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## **THE FINITE WEIL-PETERSSON DIAMETER OF RIEMANN SPACE**

SCOTT ANDREW WOLPERT

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Let  $T_g$  be the Teichmüller space and  $R_g$  the Riemann space of compact Riemann surfaces of genus  $g$  with  $g \geq 2$ . The space  $R_g$  can be realized as the quotient of  $T_g$  by a properly discontinuous group  $M_g$ , the modular group. Various metrics have been defined for  $T_g$  which are compatible with the standard topology for  $T_g$  and induce quotient metrics for  $R_g$ . Several authors have considered the Weil-Petersson metric for  $T_g$ . A length estimate derived in a previous paper is summarized; combining this with the Ahlfors Schwarz lemma, an estimate of N. Halpern and L. Keen, and an additional argument shows that the Weil-Petersson quotient metric for  $R_g$  has finite diameter. A corollary is an estimate relating the Poincaré length of the shortest closed geodesic of a compact Riemann surface to the Poincaré diameter of the surface.

For background material the reader is referred to the articles of L. Ahlfors [1] and L. Bers [3] and to the article of L. Bers [5] for a survey of related topics. T. C. Chu [7, 8] and H. Masur [12] have obtained results related to ours. The author would like to thank Professor G. Kiremidjian for his assistance.

**1. The case of an annulus.** Let  $A = \{z \mid 1 < |z| < \rho\}$  be an annulus in the plane. Let  $M(A)$  be the space of Beltrami differentials of  $A$  endowed with the  $L^\infty$  metric; let  $Q(A)$  be the space of integrable holomorphic quadratic differentials of  $A$ . An element of  $M(A)$  is a tensor of type  $(-1, 1)$  with measurable coefficient.

DEFINITION 1.1. For  $\Phi \in Q(A)$  set

$$\|\Phi\|_A = \left( \int |\Phi|^2 \lambda_A^{-2} \right)^{1/2}$$

where  $\lambda_A$  is the Poincaré metric of  $A$ . For  $\mu \in M(A)$  set

$$\|\mu\|_A = \sup_{\Phi \in Q(A)} |[\mu, \Phi]| / \|\Phi\|_A$$

where  $[\mu, \Phi] = \int_A \mu \Phi$ .

The metric  $\lambda_A$  is known to be given by the following expression

$$(\pi/\log \rho)\csc(\pi \log|z|/\log \rho)|dz/z|.$$

We consider a particular deformation of the annulus

A. For  $t \geq 1$  let  $A_t = \{z_t | 1 < |z_t| < \rho^t\}$  then the map

$$(1.1) \quad z \mapsto z|z|^{t-1} = z_t(z)$$

is quasiconformal with Beltrami differential

$$(t - 1/t + 1)(z/|z|)^2 \overline{dz}/dz.$$

By considering solutions  $\omega(z)$  of the Beltrami differential equation  $\omega_z = \mu\omega_z$  where  $\mu$  is a Beltrami differential it is seen that the curve of Riemann surfaces  $A_t$  is represented by the curve

$$(t - 1/t + 1)(z/|z|)^2 \overline{dz}/dz \subset M(A), \quad t \geq 1.$$

As described in our previous paper [16]  $(1/2t)(z_t/|z_t|)^2 \overline{dz_t}/dz_t$  is the tangent to this curve at  $A_t$  expressed as an element of  $M(A_t)$ ,  $t \geq 1$ . By Definition 1.1

$$(1.2) \quad \begin{aligned} & \| (1/2t)(z_t/|z_t|)^2 \overline{dz_t}/dz_t \|_{A_t} \\ &= \sup_{\Phi \in Q(A_t)} \left| \int_{A_t} (1/2t)(z_t/|z_t|)^2 \overline{dz_t}/dz_t \Phi \right| / \left( \int_{A_t} |\Phi|^2 \lambda_{A_t}^{-2} \right)^{1/2}. \end{aligned}$$

It is clear that the extremal  $\Phi$  is given by  $(dz_t/z_t)^2$ . The value of the quotient in (1.2) is now equal to

$$(1.3) \quad (2\pi^3/t^3 \log \rho)^{1/2}.$$

Thus the length of the curve  $A_t$ ,  $t \geq 1$  is given by the convergent integral

$$(1.4) \quad \int_1^\infty (2\pi^3/t^3 \log \rho)^{1/2} dt.$$

For a compact Riemann surface  $R$  of genus  $g$ ,  $g \geq 2$  one can identify the cotangent space at the point  $R$  of Teichmüller space with the regular quadratic differentials  $Q(R)$  of  $R$  and the tangent space at  $R$  with the Beltrami differentials  $M(R)$  modulo those which are infinitesimally trivial, [1]. In this instance the Weil-Petersson metric and cometric are given by Definition 1.1 on replacing  $A$  by  $R$ , [15].

