

# Pacific Journal of Mathematics

**RATIONAL APPROXIMATION AND THE GROWTH OF  
ANALYTIC CAPACITY**

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## RATIONAL APPROXIMATION AND THE GROWTH OF ANALYTIC CAPACITY

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Let  $X$  be a compact set in the complex plane  $\mathbb{C}$ . Denote by  $R(X)$  the closure in the supremum norm of the rational functions with poles off  $X$  and by  $A(X)$  the set of continuous functions, which are analytic on the interior of  $X$ . The analytic capacity of a set  $S$  is denoted by  $\gamma(S)$ . For the definition of  $\gamma$  see below. Let  $B_z(\delta) = \{\zeta \in \mathbb{C}; |z - \zeta| < \delta\}$  and let  $\partial X$  denote the boundary of  $X$ . Vitushkin has proved that  $R(X) = A(X)$  if

$$\liminf_{\delta \rightarrow 0} \frac{\gamma(B_z(\delta) \setminus X)}{\delta} > 0 \text{ for all } z \in \partial X.$$

Let  $\psi$  be a function from  $\mathbb{R}^+$  to  $\mathbb{R}^+$ , where  $\mathbb{R}^+ = \{x \in \mathbb{R}; x \geq 0\}$ . We now ask the following questions. If  $\lim_{\delta \rightarrow 0} \psi(\delta) = 0$ , is it possible to find a compact set  $X$  such that  $R(X) \neq A(X)$  and such that  $\gamma(B_z(\delta) \setminus X) \geq \delta \psi(\delta)$  for all  $z \in \partial X$  and for all  $\delta$ ,  $0 < \delta < \delta_z$ ? If the answer is yes, can the answer still be yes, if  $\lim_{\delta \rightarrow 0} \psi(\delta) = 0$  is replaced by  $\lim_{\delta \rightarrow 0} \psi(\delta) > 0$ ? The answers of these questions can be found in Theorem 1 and Theorem 2.

**DEFINITION.** Let  $K$  be a compact subset of  $\mathbb{C}$ . Then  $\gamma(K) = \sup |f'(\infty)|$ , where the supremum is taken over all functions  $f$  such that  $f$  is analytic on the unbounded component of  $\mathbb{C} \setminus K$ ,  $|f(z)| \leq 1$  for all  $z \in \mathbb{C}$  and  $f(\infty) = 0$ . Let  $S$  be an arbitrary subset of  $\mathbb{C}$ . Then  $\gamma(S) = \sup \gamma(K)$ , where the supremum is taken over all compact subsets of  $S$ .

For further information about this capacity see for instance [2], [3], [4] and [5].

**THEOREM 1.** Let  $\delta_n \searrow 0$  when  $n \rightarrow \infty$ . Suppose that

$$\liminf_{n \rightarrow \infty} \frac{\gamma(B_z(\delta_n) \setminus X)}{\delta_n} > 0 \text{ for all } z \in \partial X.$$

Then  $R(X) = A(X)$ .

**THEOREM 2.** Let  $\psi$  be a function from  $\mathbb{R}^+$  to  $\mathbb{R}^+$ . Suppose that  $\lim_{\delta \rightarrow 0} \psi(\delta) = 0$ . Then there exists a compact set  $X$  such that

$$(a) \quad R(X) \neq A(X)$$

and

$$(b) \quad \gamma(B_z(\delta) \setminus X) \cong \psi(\delta)\delta \text{ for all } z \in \partial X \text{ and for all } \delta, 0 < \delta < \delta_z.$$

REMARK. Theorem 1 gives the following. Let  $\psi$  be a function from  $\mathbf{R}^+$  to  $\mathbf{R}^+$ . Suppose that  $\lim_{\delta \rightarrow 0} \psi(\delta) > 0$  and suppose that  $\gamma(B_z(\delta) \setminus X) \cong \psi(\delta)\delta$  for all  $z \in \partial X$  and for all  $\delta, 0 < \delta < \delta_z$ . Then  $R(X) = A(X)$ .

**2. The proofs.** Theorem 1 can be proved in the same way as the theorem of Vitushkin mentioned in the introduction. See [4], Ch. 2, §4. We omit the proof.

In [1] A. M. Davie constructed a compact set  $X$  such that every point of  $\partial X$  is a peak point for  $R(X)$ , but  $R(X) \neq A(X)$ . Our proof of Theorem 2 is a refinement of Davie's construction. We start by formulating two lemmas. The first lemma is well-known (see for instance [2], p. 199). The second lemma is due to Carleson. For a proof see [1].

LEMMA 1. *Let  $L$  be a compact set on a line. Then*

$$\gamma(L) \cong \frac{1}{4} \{ \text{the length of } L \}.$$

LEMMA 2. *Let  $E$  be a perfect subset of the real line and  $I$  the closed interval  $[0, 1]$ . Then we can find a continuous function on  $\mathbf{C}$ , analytic outside  $I \times E$ , such that  $f(\infty) = 0$ ,  $f'(\infty) = \frac{1}{4}$  and  $|f(z)| \leq 1$  for all  $z \in \mathbf{C}$ .*

If  $x \in \mathbf{R}$ , let  $[x]$  denote the greatest integer less than or equal to  $x$ .

