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**LOCALLY BOUNDED TOPOLOGIES ON  $F(X)$**

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## LOCALLY BOUNDED TOPOLOGIES ON $F(X)$

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It is classic that, to within equivalence, the only valuations on the field  $F(X)$  of rational functions over a field  $F$  that are improper on  $F$  are the valuations  $v_p$ , where  $p$  is a prime of the principal ideal subdomain  $F[X]$  of  $F(X)$ , and the valuation  $v_\infty$ , defined by the prime  $X^{-1}$  of the principal ideal subdomain  $F[X^{-1}]$  of  $F(X)$  ([1], p. 94, Corollary 2). If  $\mathcal{T}$  is the supremum of finitely many of the associated valuation topologies, then  $\mathcal{T}$  is a Hausdorff, locally bounded ring topology on  $F(X)$  for which  $F$  is a bounded set and for which there is a nonzero topological nilpotent  $a$ . In this paper we shall show conversely that any topology on  $F(X)$  having these properties is the supremum of finitely many valuation topologies.

A subset  $S$  of a topological ring  $R$  is *bounded* if given any neighborhood  $V$  of 0, there exists a neighborhood  $U$  of 0 such that  $SU \subseteq V$  and  $US \subseteq V$ . The topology on  $R$  is *locally bounded* if there is a bounded neighborhood of 0. A bounded subfield of a Hausdorff topological ring is discrete ([2], p. 119, Exercise 13). It is easy to see that if  $\lim_{n \rightarrow \infty} x_n = 0$  and if  $\{a_n\}_{n=1}^\infty$  is a bounded sequence, then  $\lim_{n \rightarrow \infty} a_n x_n = 0$ .

An element  $c$  of a topological ring is a *topological nilpotent* if  $\lim_{n \rightarrow \infty} c^n = 0$ . Let  $\mathcal{P} = \{p \in F[X] : p \text{ is a prime polynomial}\}$ , and let  $\mathcal{P}' = \mathcal{P} \cup \{\infty\}$ . For each  $p \in \mathcal{P}'$ , we shall denote by  $\mathcal{T}_p$  the topology defined by the valuation  $v_p$ . Then for any finite subset  $L$  of  $\mathcal{P}'$ ,  $\sup_{p \in L} \mathcal{T}_p$  has a nonzero topological nilpotent. Indeed, let  $g$  be the product of the members of  $L \cap \mathcal{P}$ . If  $\infty \notin L$ ,  $g$  is a nonzero topological nilpotent for  $\sup_{p \in L} \mathcal{T}_p$ ; if  $\infty \in L$ , let  $q$  be a prime polynomial not in  $L$  and let  $r > 0$  be such that  $\deg(q^r) > \deg g$ ; then  $g q^{-r}$  is a nonzero topological nilpotent of  $\sup_{p \in L} \mathcal{T}_p$ .

We recall that a *norm*  $\|\cdot\|$  on a field  $K$  is a function to the nonnegative reals satisfying  $\|x\| = 0$  if and only if  $x = 0$ ,  $\|x - y\| \leq \|x\| + \|y\|$ , and  $\|xy\| \leq \|x\| \|y\|$  for all  $x, y \in K$ . Clearly a subset of  $K$  is bounded in norm if and only if it is bounded for the topology defined by the norm; in particular the topology given by a norm is a locally bounded topology. We shall use the following theorem of P. M. Cohn ([4], Theorem 6.1): A Hausdorff, locally bounded ring topology on a field  $K$  for which there is a nonzero topological nilpotent is defined by a norm.

