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A CLASS OF MAXIMAL TOPOLOGIES

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In this note, we characterize maximal topologies of a class of topological properties which include lightly compact spaces and *QHC*-spaces and, when restricted to completely regular spaces, pseudocompact spaces. In addition we prove some results relating maximal lightly compact and maximal pseudocompact spaces.

A. B. Raha [12] has shown that maximal lightly compact spaces are submaximal as are maximal pseudocompact spaces, and Douglas E. Cameron [6] has characterized maximal *QHC*-spaces and shown these to be submaximal. In Tychonoff spaces, lightly compact and pseudocompact are equivalent; and in Hausdorff spaces, *QHC* and *H*-closed are equivalent. We shall show that the maximal topologies of a class of topologies which include lightly compact and *QHC* are submaximal and T_1 spaces.

The topological space with topology τ on set X shall be denoted by (X, τ) , the closure of a subset A of X with respect to τ is $\text{cl}_\tau A$ and the interior of A with respect to τ is $\text{int}_\tau A$, the complement of A with respect to X is $X - A$, the relative topology of τ on A is $\tau|A$, and the product of spaces (X_α, τ_α) for $\alpha \in \mathfrak{A}$ is $(\prod_{\mathfrak{A}} X_\alpha, \prod_{\mathfrak{A}} \tau_\alpha)$.

A topological space (X, τ) with property R is *maximal* R if whenever τ' is stronger than τ ($\tau' \supset \tau$), then (X, τ') does not have property R . In [5] it was shown that for a topological property R , (X, τ) is maximal R if and only if every continuous bijection from a space (Y, τ) with property R to (X, τ) is a homeomorphism. A topological space (X, τ) for which there exists a stronger maximal R topology is said to be *strongly* R . For $A \subseteq X$ the topology $\tau(A)$ with subbase $\tau \cup \{A\}$ is the *simple expansion* of τ by A .

We shall restrict our study to topological properties which satisfy some or all of the following:

P-1: contractive (preserved by continuous surjections)

P-2: regular closed hereditary

P-3: semi-regular (A topological property R is *semi-regular* if (X, τ) has property R if and only if (X, τ_s) has property R where τ_s is the semi-regularization of τ .)

P-4: contagious (A topological property R is *contagious* if

whenever a dense subset of a space has property R , the entire space has property R [8]).

P-5: finitely unionable (If (X, τ) is a topological space, $A_i = X$, $i = 1, \dots, n$ are subsets which have property R , then $\bigcup_{i=1}^n A_i$ has property R).

DEFINITION 1. Two topologies τ and τ' on X are *ro-equivalent* if $\tau_s = \tau'_s$.

THEOREM 1. An expansion τ' of τ is *ro-equivalent* to τ if and only if $\text{cl}_\tau U = \text{cl}_{\tau'} U$ for all $U \in \tau'$ [10].

THEOREM 2. If a topological property R satisfies P-3, then a maximal R topology is submaximal.

Proof. This follows from the properties of P-3 and the fact that every topological space has a stronger submaximal space with the same semiregularization [3].

COROLLARY 1. If a topological property R satisfies P-3, then maximal R topologies are T_D .

THEOREM 3. If topological property R satisfies P-1–P-5 a submaximal space (X, τ) is maximal R if and only if for any $A \subseteq X$, such that both $X - \text{int}_\tau A$ and A have property R , then A is closed.

Proof. If (X, τ) is submaximal and not maximal R , then there is $\tau' \supset \tau$ such that $\tau'_s \neq \tau_s$ and (X, τ') has property R . Therefore there is $U \in \tau'$ such that $\text{cl}_{\tau'} U \supset \text{cl}_\tau U$ and thus $\text{cl}_\tau U$ is not τ -closed. $\text{cl}_{\tau'} U$ and $\text{cl}_\tau (X - \text{cl}_\tau U)$ are τ' regular closed and thus are τ' and τ subspaces with property R .

By P-4, $\text{cl}_\tau (\text{cl}_{\tau'} (X - \text{cl}_\tau U)) = \text{cl}_\tau (X - \text{cl}_\tau U) = X - \text{int}_\tau (\text{cl}_{\tau'} U)$ has property R with respect to τ .

If (X, τ) has property R and there is a nonclosed subset $A \subseteq X$ such that both A and $X - \text{int}_\tau A$ have property R , then the topology $\text{cl}_\tau (X - A)$ has property R . Since every dense subset of a submaximal space is open, $(X - A) \cup \text{int}_\tau A$ is τ open implying $\tau|_B = \tau(X - A)|_B$ where $\text{cl}_\tau (X - A) = B$. Also $\tau|_A = \tau(X - A)|_A$ so both A and B are $\tau(X - A)$ subspaces with property R and by P-5, $(X, \tau(X - A))$ has property R since $X = A \cup B$, thus (X, τ) is not maximal R .

