P-5: finitely unionable (If (X, τ) is a topological space, $A_i = X$, $i = 1, \cdots, n$ are subsets which have property R, then $\bigcup_{i=1}^{n} A_i$ has property R).

DEFINITION 1. Two topologies τ and τ' on X are *ro-equivalent* if $\tau_S = \tau'_S$.

THEOREM 1. *An expansion τ' of τ is ro-equivalent to τ if and only if* $\mathrm{cl}_{\tau'} U = \mathrm{cl}_\tau U$ *for all* $U \in \tau'$ [10].

THEOREM 2. *If a topological property R satisfies* P-3, *then a maximal R topology is submaximal.*

Proof. This follows from the properties of P-3 and the fact that every topological space has a stronger submaximal space with the same semiregularization [3].

COROLLARY 1. *If a topological property R satisfies* P-3, *then maximal R topologies are T_D.*

THEOREM 3. *If topological property R satisfies* P-1–P-5 *a submaximal space (X, τ) is maximal R if and only if for any $A \subseteq X$, such that both X-$\mathrm{int}_\tau A$ and A have property R, then A is closed.*

Proof. If (X, τ) is submaximal and not maximal R, then there is $\tau' \supset \tau$ such that $\tau'_S \neq \tau_S$ and (X, τ') has property R. Therefore there is $U \in \tau'$ such that $\mathrm{cl}_\tau U \supset \mathrm{cl}_{\tau'} U$ and thus $\mathrm{cl}_{\tau'} U$ is not τ-closed. $\mathrm{cl}_{\tau'} U$ and $\mathrm{cl}_{\tau'}(X - \mathrm{cl}_{\tau'} U)$ are τ' regular closed and thus are τ' and τ subspaces with property R.

By P-4, $\mathrm{cl}_\tau(\mathrm{cl}_{\tau'}(X - \mathrm{cl}_{\tau'} U)) = \mathrm{cl}_\tau(X - \mathrm{cl}_{\tau'} U) = X - \mathrm{int}_\tau(\mathrm{cl}_{\tau'} U)$ has property R with respect to τ.

If (X, τ) has property R and there is a nonclosed subset $A \subseteq X$ such that both A and $X - \mathrm{int}_\tau A$ have property R, then the topology $\mathrm{cl}_\tau(X - A)$ has property R. Since every dense subset of a submaximal space is open, $(X - A) \cup \mathrm{int}_\tau A$ is τ open implying $\tau \mid B = \tau(X - A) \mid B$ where $\mathrm{cl}_\tau(X - A) = B$. Also $\tau \mid A = \tau(X - A) \mid A$ so both A and B are $\tau(X - A)$ subspace with property R and by P-5, $(X, \tau(X - A))$ has property R since $X = A \cup B$, thus (X, τ) is not maximal R.

COROLLARY 2. *A submaximal space satisfying* P-1–P-5 *with property R in which every subspace with property R is closed is maximal R.*

THEOREM 4. *If property R satisfies* P-1–P-5 *and all one point sets have property R, then maximal R spaces are T_1.*

Proof. Let (X, τ) be submaximal R. If for $x_0 \in X, \{x_0\} \notin \tau$ then $X - \{x_0\}$ is τ-dense therefore is open and so $\{x_0\}$ is closed. If $\{x_0\} \in \tau$ and $\mathrm{cl}_\tau\{x_0\}\text{-}\mathrm{int}_\tau\ \mathrm{cl}_\tau\{x_0\} = \emptyset$ then since $\{x_0\}$ has property R, $\mathrm{cl}_\tau\{x_0\} - \{y_0\}$ has property R for $y_0 \neq x_0$ by P-4. Since $\{y_0\} \notin \tau$, $\mathrm{cl}_\tau\{y_0\} = \{y_0\}$, and the free union of $X - \mathrm{cl}_\tau\{x_0\}$, $\{y_0\}$, and $\mathrm{cl}_\tau\{x_0\} - \{y_0\}$ has property R and is finer than (X, τ) which is a contradiction since (X, τ) is maximal R. If $\mathrm{cl}_\tau\{x_0\} - \mathrm{int}_\tau\ \mathrm{cl}_\tau\{x_0\} \neq \emptyset$, choose $y_0 \in \mathrm{cl}_\tau\{x_0\} - \mathrm{int}_\tau\ \mathrm{cl}_\tau\{x_0\}$. Then $A = \mathrm{cl}_\tau\{x_0\} - \{y_0\}$ has property R and is not closed. $X - \mathrm{int}_\tau A = \mathrm{cl}_\tau(X - \mathrm{cl}_\tau A)$ is regular closed and thus has property R. By Theorem 3, A is closed, a contradiction as $\{x_0\} \subseteq A \subsetneq \mathrm{cl}_\tau\{x_0\}$.

THEOREM 5 *If property R is productive and contractive* (P-1) *and $(\pi_{\mathfrak{A}} X_\alpha, \pi_{\mathfrak{A}} \tau_\alpha)$ is maximal R, then (X_α, τ_α) is maximal R for $\alpha \in \mathfrak{A}$.*

Proof. (X_α, τ_α) has property R for $\alpha \in \mathfrak{A}$ since R is contractive; if (X_β, τ_β) is not maximal R for some $\beta \in \mathfrak{A}$, there is $\tau'_\beta \supset \tau_\beta$ such that (X_β, τ'_β) has property R. Then for $\tau'_\alpha = \tau_\alpha$ for $\alpha \neq \beta$, $(\pi_{\mathfrak{A}} X_\alpha, \pi_{\mathfrak{A}} \tau'_\alpha)$ has property R which is a contradiction.

QHC-spaces (spaces for which every open cover has a finite subcollection whose closures cover the space) have properties P-1–P-5 and have been studied in detail **[6]**. QHC-spaces which are Hausdorff are called H-closed spaces. Lightly compact spaces (spaces for which every countable open cover has a finite subcollection whose closures cover the space) satisfy P-1–P-5 (See **[2]** for P-2; **[12]** for P-3; P-1, P-4, and P-5 are proven as for QHC). Lightly compact spaces are called feebly compact in **[14, 15]**. Pseudocompact spaces satisfy P-1, P-3 **[12]**, P-4 **[8]** and P-5, but not P-2. However P-2 is satisfied for pseudocompactness in the class of completely regular spaces **[9]** and maximal pseudocompact spaces are T_1 **[7]**.

Spaces having properties $P_1 - P_5$ are not necessarily strongly R (QHC—**[6]**; lightly compact—**[12]**). However H-closed spaces are strongly H-closed **[10]** and a first countable Hausdorff space which is lightly compact is strongly lightly compact. This follows from P-3, the fact that every space is coarser than some submaximal space with the same semi-regularization, the fact that in a first countable Hausdorff space, lightly compact subsets are closed (proven similarly to the same result for first countable, T_1 countably compact spaces **[1]**) and Corollary 2. In Tychonoff spaces pseudocompactness is closed hereditary **[9]**, thus we have the following result:

THEOREM 6. *A Tychonoff space is maximal pseudocompact if and only if it is maximal lightly compact.*

Proof. In completely regular spaces, pseudocompactness is equivalent to lightly compact **[2]**; since lightly compact spaces are pseudocompact then a lightly compact maximal pseudocompact space is maximal lightly compact. If not maximal pseudocompact there is $\tau' \supset \tau$ such that (X, τ') is pseudocompact and therefore there is $A \in \tau' - \tau$ such that $(X, \tau(A))$ is pseudocompact and is completely regular **[13]**. Therefore $(X, \tau(A))$ is lightly compact.

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