**THEOREM 1.** *Let  $F$  be a field and  $x$  a transcendental element over  $F$  in some field extension. Let  $\mathcal{T}$  be a Hausdorff, locally bounded ring*

topology on  $F(x)$  for which the subfield  $F$  is bounded (and hence discrete) and for which there exists a nonzero topological nilpotent  $g_0(x) \in F[x]$ , and let  $J = \{f \in F[X]: \lim_{n \rightarrow \infty} f(x)^n = 0\}$ . Then

1.  $F[x]$  is bounded.
2.  $J$  is a proper ideal of  $F[X]$ ; its monic generator  $h$  is the product of a sequence  $p_1, \dots, p_k$  of distinct prime polynomials of  $F[X]$ .
3.  $\mathcal{T} = \sup_{1 \leq i \leq k} \mathcal{T}_{p_i(x)}$ .

*Proof.* 1. By Cohn's Theorem there exists a norm  $\|\cdot\|$  defining  $\mathcal{T}$ . Then  $\|g_0(x)^n\| < 1$  for some  $n \geq 1$ ; let  $g = g_0^n$ . Then  $g(x)$  is a topological nilpotent and  $\|g(x)\| < 1$ .

As  $F$  is bounded for  $\mathcal{T}$ , there exists a positive constant  $M$  such that  $\|a\| \leq M$ , for all  $a \in F$ . Let  $L = M \sum_{i=0}^m \|x\|^i$ , where  $m = \deg g$ . As  $F$  is discrete,  $F$  contains no nonzero topological nilpotents, so  $m \geq 1$ . For any polynomial  $f$  such that  $\deg f < m$ ,  $f(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_0$  where  $n < m$ ,  $a_0, a_1, \dots, a_n \in F$ ; so

$$\|f(x)\| \leq \sum_{i=0}^n \|a_i\| \|x\|^i \leq M \sum_{i=0}^n \|x\|^i \leq M \sum_{i=0}^m \|x\|^i = L.$$

Therefore,  $\{f(x) \in F[X]: \deg f < m\}$  is a bounded subset of  $F(x)$ .

We shall show next that  $\|f(x)\| \leq S$  for all  $f \in F[X]$ , where  $S = L/(1 - \|g(x)\|)$ . Given  $f \in F[X]$ , there exists an integer  $n \geq 0$  and polynomials  $q_0, \dots, q_n$  such that  $f = g^n q_n + \dots + g q_1 + q_0$  and for each  $i \in [0, n]$ ,  $q_i = 0$  or  $\deg q_i < m$ . Therefore,

$$\begin{aligned} \|f(x)\| &= \|g(x)^n q_n(x) + \dots + g(x) q_1(x) + q_0(x)\| \\ &\leq \sum_{i=0}^n \|g(x)^i\| \|q_i(x)\| \leq L \sum_{i=0}^n \|g(x)^i\| \\ &\leq L \sum_{i=0}^{\infty} \|g(x)^i\| \leq L \sum_{i=0}^{\infty} \|g(x)\|^i = S. \end{aligned}$$

Thus  $F[x]$  is bounded.

2. Let  $f_1, f_2 \in J$ , and let  $t \in F[X]$ . Then

$$\begin{aligned} \|(f_1 + f_2)^{2n}(x)\| &= \left\| \sum_{k=0}^{2n} \binom{2n}{k} f_1^{2n-k}(x) f_2^k(x) \right\| \\ &= \left\| f_1^n(x) \sum_{k=0}^n \binom{2n}{k} f_1^{n-k}(x) f_2^k(x) \right. \\ &\quad \left. + f_2^n(x) \sum_{k=n+1}^{2n} \binom{2n}{k} f_1^{2n-k}(x) f_2^{k-n}(x) \right\| \end{aligned}$$

$$\begin{aligned} &\cong \|f_1^n(x)\| \left\| \sum_{k=0}^n \binom{2n}{k} f_1^{n-k}(x) f_2^k(x) \right\| \\ &\quad + \|f_2^n(x)\| \left\| \sum_{k=n+1}^{2n} \binom{2n}{k} f_1^{2n-k}(x) f_2^{k-n}(x) \right\| \\ &\leq S \|f_1^n(x)\| + S \|f_2^n(x)\| \rightarrow 0. \end{aligned}$$