*Proof of Theorem 2.* We may assume that  $\psi(\delta)$  is a strictly increasing function. Put  $a_n = 16\psi(2^{-n+1})$ ,  $n = 1, 2, 3, \dots$ . Then  $a_n \searrow 0$  when  $n \rightarrow \infty$ .

Let  $f$  be an increasing function such that  $f(-2 - \log a_n) = n$ . Put

$$b_0 = 1$$

and

$$b_n = \min(e^{-f(n)}, \frac{1}{4}b_{n-1}) \quad \text{for } n \geq 1.$$

Let  $E$  be the usual Cantor set on the real axis such that the set  $E_n$  obtained in  $n$ th step consists of  $2^n$  intervals of length  $b_n$ . Let  $I = [0, 1]$ .

Let  $n$  be fixed for a moment. There exists an integer  $k_n$  such that

$$(1) \quad b_n \geq 2^{-k_n}.$$

Denote the intervals in  $E_n$  by  $I_{n,i}$ ,  $i = 1, 2, \dots, 2^n$ . In every  $I \times I_{n,i}$  choose open disjoint discs with radius  $2^{-k_n-3}e^{-n-1}$  in the following way. Every disc must not intersect  $I \times E_{n+1}$  but every disc must touch  $I \times E_{n+1}$ . Moreover, the discs are arranged such that the centres of the discs lie on two horizontal lines in every  $I_{n,i}$ . There are  $2^{k_n+3}$  centres on each line and the distance between two successive centres is  $2^{-k_n-3}$ . Call the chosen discs  $U_{n,j}$ .

Repeat the construction for all  $n$ ,  $n = 1, 2, 3, \dots$ . Put

$$X = \overline{B_0(2)} \setminus \left( \bigcup_{n,j} U_{n,j} \right),$$

where  $\overline{B_0(2)}$  denotes the closure of  $B_0(2)$ .  $X$  is a compact set and

$$\partial X = \partial B_0(2) \cup \left( \bigcup_{n,j} \partial U_{n,j} \right) \cup (I \times E).$$

It is easy to see that  $\sum_{n,j} \text{diam } U_{n,j} < \infty$ . Lemma 2 and a standard argument give

$$R(X) \neq A(X).$$

See [2], p. 220.

(i) Let

$$z \in \partial B_0(2) \cup \left( \bigcup_{n,j} \partial U_{n,j} \right).$$

Lemma 1 gives for all  $m \geq m_z$

$$\gamma(B_z(2^{-m}) \setminus X) \geq \frac{1}{4} 2^{-m} \geq \frac{1}{4} a_m 2^{-m}.$$

(ii) Let  $z \in I \times E$ . Let  $m$  be a positive integer such that  $a_m < e^{-2}$ . The definition of  $f$  gives  $f(-2 - \log a_m) = m$ . Fix  $n$  such that  $n = \lceil -\log a_m \rceil - 1$ . If we use that  $f$  is an increasing function and the definition of  $b_n$ , we obtain

$$2^{-m} = e^{-f(-2-\log a_m)} \geq e^{-f(-1+\lceil -\log a_m \rceil)} = e^{-f(n)} \geq b_n.$$

Thus

$$(2) \quad 2^{-m} \geq b_n.$$

One now easily shows that  $B_z(2^{-m})$  contains disjoint discs  $U_{n,j_i}$ ,  $i = 1, 2, \dots, 2^{k_n+2}2^{-m} - 2$ , such that their centres are on one straight line. Lemma 1, (1) and (2) give

$$\begin{aligned} \gamma(B_z(2^{-m}) \setminus X) &\cong \gamma\left(\bigcup_i U_{n,i}\right) \cong \frac{1}{4}\{2^{k_n+2}2^{-m} - 2\}2^{-k_n-2}e^{-n-1} \\ &= \frac{1}{4}e^{-n-1}\{2^{-m} - 2^{-k_n-1}\} \cong \frac{1}{4}e^{-n-1}\{2^{-m} - \frac{1}{2}b_n\} \\ &\cong \frac{1}{4}e^{-n-1}\{2^{-m} - \frac{1}{2}2^{-m}\} = \frac{1}{8}2^{-m}e^{-n-1}. \end{aligned}$$

Thus

$$\gamma(B_z(2^{-m}) \setminus X) \cong \frac{1}{8}2^{-m}e^{-n-1}.$$

If we use that  $n = \lceil -\log a_m \rceil - 1$ , we obtain

$$e^{-n-1} = e^{-\lceil -\log a_m \rceil} \geq e^{\log a_m} = a_m.$$

Thus

$$\gamma(B_z(2^{-m}) \setminus X) \cong \frac{1}{8}a_m 2^{-m}.$$

Now (i) and (ii) give that for all  $z \in \partial X$  there is a constant  $m_z$  such that

$$\gamma(B_z(2^{-m}) \setminus X) \cong \frac{1}{8}a_m 2^{-m} \text{ for all } m \geq m_z.$$

The definition of  $a_m$  gives for all  $z \in \partial X$  and for all  $m \geq m_z$

$$\gamma(B_z(2^{-m}) \setminus X) \cong 2\psi(2^{-m+1})2^{-m}.$$

If we use that  $\psi$  is increasing, we get

$$\gamma(B_z(\delta) \setminus X) \cong \psi(\delta)\delta$$

for all  $z \in \partial X$  and for all  $\delta$ ,  $0 < \delta < \delta_z$ .

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