**2. Finite diameter of Riemann space.** The Riemann space  $R_g$  of genus  $g$ ,  $g \geq 2$  is the space of conformal equivalence classes of similarly oriented compact Riemann surfaces of genus  $g$ , [14]. A natural projection  $\pi_g$  of  $T_g$  to  $R_g$  exists; this projection can be given by the action of a properly discontinuous group  $M_g$ , the modular group, [6]. S. Kravetz showed that every metric  $d(\cdot, \cdot)$  for  $T_g$  compatible with the topology of  $T_g$  induces a quotient metric  $\tilde{d}(\cdot, \cdot)$  for  $R_g$  defined as

$$\tilde{d}(\tilde{x}, \tilde{y}) = \inf_{\substack{\pi_g(x)=\tilde{x} \\ \pi_g(y)=\tilde{y}}} d(x, y)$$

for  $x, y \in T_g$  and  $\tilde{x}, \tilde{y} \in R_g$ , [11].

DEFINITION 2.1. For  $\tilde{x}, \tilde{y} \in R_g$  let

$$\omega(\tilde{x}, \tilde{y}) = \inf_{\substack{\pi_g(x)=\tilde{x} \\ \pi_g(y)=\tilde{y}}} d_{w-p}(x, y)$$

where  $d_{w-p}(\cdot, \cdot)$  is the Weil-Petersson metric for  $T_g$ .

Let  $H = \{z \mid \text{Im } z > 0\}$  denote the upper half plane and  $\Delta = \partial^2/\partial x^2 + \partial^2/\partial y^2$  the Laplacian. The following definition and theorem are due to L. Ahlfors, [2].

DEFINITION 2.2. A metric  $\rho |dz|$ ,  $\rho \geq 0$  is said to be ultrahyperbolic in  $H$  if it has the following properties:

- (i)  $\rho$  is upper semicontinuous;
- (ii) at every  $z_0 \in H$  with  $\rho(z_0) > 0$  there exists a  $\rho_0$  defined and of class  $C^2$  in a neighborhood  $V$  of  $z_0$  such that  $\Delta \log \rho_0 \geq \rho_0^2$  and  $\rho \geq \rho_0$  in  $V$  while  $\rho(z_0) = \rho_0(z_0)$ .

The Poincaré metric of  $H$  is  $|dz|/y$ .

THEOREM 2.3. Let  $\rho |dz|$  be an ultrahyperbolic metric for  $H$ . Then  $\rho |dz| \leq |dz|/y$ .

The following theorem is due to L. Bers, [4] and D. Mumford, [13].

THEOREM 2.4. For  $c > 0$ , let  $K_c \subset R_g$ ,  $g \geq 2$  consist of those Riemann surfaces  $R$  for which each closed Poincaré geodesic has length at least  $c$ . Then  $K_c$  is a compact set.

THEOREM 2.5.  $R_g$  has finite diameter for the  $\omega(\cdot, \cdot)$  metric.

*Proof.* Consider the following regions in  $H$   $C(l, \theta_0) = \{z \mid \text{Im } z >$