Thus,  $(f_1 + f_2)^2(x)$  is a topological nilpotent, whence  $(f_1 + f_2)(x)$  is. Also  $\|(f_1 t)^n(x)\| \cong \|f_1^n(x)\| \|t^n(x)\| \leq S \|f_1^n(x)\| \rightarrow 0$ , so  $f_1 t \in J$ . Hence as  $1 \notin J$ ,  $J$  is a proper nonzero ideal of  $F[X]$ , a principal ideal domain. Let  $h$  be its monic generator, and let  $h = \prod_{i=1}^k p_i^{r_i}$  where  $p_1, \dots, p_k$  are distinct prime polynomials. Let  $h_0 = \prod_{i=1}^k p_i$ , and let  $r = \max\{r_1, \dots, r_k\}$ . Then  $h | h_0^r$ , so  $h_0^r \in J$  and hence,  $h_0(x)$  is a topological nilpotent. Therefore,  $h_0$  belongs to  $J$ . So as  $h_0 | h$ ,  $h_0 = h$ .

3. We shall first show that the topology induced on  $F[x]$  by  $\mathcal{T}$  is weaker than that induced on  $F[x]$  by  $\sup_{1 \leq i \leq k} \mathcal{T}_{p_i(x)}$ .

For all  $n \geq 1$ , let  $U_n = \{f(x) \in F[x] : p_i^n | f, 1 \leq i \leq k\}$ . Then  $(U_n)_{n \geq 1}$  is a fundamental system of neighborhoods of zero for the topology induced on  $F[x]$  by  $\sup_{1 \leq i \leq k} \mathcal{T}_{p_i(x)}$ . But clearly  $p_i^n | f$  for all  $i \in [1, k]$  if and only if  $h^n | f$  as  $h = p_1 \cdots p_k$ . Thus  $U_n = F[x]h^n(x)$ .

For each  $\epsilon > 0$ , let  $B_\epsilon = \{y \in F(x) : \|y\| < \epsilon\}$ . It suffices to show that there exists  $N \in \mathbb{N}$  such that  $F[x]h^N(x) \subseteq B_\epsilon \cap F[x]$ .

Let  $N$  be such that  $\|h(x)^N\| < \epsilon/S$ . Then for any  $g(x) \in F[x]$ ,

$$\|g(x)h^N(x)\| \cong \|g(x)\| \|h^N(x)\| < S \cdot \frac{\epsilon}{S} = \epsilon.$$

Hence  $F[x]h^N(x) \subseteq B_\epsilon \cap F[x]$ .

We next show that  $\mathcal{T} \subseteq \sup_{1 \leq i \leq k} \mathcal{T}_{p_i(x)}$ , that is, that given  $\epsilon > 0$ , there exists  $R \geq 0$  such that for every  $y \in F(x)$ , if  $v_{p_i(x)}(y) > R$  for all  $i \in [1, k]$ , then  $\|y\| < \epsilon$ .

Let  $\epsilon > 0$ . As the mapping  $(y, z) \rightarrow yz$  is continuous at  $(1, 0)$ , there exists  $\delta, \gamma > 0$  such that  $(B_\delta + 1)B_\gamma \subseteq B_1$ . By 2, there exists  $k_0$  such that  $\|h(x)^{k_0}\| < \gamma$ . Then  $(B_\delta + 1) \subseteq h(x)^{-k_0}B_1$  and so  $h(x)^{-k_0}B_1$  is a neighborhood of 1. As the topology  $\mathcal{T}$  is given by a norm,  $y \rightarrow y^{-1}$  is continuous for  $\mathcal{T}$  on the set of nonzero elements of  $F(x)$  (the proof is the same as for normed algebras found in [3], p. 75, Proposition 13), so there exists  $\eta$  such that  $0 < \eta < 1$  and  $(B_\eta + 1)^{-1} \subseteq h(x)^{-k_0}B_1$ . Since the topology induced on  $F[x]$  by  $\mathcal{T}$  is weaker than that induced on  $F[x]$  by  $\sup_{1 \leq i \leq k} \mathcal{T}_{p_i(x)}$ , there exists  $N$  such that  $F[x]h(x)^N \subseteq B_\eta$ . Then for all  $t \in F[X]$ ,

