RATIONAL APPROXIMATION AND THE GROWTH OF ANALYTIC CAPACITY

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Let \( X \) be a compact set in the complex plane \( \mathbb{C} \). Denote by \( R(X) \) the closure in the supremum norm of the rational functions with poles off \( X \) and by \( A(X) \) the set of continuous functions, which are analytic on the interior of \( X \). The analytic capacity of a set \( S \) is denoted by \( \gamma(S) \). For the definition of \( \gamma \) see below. Let \( B_z(\delta) = \{ \zeta \in \mathbb{C}; |z - \zeta| < \delta \} \) and let \( \partial X \) denote the boundary of \( X \). Vitushkin has proved that \( R(X) = A(X) \) if

\[
\lim_{\delta \to 0} \frac{\gamma(B_z(\delta) \setminus X)}{\delta} > 0 \text{ for all } z \in \partial X.
\]

Let \( \psi \) be a function from \( \mathbb{R}^+ \) to \( \mathbb{R}^+ \). We now ask the following questions. If \( \lim_{\delta \to 0} \psi(\delta) = 0 \), is it possible to find a compact set \( X \) such that \( R(X) \neq A(X) \) and such that \( \frac{\gamma(B_z(\delta) \setminus X)}{\delta} \leq \psi(\delta) \) for all \( z \in \partial X \) and for all \( \delta, 0 < \delta < \delta_z \)? If the answer is yes, can the answer still be yes, if \( \lim_{\delta \to 0} \psi(\delta) = 0 \) is replaced by \( \lim_{\delta \to 0} \psi(\delta) > 0 \)? The answers of these questions can be found in Theorem 1 and Theorem 2.

**Definition.** Let \( K \) be a compact subset of \( \mathbb{C} \). Then \( \gamma(K) = \sup |f'(\infty)| \), where the supremum is taken over all functions \( f \) such that \( f \) is analytic on the unbounded component of \( \mathbb{C} \setminus K \), \( |f(z)| \leq 1 \) for all \( z \in \mathbb{C} \) and \( f(\infty) = 0 \). Let \( S \) be an arbitrary subset of \( \mathbb{C} \). Then \( \gamma(S) = \sup \gamma(K) \), where the supremum is taken over all compact subsets of \( S \).

For further information about this capacity see for instance [2], [3], [4] and [5].

**Theorem 1.** Let \( \delta_n \searrow 0 \) when \( n \to \infty \). Suppose that

\[
\lim_{n \to \infty} \frac{\gamma(B_z(\delta_n) \setminus X)}{\delta_n} > 0 \text{ for all } z \in \partial X.
\]

Then \( R(X) = A(X) \).

**Theorem 2.** Let \( \psi \) be a function from \( \mathbb{R}^+ \) to \( \mathbb{R}^+ \). Suppose that \( \lim_{\delta \to 0} \psi(\delta) = 0 \). Then there exists a compact set \( X \) such that
(a) \( R(X) \neq A(X) \)

and

(b) \( \gamma(B_z(\delta) \setminus X) \geq \psi(\delta) \delta \) for all \( z \in \partial X \) and for all \( \delta, 0 < \delta < \delta_z \).

**Remark.** Theorem 1 gives the following. Let \( \psi \) be a function from \( \mathbb{R}^+ \) to \( \mathbb{R}^+ \). Suppose that \( \lim_{\delta \to 0} \psi(\delta) > 0 \) and suppose that \( \gamma(B_z(\delta) \setminus X) \geq \psi(\delta) \delta \) for all \( z \in \partial X \) and for all \( \delta, 0 < \delta < \delta_z \). Then \( R(X) = A(X) \).

2. **The proofs.** Theorem 1 can be proved in the same way as the theorem of Vitushkin mentioned in the introduction. See [4], Ch. 2, §4. We omit the proof.

In [1] A. M. Davie constructed a compact set \( X \) such that every point of \( \partial X \) is a peak point for \( R(X) \), but \( R(X) \neq A(X) \). Our proof of Theorem 2 is a refinement of Davie’s construction. We start by formulating two lemmas. The first lemma is well-known (see for instance [2], p. 199). The second lemma is due to Carleson. For a proof see [1].

**Lemma 1.** Let \( L \) be a compact set on a line. Then

\[ \gamma(L) \geq \frac{1}{4} \{ \text{the length of } L \} \]

**Lemma 2.** Let \( E \) be a a perfect subset of the real line and \( I \) the closed interval \([0,1]\). Then we can find a continuous function on \( \mathbb{C} \), analytic outside \( I \times E \), such that \( f(\infty) = 0 \), \( f'(\infty) = \frac{1}{4} \) and \( |f(z)| \leq 1 \) for all \( z \in \mathbb{C} \).

If \( x \in \mathbb{R} \), let \([x]\) denote the greatest integer less than or equal to \( x \).

**Proof of Theorem 2.** We may assume that \( \psi(\delta) \) is a strictly increasing function. Put \( a_n = 16\psi(2^{-n+1}) \), \( n = 1, 2, 3, \ldots \). Then \( a_n \searrow 0 \) when \( n \to \infty \).