$0, 1 < |z| < \exp l, \quad \theta_0 < \arg z < \pi - \theta_0\}$  and  $\theta_1 < \theta_2$   $C(l, \theta_1, \theta_2) = C(l, \theta_1) - C(l, \theta_2)$ . The Poincaré area of  $C(l, \theta_0)$  (resp.  $C(l, \theta_1, \theta_2)$ ) is  $2l \cot \theta_0$  (resp.  $2l(\cot \theta_1 - \cot \theta_2)$ ). The self map of  $H z \mapsto z \exp l$  identifies the boundaries of  $C(l, \theta_0)$  such that the quotient  $A(l, \theta_0) = C(l, \theta_0)/\{z \mapsto z \exp l\}$  is conformally an annulus. Let  $\tilde{C}(l, \theta_1, \theta_2)$  denote  $C(l, \theta_1, \theta_2)$  with the boundaries  $\tilde{C}(l, \theta_1, \theta_2) \cap \{z \mid \arg z = \theta_2\}$  and  $\tilde{C}(l, \theta_1, \theta_2) \cap \{z \mid \arg z = \pi - \theta_2\}$  identified by the map  $z \mapsto z \exp i(\pi - 2\theta_2)$ ; the quotient  $A(l, \theta_1, \theta_2) = \tilde{C}(l, \theta_1, \theta_2)/\{z \mapsto z \exp l\}$  is conformally an annulus. Let  $\alpha(\theta)$  (resp.  $\beta(\theta)$ ) denote the projection to  $A(l, \theta_0)$  (resp.  $A(l, \theta_1, \theta_2)$ ) of the curve  $z = r \exp i\theta$ ,  $1 \leq r \leq \exp l$  provided  $\theta_0 \leq \theta \leq \pi - \theta_0$  (resp.  $\theta_1 \leq \theta \leq \theta_2$ ). A quotient metric for  $A(l, \theta_0)$  (resp.  $A(l, \theta_1, \theta_2)$ ) is obtained from the restriction to  $C(l, \theta_0)$  (resp.  $C(l, \theta_1, \theta_2)$ ) of the line element  $|dz|/y$ . The distance between the boundaries of  $A(l, \theta_0)$  (resp.  $A(l, \theta_1, \theta_2)$ ) in the quotient metric will be referred to as the width of  $A(l, \theta_0)$  (resp.  $A(l, \theta_1, \theta_2)$ ). Since each curve  $z = r \exp i\theta \subset H$   $0 < \theta < \pi$  is a Poincaré geodesic it follows that the width of  $A(l, \theta_0)$  is given by the integral  $\int_{\theta_0}^{\pi - \theta_0} r d\theta / r \sin \theta = 2 \ln(\cot \theta_0 + \csc \theta_0)$ . The induced quotient metric for  $A(l, \theta_1, \theta_2)$  is not differentiable on the curve  $\beta(\theta_2)$ ; nevertheless, it is straightforward that the width of  $A(l, \theta_1, \theta_2)$  is  $2 \ln(\cot \theta + \csc \theta)|_{\theta_2}^{\theta_1}$ . The curve  $\beta(\theta_2)$  has length  $\int_1^{\exp l} dr / r \sin \theta_2 = l \csc \theta_2$ .

The following lemmas of N. Halpern [9] and L. Keen [10] are essential to our argument.

LEMMA 2.6. *Let  $R$  be a compact Riemann surface. For every  $c_1 > 0$  there exists a  $c_2 > 0$  such that for  $\gamma$  a simple closed Poincaré geodesic of length  $l$  at most  $c_1$ , the region  $A(l, \theta_1)$ ,  $\theta_1 = \cot^{-1}(c_2/2l)$ , can be isometrically imbedded into  $R$  with  $\alpha(\pi/2)$  realizing  $\gamma$ .*

Observe that  $2l \cot \theta_1$  represents the area of  $A(l, \theta_1)$ .

LEMMA 2.7. *Let  $R$  be a compact Riemann surface of genus  $g$ ,  $g \geq 2$ . There exists a constant  $c_3 > 0$  such that there are at most  $3g - 3$  simple closed Poincaré geodesics of length at most  $c_3$ .*

*Proof of Lemma 2.7.* By Lemma 2.6 one can choose  $c_3 < c_1$  such that the width of  $A(l, \theta_1)$  for  $l \leq c_3$  is at least  $c_3$ . The conclusion now follows since there are at most  $3g - 3$  mutually disjoint, homotopically nontrivial, simple closed curves on  $R$  which are mutually not freely homotopic.