$$\left\| \frac{1}{t(x)h(x)^N + 1} \right\| \cong \|h(x)^{-k_0}\|.$$

Choose  $R \in \mathbf{N}$  such that  $\|h(x)^R\| < \epsilon / \|h(x)^{-k_0}\|S$ . Suppose  $v_{p_i}(y) \geq R$  for all  $i = 1, 2, \dots, k$ . Then

$$y = p_1^R(x) \cdots p_k^R(x) r(x) \frac{f(x)}{q(x)} = h^R(x) r(x) \frac{f(x)}{q(x)}$$

where  $r, f, q \in F[X]$ , and in  $F[X]$ ,  $p_i \nmid f, p_i \nmid q, i = 1, 2, \dots, k$ . Thus  $h^N$  and  $q$  are relatively prime, so there exists polynomials  $s, t \in F[X]$  such that  $q(x)s(x) = h^N(x)t(x) + 1$ .

Thus,

$$\begin{aligned} y &= h^R(x) r(x) \frac{f(x)}{q(x)} \\ &= h^R(x) \frac{r(x)s(x)f(x)}{h^N(x)t(x) + 1} \end{aligned}$$

and therefore

$$\begin{aligned} \|y\| &\leq \|h^R(x)\| \|r(x)s(x)f(x)\| \|(h^N(x)t(x) + 1)^{-1}\| \\ &< \frac{\epsilon}{\|h(x)^{-k_0}\|S} S \|h(x)^{-k_0}\| = \epsilon. \end{aligned}$$

To complete the proof of the theorem it remains to show that  $\sup_{1 \leq i \leq k} \mathcal{T}_{p_i(x)} \subseteq \mathcal{T}$ . For this, as both  $\mathcal{T}$  and  $\sup_{1 \leq i \leq k} \mathcal{T}_{p_i(x)}$  are locally bounded topologies, it suffices to show that  $B_1 \subseteq \{y \in F(x) : v_{p_i(x)}(y) \geq 1, 1 \leq i \leq k\}$ . Let  $y \in B_1$  and let  $y = p_1(x)^{e_1} \cdots p_k(x)^{e_k} f(x)/q(x)$  where  $e_i \in \mathbf{Z}, f, q \in F[X]$ , and  $p_i \nmid f, p_i \nmid q$  for all  $i \in [1, k]$ , and  $f$  and  $q$  are relatively prime. Let  $I = \{i \in [1, k] : e_i \leq 0\}$ . Then  $yq(x) \prod_{i \in I} p_i(x)^{-e_i}$  is a topological nilpotent in  $\mathcal{T}$  as  $F[x]$  is bounded. But

$$yq(x) \prod_{i \in I} p_i(x)^{-e_i} = f(x) \prod_{i \notin I} p_i(x)^{e_i} \in F[x],$$

so  $f \prod_{i \notin I} p_i^{e_i} \in J = (h)$ . Hence as  $h = p_1 \cdots p_k, [1, k] \sim I = [1, k]$ , i.e.,  $I = \emptyset$ . So  $e_i \geq 1$  for all  $i \in [1, k]$ , and thus  $v_{p_i(x)}(y) \geq 1$  for all  $i \in [1, k]$ .

