ON THE MEASURABILITY OF CONDITIONAL EXPECTATIONS

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It is shown that for a measurable stochastic process \( V \) and a nondecreasing family of \( \sigma \)-algebras \( \mathcal{A} \), there exists a measurable stochastic process \( V^* \) such that \( V^*(t, \cdot) \) is a version of \( E(V(t, \cdot) \mid \mathcal{A}_t) \) for all \( t \).

Let \((\Omega, \mathcal{A}, P)\) be a probability space (not necessarily complete), \( T \) an interval (bounded or unbounded) of the real line and \( V \) a real-valued stochastic process defined on \( T \times \Omega \) which is a measurable process, see Doob [3, p.60]. Let \( \mathcal{A}_t, t \in T, \mathcal{A}_t \subset \mathcal{A} \) form a nondecreasing family of \( \sigma \)-algebras. We shall prove in this note that under some boundedness condition on \( V \) the conditional expectations with respect to \( P, E(V(t, \cdot) \mid \mathcal{A}_t) \) can be chosen as to define a measurable process on \( T \times \Omega \). A similar statement appears in a paper by Brooks [1] but there it is additionally assumed that the family of \( \sigma \)-algebras is left-continuous, and the proof given there does not seem to carry over to a general nondecreasing family.

**Theorem.** Suppose for each \( t \in T \): \( V(t, \cdot) \geq 0 \) \( P \)-a.s. or \( \int |V(t, \cdot)| \, dP < \infty \). Then there exists a measurable process \( V^* \) such that for each \( t \in T \), \( V^*(t, \cdot) \) is a version of \( E(V(t, \cdot) \mid \mathcal{A}_t) \).

**Proof.** Since for any \( t \in T \)

\[
E(V(t, \cdot) \mid \mathcal{A}_t) = E(V(t, \cdot)^+ \mid \mathcal{A}_t) - E(V(t, \cdot)^- \mid \mathcal{A}_t)
\]

we may assume without loss of generality that for each \( t \in T \) \( V(t, \cdot) \geq 0 \) \( P \)-a.s. Using the linearity and monotone convergence property of conditional expectations the theorem now is easily reduced to the case that \( V \) is the characteristic function \( I_D \) of some subset \( D = B \times A \) of \( T \times \Omega \) with \( A \in \mathcal{A} \) and \( B \) belonging to the Borel sets of \( T \).

Since \( E(I_D(t, \cdot) \mid \mathcal{A}_t) = I_B(t)E(I_A \mid \mathcal{A}_t) \) holds it is enough to show that \( E(I_A \mid \mathcal{A}_t) \) can be chosen to form a measurable process. Let \( \mathcal{M} \) denote the set of all random variables on \((\Omega, \mathcal{A}, P)\) taking values in \([0,1]\) with random variables that are equal \( P \)-a.e. identified. Then \( \mathcal{M} \) is a metrizable topological space under the topology of convergence in
probability. By Theorem 3 in Cohn [2] it is now sufficient to show that the mapping $E_A: T \to M$ with $E_A(t) = E(I_A \mid \mathcal{A}_t)$ has separable range and is measurable with respect to the Borel sets of $M$. $E(I_A \mid \mathcal{A}_t)$, $t \in T$, forms a uniformly integrable martingale and so it follows from Theorem 11.2 in Doob [3], p. 358, that $E_A$ is continuous at all but countably many points of $T$. This yields at once that $E_A$ is measurable and furthermore—since $T$ is separable—that the range of $E_A$ is separable. This concludes the proof.

If the condition $\forall (t, \cdot)^0 \geq 0 \ P\text{-}a.s. \ or \ \int |V(t, \cdot)|dP < \infty$ is only required to hold for $\mu$-a.a. $t \in T$, $\mu$ being any measure on the Borel sets of $T$, then obviously there exists a measurable process $V^*$ which is a version of $E(V(t, \cdot) \mid \mathcal{A}_t)$ for $\mu$-a.a. $t \in T$.

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Received February 25, 1975 and in revised form February 22, 1977.

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