Let  $\Phi_l = \cot^{-1}(c_2/4l)$  and consider the domain  $A(l, \theta, \Phi_l)$ . The width of  $A(l, \theta, \Phi_l)$  is  $2 \ln(\cot \theta + \csc \theta)|_{\Phi_l}^{\theta}$ , which is bounded from below for  $l \leq c_3$  provided there exists a constant  $c > 0$  such that

$$(\cot \theta_l + \csc \theta_l)/(\cot \Phi_l + \csc \Phi_l) \geq c \quad \text{for } l \leq c_3.$$

For  $c_3$  sufficiently small  $\csc \Phi_l \leq 2 \cot \Phi_l$  thus

$$(2.1) \quad (\cot \theta_l + \csc \theta_l)/(\cot \Phi_l + \csc \Phi_l) \geq \cot \theta_l/3 \cot \Phi_l \geq 2/3.$$

The length of  $\beta(\Phi_l)$  is

$$(2.2) \quad l \csc(\cot^{-1}(c_2/4l)) \geq l \cot(\cot^{-1}(c_2/4l)) = c_2/4.$$

For an annulus  $A = \{z \mid 1 < |z| < r\}$  we make the following definition.

DEFINITION 2.8. The extremal length  $E(A)$  of  $A$  is given by  $E(A) = 2\pi/\log r$ .

Now the extremal length of  $A(l, \theta_b, \Phi_l)$  is  $E(A(l, \theta_b, \Phi_l)) = l/2(\Phi_l - \theta_l) = l/2(\cot^{-1}(c_2/4l) - \cot^{-1}(c_2/2l))$  where by l'Hopital's rule

$$(2.3) \quad \lim_{l \rightarrow 0} l/2(\cot^{-1}(c_2/4l) - \cot^{-1}(c_2/2l)) = c_2/4.$$

It is now clear that  $c', 0 < c' < c_3$  can be chosen such that for  $l \leq c'$

$$(2.4) \quad 2 \ln(\cot \theta + \csc \theta)|_{\theta_l}^{\theta_b} \geq c'$$

$$(2.5) \quad l \csc \Phi_l \geq c'$$

and

$$(2.6) \quad l/2(\Phi_l - \theta_l) \leq c_2.$$

These inequalities will now be used to estimate the diameter of  $R_g$ . The region  $K_{c'} \subset R_g$  is compact and thus has finite  $\omega$  diameter. Let a Riemann surface  $R$  represent a point in  $T_g$  such that  $\pi_g(R) \notin K_{c'}$  with  $\gamma_1, \dots, \gamma_n$  the geodesics of  $R$  of length less than  $c'$ . The object is to "fatten"  $R$  in a neighborhood of each of  $\gamma_1, \dots, \gamma_n$  thereby obtaining a surface in  $K_{c'}$ . By Lemma 2.6 a region  $A(l, \theta_l)$  can be considered as a coordinate neighborhood of  $\gamma_1$  where  $l$  is the length of  $\gamma_1$ . A new surface  $R^*$  can be formed by removing the part of  $A(l, \theta_l)$  corresponding to  $A(l, \Phi_l)$  and identifying the boundaries by the map  $z \mapsto z \exp i(\pi - 2\Phi_l)$ . Thus  $A(l, \theta_b, \Phi_l)$  represents a coordinate patch in a neighborhood of the gluing and the original coordinates are chosen otherwise. In a neighborhood of the gluing  $\lambda_R|_{R^*}$ , the Poincaré metric of  $R$  restricted to  $R^*$ , is defined in terms of the coordinate patch  $A(l, \theta_b, \Phi_l)$ ; for coordinate patches disjoint from the gluing  $\lambda_R|_{R^*} = \lambda_R$ . Assuming that  $\lambda_R|_{R^*}$  is