**COROLLARY.** *Let  $\mathcal{T}$  be a Hausdorff ring topology on  $F(x)$  for which the subfield  $F$  is bounded and let  $p$  be a prime polynomial in  $F[X]$ . Then  $\mathcal{T} = \mathcal{T}_{p(x)}$  if and only if  $\mathcal{T}$  is locally bounded and  $\lim_{n \rightarrow \infty} p^n(x) = 0$ .*

**THEOREM 2.** *Let  $F$  be a field,  $x$  a transcendental element over  $F$ . Let  $\mathcal{T}$  be a Hausdorff, locally bounded ring topology on  $F(x)$  for which the subfield  $F$  is bounded and for which there is a nonzero topological*

nilpotent  $y$ . Then there exists a finite subset  $S$  of  $\mathcal{P}'$  such that  $\mathcal{T} = \sup_{s \in S} \mathcal{T}_s$ .

*Proof.* As each valuation  $v$  on  $F(x)$  that is improper on  $F$  is equivalent to  $v_s$  for exactly one  $s \in \mathcal{P}'$  ([1], p. 94, Corollary 2) it suffices to show that  $\mathcal{T}$  is the supremum of finitely many valuation topologies where each valuation is improper on  $F$ .

Let  $y = g(x)/h(x) \neq 0$  where  $(g(x), h(x)) = 1$ . By Cohn's Theorem, there is a norm  $\|\cdot\|$  defining the topology on  $F(x)$ .

*Case 1.*  $\deg h < \deg g$ . Let  $n = \deg h$ ,  $n + r = \deg g$ . We shall show that

$$F[x] \subseteq F[y] + F[y]x + \cdots + F[y]x^{n+r-1},$$

that is for each  $f \in F[X]$  there exists  $Q \in F[y][X]$  of degree  $< n + r$  such that  $f(x) = Q(x)$ . For this it clearly suffices to show that for each  $k \geq 0$ ,  $x^{n+r+k} = Q_k(x)$  for some  $Q_k \in F[Y][X]$  of degree  $< n + r$ .

Let  $g(x) = a_{n+r}x^{n+r} + \cdots + a_1x + a_0$ , where each  $a_i \in F$ , and let  $h(x) = b_nx^n + \cdots + b_1x + b_0$ , where each  $b_i \in F$ . Then  $yb_nx^n + yb_{n-1}x^{n-1} + \cdots + yb_0 = a_{n+r}x^{n+r} + \cdots + a_1x + a_0$ . So  $x^{n+r} = Q_0(x)$  where

$$Q_0(X) = \sum_{j=0}^n a_{n+r}^{-1}(b_j y - a_j)X^j - \sum_{j=n+1}^{n+r-1} a_{n+r}^{-1}b_j X^j.$$

Assume  $x^{n+r+k} = Q_k(x)$  where  $Q_k$  is a polynomial of degree  $\leq n + r - 1$  over  $F[y]$ ; let  $Q_k = g_0(y)x^{n+r-1} + P_k$  where  $g_0 \in F[X]$  and  $P_k$  is a polynomial in  $F[y][X]$  of degree  $\leq n + r - 2$ . Then

$$\begin{aligned} x^{n+r+k+1} &= g_0(y)x^{n+r} + xP_k(x) \\ &= g_0(y)Q_0(x) + xP_k(x) = Q_{k+1}(x) \end{aligned}$$

where  $Q_{k+1}(X) = g_0(y)Q_0(X) + XP_k(X)$ , a polynomial over  $F[y]$  of degree  $\leq n + r - 1$ . Therefore

$$F[x] \subseteq F[y] + F[y]x + \cdots + F[y]x^{n+r-1}.$$

By 1 of Theorem 1, applied to  $F(y)$  with its induced nondiscrete topology which is given by the induced norm,  $F[y]$  is bounded in norm, and hence  $F[x]$  is also. Thus  $F[x]$  is a bounded subset of  $F(x)$ . Consequently,  $h(x)y$  is a topological nilpotent; but  $h(x)y = g(x)$ , so  $g(x)$  is a topological nilpotent. Then by Theorem 1,  $\mathcal{T}$  is the supremum of  $p_i$ -adic topologies for some finite sequence of primes in  $F[x]$ .

