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THE LENGTH OF THE PERIOD OF THE SIMPLE CONTINUED FRACTION OF $d^{1/2}$

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Let p(d) denote the length of the period of the simple continued fraction for $d^{1/2}$ and ε the fundamental unit in the ring $Z[d^{1/2}]$. We prove that as $d \to \infty$,

 THEOREM 1.
 $p(d) \leq 7/2\pi^{-2}d^{1/2}\log d + O(d^{1/2}).$

 THEOREM 2.
 $\log \varepsilon \leq 3\pi^{-2}d^{1/2}\log d + O(d^{1/2}).$

 THEOREM 3.
 $p(d) \neq o(d^{1/2}/\log \log d).$

 THEOREM 4.
 If $\log \varepsilon \neq o(d^{1/2}\log d)$ then also $p(d) \neq o(d^{1/2}\log d).$

Recently Hickerson [1] has proved that $p(d) = O(d^{1/2+\delta})$ for every $\delta > 0$, and in fact a result somewhat more precise than this. Lehmer [2] has suggested that for arbitrarily large d, p(d) might be as large as $0.30d^{1/2}\log d$, and if this is indeed the case then Theorem 1 is almost the best possible result. In fact it is easy to show that $p(d) = O(d^{1/2}\log d)$ using known results regarding $\log \varepsilon$, but the constant in Theorem 1 improves the best obtainable in this way.

Let ε_0 denote the fundamental unit in the field $Q(d^{1/2})$, $[a_0, \overline{a_1, a_2}, \overline{\cdots a_{p(d)-1}, 2a_0}]$ the continued fraction for $d^{1/2}$ and P_r/Q_r , its rth convergent. Then as is well known $\varepsilon = \varepsilon_0$ or ε_0^3 . Thus by the result of Stephens [3],

$$\log arepsilon \leq 3\log arepsilon_{\scriptscriptstyle 0} \leq rac{3}{2}(1-e^{_{-1/2}}+\delta)d^{_{1/2}}\log d$$
 .

Now $Q_0 = 1$, $Q_1 = a_1 \ge 1$ and $Q_{r+2} = a_{r+2}Q_{r+1} + Q_r \ge Q_{r+1} + Q_r$ and so by induction $Q_r \ge u_{r+1}$, the Fibonacci number, for $r \ge 0$. Now

$$egin{aligned} arepsilon &= {P}_{p(d)-1} + {Q}_{p(d)-1} d^{1/2} \ &> 2 d^{1/2} {Q}_{p(d)-1} - 1 \ &\geq 2 d^{1/2} {u}_{p(d)} - 1 \ &> \left\{ rac{1 + \sqrt{5}}{2}
ight\}^{p(d)} , \end{aligned}$$

and so $p(d) < Ad^{1/2} \log d$ where A is approximately 5/4.

In exactly the same way, using $a_r < d^{1/2}$ for $0 \leq r < p(d)$ it is possible to show that $p(d) \gg \log \varepsilon / \log d$. Since $d = 2^{2k+1}$ gives $\varepsilon = (1 + \sqrt{2})^{2^k}$, we find that for arbitrarily large d it is possible for $p(d) \gg d^{1/2} / \log d$, and it will be shown that this can be improved at least by replacing the $\log d$ by $\log \log d$. Theorems 1 and 3 together show that the scope for sharpening the results is somewhat limited; nevertheless the remaining problem is important and worthy of further study, for as we mention in the concluding remarks, if it could be proved that $p(d) = o(d^{1/2} \log d)$ this would imply also that $\log \varepsilon = o(d^{1/2} \log d)$ a result which has been sought in vain for many years.

Throughout we use ε_1 to denote the fundamental unit in $Z[d^{1/2}]$ with norm + 1; then $\varepsilon_1 = \varepsilon$ or ε^2 . In accordance with established practice, if for given integers d and N there exist integers X and Y with $X^2 - dY^2 = N$, then we say that $X + Yd^{1/2}$ is a solution of the equation $x^2 - dy^2 = N$. Given one such solution, all the members of the set $\pm (X + Yd^{1/2})\varepsilon_1^n$ are also solutions, and this set is called a *class* of solutions. A given equation may well have more than one such class of solutions, but it is well known that the number of such classes is finite.