COROLLARY 2. A submaximal space satisfying P-1–P-5 with property R in which every subspace with property R is closed is maximal R .

THEOREM 4. *If property R satisfies P-1–P-5 and all one point sets have property R , then maximal R spaces are T_1 .*

Proof. Let (X, τ) be submaximal R . If for $x_0 \in X, \{x_0\} \notin \tau$ then $X - \{x_0\}$ is τ -dense therefore is open and so $\{x_0\}$ is closed. If $\{x_0\} \in \tau$ and $\text{cl}_\tau \{x_0\} - \text{int}_\tau \text{cl}_\tau \{x_0\} = \emptyset$ then since $\{x_0\}$ has property R , $\text{cl}_\tau \{x_0\} - \{x_0\}$ has property R for $y_0 \neq x_0$ by P-4. Since $\{y_0\} \notin \tau$, $\text{cl}_\tau \{y_0\} = \{y_0\}$, and the free union of $X - \text{cl}_\tau \{x_0\}$, $\{y_0\}$, and $\text{cl}_\tau \{x_0\} - \{y_0\}$ has property R and is finer than (X, τ) which is a contradiction since (X, τ) is maximal R . If $\text{cl}_\tau \{x_0\} - \text{int}_\tau \text{cl}_\tau \{x_0\} \neq \emptyset$, choose $y_0 \in \text{cl}_\tau \{x_0\} - \text{int}_\tau \text{cl}_\tau \{x_0\}$. Then $A = \text{cl}_\tau \{x_0\} - \{y_0\}$ has property R and is not closed. $X - \text{int}_\tau A = \text{cl}_\tau (X - \text{cl}_\tau A)$ is regular closed and thus has property R . By Theorem 3, A is closed, a contradiction as $\{x_0\} \subseteq A \not\subseteq \text{cl}_\tau \{x_0\}$.

THEOREM 5 *If property R is productive and contractive (P-1) and $(\pi_{\mathfrak{A}} X_\alpha, \pi_{\mathfrak{A}} \tau_\alpha)$ is maximal R , then (X_α, τ_α) is maximal R for $\alpha \in \mathfrak{A}$.*

Proof. (X_α, τ_α) has property R for $\alpha \in \mathfrak{A}$ since R is contractive; if (X_β, τ_β) is not maximal R for some $\beta \in \mathfrak{A}$, there is $\tau'_\beta \supset \tau_\beta$ such that (X_β, τ'_β) has property R . Then for $\tau'_\alpha = \tau_\alpha$ for $\alpha \neq \beta$, $(\pi_{\mathfrak{A}} X_\alpha, \pi_{\mathfrak{A}} \tau'_\alpha)$ has property R which is a contradiction.

QHC -spaces (spaces for which every open cover has a finite subcollection whose closures cover the space) have properties P-1–P-5 and have been studied in detail [6]. QHC -spaces which are Hausdorff are called H -closed spaces. Lightly compact spaces (spaces for which every countable open cover has a finite subcollection whose closures cover the space) satisfy P-1–P-5 (See [2] for P-2; [12] for P-3; P-1, P-4, and P-5 are proven as for QHC). Lightly compact spaces are called feebly compact in [14, 15]. Pseudocompact spaces satisfy P-1, P-3 [12], P-4 [8] and P-5, but not P-2. However P-2 is satisfied for pseudocompactness in the class of completely regular spaces [9] and maximal pseudocompact spaces are T_1 [7].

Spaces having properties $P_1 - P_5$ are not necessarily strongly R (QHC —[6]; lightly compact—[12]). However H -closed spaces are strongly H -closed [10] and a first countable Hausdorff space which is lightly compact is strongly lightly compact. This follows from P-3, the fact that every space is coarser than some submaximal space with the same semi-regularization, the fact that in a first countable Hausdorff space, lightly compact subsets are closed (proven similarly to the same result for first countable, T_1 countably compact spaces [1]) and Corollary 2. In Tychonoff spaces pseudocompactness is closed hereditary [9], thus we have the following result:

THEOREM 6. *A Tychonoff space is maximal pseudocompact if and only if it is maximal lightly compact.*

Proof. In completely regular spaces, pseudocompactness is equivalent to lightly compact [2]; since lightly compact spaces are pseudocompact then a lightly compact maximal pseudocompact space is maximal lightly compact. If not maximal pseudocompact there is $\tau' \supset \tau$ such that (X, τ') is pseudocompact and therefore there is $A \in \tau' - \tau$ such that $(X, \tau(A))$ is pseudocompact and is completely regular [13]. Therefore $(X, \tau(A))$ is lightly compact.

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 AKRON, OH 44325
 AND
 OHIO UNIVERSITY (visiting)
 ATHENS, OH 45701

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