Case 2.  $\deg h = \deg g$ .

Choose  $N$  such that

$$\left\| \left( \frac{g(x)}{h(x)} \right)^N \right\| < \frac{1}{\|x\|}. \quad \text{Then } x \left( \frac{g(x)}{h(x)} \right)^N$$

is a topological nilpotent. Since  $\deg g = \deg h$ ,  $\deg (Xg^N) > \deg h^N$ . By Case 1,  $\mathcal{T} = \sup_{1 \leq i \leq k} \mathcal{T}_{p_i}$  for some sequence  $p_1, \dots, p_k$  of primes in  $F[x]$ .

Case 3.  $\deg g < \deg h$  and there exists  $a_0 \in F$  such that  $X - a_0 \nmid g$ . Since the substitution mapping  $f \rightarrow f(x - a_0)$  is an automorphism of  $F[x]$ , let  $g(x) = g_1(x - a_0)$ ,  $h(x) = h_1(x - a_0)$  where  $g_1, h_1 \in F[X]$ . Then  $\deg g_1 < \deg h_1$  and as  $X - a_0 \nmid g$ ,  $X \nmid g_1$ . Let  $g_1 = C_n X^n + \dots + C_0$ ,  $h_1 = b_{n+r} X^{n+r} + \dots + b_0$ . Then  $C_0 \neq 0$  as  $X \nmid g_1$ . Hence

$$\begin{aligned} \frac{g(x)}{h(x)} &= \frac{C_n(x - a_0)^n + \dots + C_0}{b_{n+r}(x - a_0)^{n+r} + \dots + b_0} \\ &= \frac{(x - a_0)^{n+r} C_n(x - a_0)^{-r} + \dots + C_0(x - a_0)^{-(n+r)}}{(x - a_0)^{n+r} b_{n+r} + \dots + b_0(x - a_0)^{-(n+r)}} \\ &= \frac{g_0((x - a_0)^{-1})}{h_0((x - a_0)^{-1})} \end{aligned}$$

where  $g_0 = C_0 X^{n+r} + \dots + C_n X^r$ , a polynomial of degree  $n + r$  as  $C_0 \neq 0$ , and  $h_0 = b_0 X^{n+r} + \dots + b_{n+r}$ , a polynomial of degree  $\leq n + r$ . Let  $z = (x - a_0)^{-1}$ . Then  $F(z) = F(x)$  and  $g_0(z)/h_0(x)$  is a topological nilpotent. By Cases 1 and 2,  $\mathcal{T} = \sup_{1 \leq i \leq k} \mathcal{T}_{p_i}$  for some sequence of primes in  $F[(x - a_0)^{-1}]$ .

Case 4.  $\deg g < \deg h$  and for all  $a \in F$ ,  $X - a \mid g$ .

In this case  $F$  is a finite field; let  $p$  be the characteristic of  $F$  and let  $q$  be its order.

By the corollary to Theorem 1 applied to the prime polynomial  $Y$  of  $F[Y]$ , the topology induced on  $F(y)$  is the  $y$ -adic topology. Moreover,  $[F(x) : F(y)] \leq \deg h$ .

Let  $K$  be a maximal subfield of  $F(x)$  such that  $K$  contains  $F(y)$  and the topology induced on  $K$  by  $\mathcal{T}$  is the supremum of finitely many valuation topologies, each of which is discrete on  $F$ . Let  $K_0 = \{z \in F(x) : z \text{ is separable over } K\}$ . Then by Theorem 5 of ([5]), the topology induced on  $K_0$  is the supremum of finitely many valuation topologies. Hence  $K = K_0$ , that is  $F(x)$  is a purely inseparable extension of  $K$ . Let  $n$  be such that  $x^{p^n} \in K$ .