LEMMA 1. For each r, $|P_r^2 - dQ_r^2| < 2d^{1/2}$.

This is well known.

LEMMA 2. For a class K of solutions of $x^2 - dy^2 = N$, the g.c.d., (x, y) depends only upon K.

For if $x_1 + y_1 d^{1/2}$ and $x_2 + y_2 d^{1/2}$ belong to the same class, then for some integer n,

$$egin{aligned} &x_1+y_1d^{1/2}=\pm(x_2+y_2d^{1/2})arepsilon_1^n\ &=\pm(x_2+y_2d^{1/2})(a_n+b_nd^{1/2})\ , \end{aligned}$$

say. Thus $(x_2, y_2)|(x_1, y_1)$ and similarly conversely.

A class K for which (x, y) = 1 is called a *primitive* class. The main result used in the proof of the theorems is

LEMMA 3. The number of primitive classes, f(N; d), of $x^2 - dy^2 = N$ does not exceed $2^{\omega(|N|)}$. In the special case 2 || N, $f(N; d) \leq 2^{\omega(|N|)-1}$. Here $\omega(N)$ denotes the number of distinct prime factors of N.

Proof. In the first place it suffices to consider the case in which (N, d) is square-free. For if $(N, d) = k_1^2 k_2$ where k_2 is square-free, (x, y) = 1 and $x^2 - dy^2 = N$ then $k_1 | x$ and so if $x_1 = x/k_1$, $N_1 = N/k_1^2$ and $d_1 = d/k_1^2$ then $x_1^2 - d_1y^2 = N_1$ with $(x_1, y) = 1$. For the latter equation we now have $(N_1, d_1) = k_2$ which is square-free and so the total number of classes of primitive solutions of the given equation

does not exceed $2^{\omega(|N|)} \leq 2^{\omega(|N|)}$ in the general case, or $2^{\omega(|N|)^{-1}} \leq 2^{\omega(|N|)^{-1}}$ in the special case 2||N| since in this case $2||N_1|$ also. We suppose therefore from now on that (N, d) is square-free.

Let p denote any prime dividing N, and suppose that $p^* || N$;

(i) if $p \mid d$ then $p \mid x$, whence $p^2 \nmid dy^2$ otherwise we should find, since $p \nmid y$ that $p^2 \mid d$ and $p^2 \mid N$. Hence s = 1 and so $xy^{-1} \equiv 0 \pmod{p^s}$.

(ii) if $p \nmid d$ then p can divide neither x nor y, otherwise it would have to divide them both. Thus $(xy^{-1})^2 \equiv d \pmod{p^s}$ and so if p is odd, $xy^{-1} \equiv \pm a_p \pmod{p^s}$.

(iii) if $p \nmid d$, p = 2 then $(xy^{-1})^2 \equiv d \pmod{p^s}$ gives

(a) if $s = 1, xy^{-1} \equiv d \pmod{2}$, i.e., $xy^{-1} \equiv d \pmod{p^s}$

(b) if s = 2, since $x^2 - dy^2 \equiv 0 \pmod{4}$ and both x and y are odd, $d \equiv 1 \pmod{4}$ whence $(xy^{-1})^2 \equiv 1 \pmod{4}$, i.e., $xy^{-1} \equiv \pm 1 \pmod{4}$, i.e., $xy^{-1} \equiv \pm 1 \pmod{p^s}$

(c) if $s \ge 3$, then $d \equiv 1 \pmod{8}$ and now $(xy^{-1})^2 \equiv d \pmod{2^s}$ gives $xy^{-1} \equiv \pm a \pmod{2^{s-1}}$.