ultrahyperbolic Theorem 2.3 implies that  $\lambda_R|_{R^*} \leq \lambda_{R^*}$  where  $\lambda_{R^*}$  is the Poincaré metric of  $R^*$ . To show that  $\lambda_R|_{R^*}$  is ultrahyperbolic it suffices to consider the metric in a neighborhood of the gluing. Define the metric  $\tilde{\lambda}(z)|dz|$  on  $\tilde{C}(l, \theta_b, \Phi_l)$  by setting  $\tilde{\lambda}(z)|dz| = |dz|/\text{Im } z$  for  $1 < |z| < \exp l$ ,  $\theta_l < \arg z < \Phi_l$  and  $\tilde{\lambda}(z)|dz| = |dz|/\text{Im}(z \exp i(2\Phi_l - \pi))$  for  $1 < |z| < \exp l$ ,  $\pi - \Phi_l < \arg z < \pi - \theta_l$ ; that  $\tilde{\lambda}(z)|dz|$  satisfies (ii) of Definition 2.2 relative to the quotient metric of  $\tilde{C}(l, \theta_b, \Phi_l)$  is clear. The objective is to show that  $R^*$  is “fat” in the free homotopy class of  $\gamma_1$  and that no new (i.e., other than  $\gamma_2, \dots, \gamma_n$ ) “pinched” free homotopy classes were introduced. Let  $\gamma_0^* \subset R^*$  be a simple closed  $\lambda_R$ -geodesic of length less than  $c'$ . If  $\gamma_0^*$  does not intersect the gluing then  $\gamma_0^*$  can also be considered as a curve  $\gamma_0$  on  $R$ . Since  $\lambda_R|_{R^*} \leq \lambda_{R^*}$  the length of  $\gamma_0$  is also less than  $c'$ . If  $\gamma_0$  is freely homotopic to  $\gamma_1$  then  $\gamma_0$  can be lifted to the universal cover  $H$  of  $R$  with initial point  $z_0$  and end point  $z_1$  such that  $|z_0| = 1$  and  $|z_1| = \exp l$ . By the assumption that  $\gamma_0^*$  is disjoint from the gluing the lift of  $\gamma_0$  is disjoint from the domain  $A(l, \Phi_l)$  and thus by estimate (2.5) has length at least  $c'$ , a contradiction. By Lemma 2.7  $\gamma_0^*$  cannot intersect and yet be distinct from the geodesics  $\gamma_2, \dots, \gamma_n$ . Thus  $\gamma_0$  must be freely homotopic to one of  $\gamma_2, \dots, \gamma_n \subset R$  or  $\gamma_0^*$  intersects the gluing. If  $\gamma_0^*$  is contained in  $A(l, \theta_b, \Phi_l)$  then it must be freely homotopic to  $\gamma_1$  a case considered above; otherwise  $\gamma_0^*$  intersects the gluing and the boundaries of  $A(l, \theta_b, \Phi_l)$  hence crosses the domain. By estimate (2.4)  $\gamma_0^*$  has length at least  $c'$  in terms of the  $\lambda_R|_{R^*} \leq \lambda_{R^*}$  metric, a contradiction. Thus  $\gamma_0^*$  is freely homotopic to one of  $\gamma_2, \dots, \gamma_n$ . The deformation corresponding to the replacing of  $A(l, \theta_l)$  by  $A(l, \theta_b, \Phi_l)$  can be realized in terms of quasiconformal maps. For  $A = A(l, \theta_b, \Phi_l) = \{z \mid 1 < |z| < \rho\}$  the domain  $A(l, \theta_l)$  corresponds to the deformation of  $A$  given by the element  $(t - 1/t + 1)(z/|z|)^2 dz/dz \in M(A(l, \theta_b, \Phi_l))$  where  $t = (\pi - 2\theta_l)/2(\Phi_l - \theta_l)$ . We consider  $(\tau - 1/\tau + 1)(z/|z|)^2 \overline{dz}/dz$  restricted to  $A(l, \theta_b, \Phi_l) \subset R^*$   $1 \leq \tau \leq t$  as a curve in  $M(R^*)$ . The estimate for an annulus given by (1.4) can be now applied upon noting that  $\lambda_R|_A \leq \lambda_A$  and  $Q(R)|_A \subset Q(A)$ , [16]. The Weil–Petersson length of this curve is seen to be bounded in terms of  $E(A(l, \theta_b, \Phi_l))^{1/2}$ . Estimate (2.6) bounds the latter quantity by the constant  $c_2^{1/2}$ . Repeating this “fattening” process  $n$  times a surface  $\tilde{R} \in K_c$  is obtained. By Lemma 2.7  $n \leq 3g - 3$ ; the above remarks now yield  $\omega(R, \tilde{R}) \leq (3g - 3)c_2^{1/2}$ . The proof is complete.