Let  $v_1, \dots, v_r$  be valuations on  $K$  such that  $\mathcal{T}|_K = \sup_{1 \leq i \leq r} \mathcal{T}_{v_i}$ . Since every valuation on  $K$  admits an extension to  $F(x)$  ([1], p. 105, Proposition 5), we may assume each  $v_i = v_{s_i}|_K$  for some  $s_i \in \mathcal{P}'$ .

By Theorem 4 of ([5]) there exist ring topologies  $\mathcal{T}_1, \dots, \mathcal{T}_r$  on  $F(x) = K[x]$  such that  $\mathcal{T}_i|_K = \mathcal{T}_{v_i}$  and  $\mathcal{T} = \sup_{1 \leq i \leq r} \mathcal{T}_i$ . Furthermore, as  $\mathcal{T}$  is a locally bounded topology, by the proof of Theorem 4 of ([5]), each  $\mathcal{T}_i$  is locally bounded. Therefore it suffices to show that for  $1 \leq i \leq r$ ,  $\mathcal{T}_i$  is the supremum of finitely many valuation topologies, each of which is discrete on  $F$ .

Let  $1 \leq i \leq r$ . Suppose that  $v_i = v_s|_K$  where  $s$  is a prime polynomial of  $F[X]$ . As  $g(x)/h(x)$  is a topological nilpotent for the given topology it is also a topological nilpotent for the weaker topology induced on  $K$  by  $v_s$ , and hence  $v_s(g(x)/h(x)) > 0$ . Furthermore by our assumption on  $v_s$ ,  $v_s(x) \geq 0$ . Let  $m$  be such that  $p^m > \deg h$ . Then  $v_s(x^{p^m}g(x)/h(x)) > 0$ , so  $x^{p^m}g(x)/h(x)$  is a nonzero topological nilpotent for  $\mathcal{T}_{v_s}$ , and hence for  $\mathcal{T}_i$ . Furthermore,  $\deg X^{p^m}g > \deg h$ . As  $F$  is a finite field,  $F$  is a bounded set for  $\mathcal{T}_i$  and therefore by Case 1,  $\mathcal{T}_i$  is the supremum of finitely many valuation topologies on  $F(x)$ , each of which is discrete on  $F$ .

So we may assume that  $v_i = v_\infty|_K$ . Let  $t$  be the highest power of  $X$  occurring in the factorization of the polynomial  $g$ . Choose  $m$  such that  $mp^m > t$ . Let  $g_0, h_0$  be relatively prime polynomials such that  $g_0/h_0 = g/h \cdot 1/X^{p^m}$ . Since  $v_\infty(g(x)/h(x)) > 0$  and  $v_\infty(1/X^{p^m}) > 0$ ,  $g_0(x)/h_0(x)$  is a nonzero topological nilpotent for  $\mathcal{T}_{v_i}$ , and hence for  $\mathcal{T}_i$ . Moreover,  $X \nmid g_0$ . Therefore by Case 3,  $\mathcal{T}_i$  is the supremum of finitely many valuation topologies, each of which is discrete on  $F$ . This completes the proof of the theorem.

**COROLLARY.** *Let  $F$  be a field,  $x$  a transcendental element over  $F$ . Let  $\mathcal{T}$  be a Hausdorff locally bounded ring topology on  $F(x)$  for which the subfield  $F$  is bounded and for which there is a nonzero topological nilpotent. Assume further that the completion of  $F(x)$  for  $\mathcal{T}$  is a field. Then  $\mathcal{T}$  is either the  $p$ -adic topology for some prime  $p$  in  $F[x]$  or  $\mathcal{T}$  is  $\mathcal{T}_\infty$ .*

*Proof.* By Theorem 2, there exists a nonempty finite subset  $S$  of  $\mathcal{P}'$  such that  $\mathcal{T} = \sup_{s \in S} \mathcal{T}_s$ . As the completion of  $F(x)$  for  $\mathcal{T}$  is a field, the cardinality of  $S$  is 1 by the Approximation Theorem ([1], p. 136, Theorem 2).

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