Let \( f \) be an increasing function such that \( f(-2 - \log a_n) = n \). Put

\[ b_0 = 1 \]

and

\[ b_n = \min \left( e^{-f(n)}, \frac{1}{4}b_{n-1} \right) \quad \text{for} \quad n \geq 1. \]

Let \( E \) be the usual Cantor set on the real axis such that the set \( E_n \) obtained in \( n \)th step consists of \( 2^n \) intervals of length \( b_n \). Let \( I = [0,1] \).

Let \( n \) be fixed for a moment. There exists an integer \( k_n \) such that

(1) \[ b_n \geq 2^{-k_n}. \]
Denote the intervals in $E_n$ by $I_{n,i}$, $i = 1, 2, \ldots, 2^n$. In every $I \times I_{n,i}$ choose open disjoint discs with radius $2^{-k_n-3}e^{-n-1}$ in the following way. Every disc must not intersect $I \times E_{n+1}$ but every disc must touch $I \times E_{n+1}$. Moreover, the discs are arranged such that the centres of the discs lie on two horizontal lines in every $I_{n,i}$. There are $2^{k_n+3}$ centres on each line and the distance between two successive centres is $2^{-k_n-3}$. Call the chosen discs $U_{n,i}$.

Repeat the construction for all $n$, $n = 1, 2, 3, \cdots$. Put

$$X = \overline{B_0(2)} \backslash \left( \bigcup_{n,j} U_{n,j} \right),$$

where $\overline{B_0(2)}$ denotes the closure of $B_0(2)$. $X$ is a compact set and

$$\partial X = \partial B_0(2) \cup \left( \bigcup_{n,j} \partial U_{n,j} \right) \cup (I \times E).$$

It is easy to see that $\Sigma_{n,j} \text{diam } U_{n,j} < \infty$. Lemma 2 and a standard argument give

$$R(X) \neq A(X).$$

See [2], p. 220.

(i) Let

$$z \in \partial B_0(2) \cup \left( \bigcup_{n,j} \partial U_{n,j} \right).$$

Lemma 1 gives for all $m \geq m_z$

$$\gamma \left( B_z(2^{-m}) \backslash X \right) \geq \frac{1}{4} 2^{-m} \geq \frac{1}{4} a_m 2^{-m}.$$

(ii) Let $z \in I \times E$. Let $m$ be a positive integer such that $a_m < e^{-2}$. The definition of $f$ gives $f(-2 - \log a_m) = m$. Fix $n$ such that $n = \lfloor -\log a_m \rfloor - 1$. If we use that $f$ is an increasing function and the definition of $b_n$, we obtain

$$2^{-m} = e^{-f(-2 - \log a_m)} \geq e^{-f(1 + \lfloor -\log a_m \rfloor)} = e^{-f(n)} \geq b_n.$$

Thus

$$2^{-m} \geq b_n.$$

One now easily shows that $B_z(2^{-m})$ contains disjoint discs $U_{n,j}$, $i = 1, 2, \cdots, 2^{k_n+2}2^{-m} - 2$, such that their centres are on one straight line. Lemma 1, (1) and (2) give
\( \gamma(B_z(2^{-m}) \setminus X) \geq \gamma \left( \bigcup_{i} U_{n,i} \right) \geq \frac{1}{4} \{2^{k_n+2^{-m}} - 2\} 2^{-k_n} e^{-n-1} \)
\[ = \frac{1}{4} e^{-n-1} \{2^{-m} - 2^{-k_n-1}\} \geq \frac{1}{4} e^{-n-1} \{2^{-m} - \frac{1}{2} b_n\} \]
\[ \geq \frac{1}{4} e^{-n-1} \{2^{-m} - \frac{1}{2} 2^{-m}\} = \frac{1}{8} 2^{-m} e^{-n-1}. \]

Thus
\[ \gamma(B_z(2^{-m}) \setminus X) \geq \frac{1}{8} 2^{-m} e^{-n-1}. \]

If we use that \( n = \lceil -\log a_m \rceil - 1 \), we obtain
\[ e^{-n-1} = e^{-\lceil -\log a_m \rceil} \geq e^{\log a_m} = a_m. \]

Thus
\[ \gamma(B_z(2^{-m}) \setminus X) \geq \frac{1}{8} a_m 2^{-m}. \]

Now (i) and (ii) give that for all \( z \in \partial X \) there is a constant \( m_z \) such that
\[ \gamma(B_z(2^{-m}) \setminus X) \geq \frac{1}{8} a_m 2^{-m} \text{ for all } m \geq m_z. \]

The definition of \( a_m \) gives for all \( z \in \partial X \) and for all \( m \geq m_z \)
\[ \gamma(B_z(2^{-m}) \setminus X) \geq 2\psi(2^{-m+1}) 2^{-m}. \]

If we use that \( \psi \) is increasing, we get
\[ \gamma(B_z(\delta) \setminus X) \geq \psi(\delta) \delta \]
for all \( z \in \partial X \) and for all \( \delta, 0 < \delta < \delta_z. \)

**References**


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