**3. The Poincaré diameter and length of the shortest closed geodesic.** Let  $R$  be a compact Riemann surface of genus  $g$ ,  $g \geq 2$ . Let  $l(R)$  denote the length of the shortest closed Poincaré geodesic and  $d(R)$  the Poincaré diameter of  $R$ . The following lemma is a consequence of the considerations of 2.

LEMMA 3.1. *There exist constants  $\bar{c}_1$  and  $\bar{c}_2$  depending only on the genus such that*

$$\ln(\bar{c}_1/l(R)) \leq d(R) \leq 6g \ln(\bar{c}_2/l(R)).$$

*Proof.* Maintaining the constants  $c_1, c_2, c_3$  and  $c'$  of §2 we consider a surface  $R \in K_{c'}$ . As  $K_{c'}$  is compact  $l(R)$  and  $d(R)$  are bounded above and below hence constants  $\bar{c}_1, \bar{c}_2$  exist to yield

$$\ln(\bar{c}_1/l(R)) \leq d(R) \leq 2 \ln(\bar{c}_2/l(R))$$

for surfaces in  $K_{c'}$ . Now let  $R \notin K_{c'}$  then clearly  $d(R)$  is bounded below by one-half the width of  $A(l, \theta_l) \subset R$  where  $l = l(R)$ . Thus

$$(3.1) \quad \ln(c_2/2l) \leq \ln(\cot \theta_l + \csc \theta_l) \leq d(R).$$

Setting  $\bar{c}_2 = \min\{c_2, \bar{c}_2\}$  the lower bound is established. Assume that  $R \notin K_{c'}$  and has only one closed Poincaré geodesic of length less than  $c'$ . Forming the surface  $R^*$  as in 2. by removing  $A(l, \Phi_l)$  from  $A(l, \theta_l) \subset R$  where  $l = l(R)$  we have that  $d(R)$  is bounded by the sum of the width of  $A(l, \theta_l), l/2$  and  $d(R^*)$ . Specifically for two points  $x, y$  of  $R^*$  we connect them with a  $\lambda_{R^*}$  length minimizing curve  $\gamma_{x,y}$ . If this curve intersects the gluing a new curve is formed as the union of the shortest segment of  $\gamma_{x,y}$  from  $x$  to the gluing, a segment along the gluing and the shortest segment of  $\gamma_{x,y}$  from the gluing to  $y$ . Now taking account of the relation of  $R$  to  $R^*$   $d(R)$  is seen to be bounded by

$$2 \ln(\bar{c}_2/l(R)) + c' + 2 \ln(\bar{c}_2/l(R^*))$$

where  $\bar{c}_2$  has been appropriately modified. A constant  $\bar{c}_2$  can now be chosen to bound this last quantity by  $4 \ln(\bar{c}_2/l(R))$ . In general let  $S$  be a surface with exactly  $n$  closed Poincaré geodesics of length less than  $c'$ . We claim that  $d(S) \leq 2(n+1) \ln(\bar{c}_2/l(S))$  for an appropriate  $\bar{c}_2$ . Proceeding by induction on  $n$  it remains only to consider the induction step. Let  $R \notin K_{c'}$  have exactly  $n+1$  closed Poincaré geodesics of length less than  $c'$ . Forming the surface  $R^*$  and arguing as above  $d(R)$  is bounded by the sum of the width of  $A(l, \theta_l) \subset R, l/2$  and  $d(R^*)$  where  $l = l(R)$ . Using the induction hypothesis this is bounded by

$$2 \ln(\bar{c}_2/l(R)) + c' + 2(n+1) \ln(\bar{c}_2/l(R^*))$$

which in turn is bounded by

$$(3.2) \quad 2(n+2) \ln(\bar{c}_2/l(R)).$$



Observing that  $n$  is at most  $3g - 3$  the upper bound is now established.

In contrast to the present lemma the constructive estimate

$$(3.3) \quad d(R) \leq (g - 1)l(R)/\sinh^2(l(R)/2)$$

where

$$l(R)/\sinh^2(l(R)/2) \approx 4/l(R)$$

for  $l(R)$  sufficiently small was given by L. Bers, [4].

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UNIVERSITY OF MARYLAND  
COLLEGE PARK, MD 20742



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