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ON THE MEASURABILITY OF CONDITIONAL EXPECTATIONS

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It is shown that for a measurable stochastic process V and a nondecreasing family of σ -algebras \mathcal{A}_t , there exists a measurable stochastic process V^* such that $V^*(t, \cdot)$ is a version of $E(V(t, \cdot) | \mathcal{A}_t)$ for all t .

Let (Ω, \mathcal{A}, P) be a probability space (not necessarily complete), T an interval (bounded or unbounded) of the real line and V a real-valued stochastic process defined on $T \times \Omega$ which is a measurable process, see Doob [3, p. 60]. Let $\mathcal{A}_t, t \in T, \mathcal{A}_t \subset \mathcal{A}$ form a nondecreasing family of σ -algebras. We shall prove in this note that under some boundedness condition on V the conditional expectations with respect to $P, E(V(t, \cdot) | \mathcal{A}_t)$ can be chosen as to define a measurable process on $T \times \Omega$. A similar statement appears in a paper by Brooks [1] but there it is additionally assumed that the family of σ -algebras is left-continuous, and the proof given there does not seem to carry over to a general nondecreasing family.

THEOREM. *Suppose for each $t \in T: V(t, \cdot) \geq 0$ P -a.s. or $\int |V(t, \cdot)| dP < \infty$. Then there exists a measurable process V^* such that for each $t \in T, V^*(t, \cdot)$ is a version of $E(V(t, \cdot) | \mathcal{A}_t)$.*

Proof. Since for any $t \in T$

$$E(V(t, \cdot) | \mathcal{A}_t) = E(V(t, \cdot)^+ | \mathcal{A}_t) - E(V(t, \cdot)^- | \mathcal{A}_t)$$

we may assume without loss of generality that for each $t \in T$ $V(t, \cdot) \geq 0$ P -a.s. Using the linearity and monotone convergence property of conditional expectations the theorem now is easily reduced to the case that V is the characteristic function I_D of some subset $D = B \times A$ of $T \times \Omega$ with $A \in \mathcal{A}$ and B belonging to the Borel sets of T .

Since $E(I_D(t, \cdot) | \mathcal{A}_t) = I_B(t)E(I_A | \mathcal{A}_t)$ holds it is enough to show that $E(I_A | \mathcal{A}_t)$ can be chosen to form a measurable process. Let \mathcal{M} denote the set of all random variables on (Ω, \mathcal{A}, P) taking values in $[0, 1]$ with random variables that are equal P -a.e. identified. Then \mathcal{M} is a metrizable topological space under the topology of convergence in

probability. By Theorem 3 in Cohn [2] it is now sufficient to show that the mapping $E_A: T \rightarrow \mathcal{M}$ with $E_A(t) = E(I_A | \mathcal{A}_t)$ has separable range and is measurable with respect to the Borel sets of \mathcal{M} . $E(I_A | \mathcal{A}_t)$, $t \in T$, forms a uniformly integrable martingale and so it follows from Theorem 11.2 in Doob [3], p. 358, that E_A is continuous at all but countably many points of T . This yields at once that E_A is measurable and furthermore—since T is separable—that the range of E_A is separable. This concludes the proof.

If the condition ' $V(t, \cdot) \geq 0$ P -a.s. or $\int |V(t, \cdot)| dP < \infty$ ' is only required to hold for μ -a.a. $t \in T$, μ being any measure on the Borel sets of T , then obviously there exists a measurable process V^* which is a version of $E(V(t, \cdot) | \mathcal{A}_t)$ for μ -a.a. $t \in T$.

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