Combining (i), (ii), and (iii) and using the Chinese Remainder Theorem, we see that xy^{-1} is congruent to one of at most

 $egin{array}{lll} 2^{\omega(N)-1} & ext{residues modulo } N ext{ if } 2 || N \ 2^{\omega(|N|)} & ext{residues modulo } N ext{ unless } 8 | N \ 2^{\omega(|N|)} & ext{residues modulo } rac{1}{2} N ext{ if } 8 | N \ . \end{array}$

Next we prove that if $x^2 - dy^2 = X^2 - dY^2 = N$ and if $xy^{-1} \equiv XY^{-1} \pmod{N}$ then $x + yd^{1/2}$ and $X + Yd^{1/2}$ belong to the same class K. For

$$rac{x+yd^{1/2}}{X+Yd^{1/2}} = rac{(x+yd^{1/2})(X-Yd^{1/2})}{X^2-d\,Y^2} = rac{xX-dy\,Y}{N} + rac{-xY+Xy}{N}d^{1/2} = A+Bd^{1/2}$$
, say.

Now B is an integer and A rational, and since $A^2 - dB^2 = 1$ it follows that A too is an integer, and so that result of the lemma follows, except if 8|N.

Finally, if 8 | N then we find that if $xy^{-1} \equiv XY^{-1} \pmod{1/2N}$ then $x + yd^{1/2}$ and $X + Yd^{1/2}$ belong to the same class; for if as above $A + Bd^{1/2}$ denote their quotient, we find that B equals either an integer or else half an odd integer. In the former case the result follows as above. In the latter case we find $(2A)^2 = d(2B)^2 + 4$ and since now 2B is an odd integer and $4 \nmid d$, 2A is also an odd integer, whence $d \equiv 5 \pmod{8}$. But this is inconsistent with $x^2 - dy^2 \equiv 0 \pmod{8}$ where (x, y) = 1 and so this latter case does not arise. This concludes the proof.

LEMMA 4. If $N(\varepsilon) = 1$, then

$$f(N, d) + f(-N, d) \leq 2^{\omega(|N|)}$$

and

$$f(N, d) + f(-N, d) \leq 2^{\omega(|N|)-1} \quad if \quad 2||N|.$$

Proof. After Lemma 3, it merely remains to prove that $x^2 - dy^2 = N$ and $X^2 - dY^2 = -N$ with $xy^{-1} \equiv XY^{-1} \pmod{N}$, or even $(\mod 1/2N)$ if $8 \mid N$, is impossible. For we should obtain if

$$A + Bd^{_{1/2}} = (x + yd^{_{1/2}})(X + Yd^{_{1/2}})^{_{-1}}$$

that $A^2 - dB^2 = -1$ with either A and B both integers, or else both half integers. Both cases are impossible if $N(\varepsilon) = +1$.

LEMMA 5. (1) If
$$N(\varepsilon) = 1$$
 then

$$p(d) \leq \sum_{0 < N < 2d^{1/2}} \{f(N; d) + f(-N; d)\}.$$
(2) If $N(\varepsilon) = -1$ then

$$p(d) \leq \sum_{0 < N < 2d^{1/2}} f(N; d).$$

Proof. If $0 \le m < n \le p(d) - 1$ then $P_m + Q_m d^{1/2}$ and $P_n + Q_n d^{1/2}$ are primitive solutions in distint classes; they are primitive since $(P_r, Q_r) = 1$ and are in distinct classes since

$$1 < {P}_{{m}} + {Q}_{{m}} d^{\scriptscriptstyle 1/2} < {P}_{{n}} + {Q}_{{n}} d^{\scriptscriptstyle 1/2} \leqq arepsilon_{{}_{1}}$$
 .

Hence using Lemma 1,

$$p(d) \leq ext{the number of distinct primitive classes of all} \ ext{equations } x^2 - dy^2 = N ext{ with } -2d^{1/2} < N < 2d^{1/2} \ = \sum_{-2d^{1/2} < N < 2d^{1/2}} f(N; d), ext{ which gives (1) }.$$

If $N(\varepsilon) = -1$ then the above reasoning applies if $0 \le m < n \le 2p(d) - 1$ and so (2) follows, since if $N(\varepsilon) = -1$, f(N; d) = f(-N; d).

We remark that this result is best possible for example for the values d = 7, 13 respectively.

LEMMA 6. As $x \to \infty$

(1)
$$F(x) = \sum_{1 \leq N \leq x} 2^{\omega(N)} = cx \log x + O(x)$$
,

(2)
$$A(x) = \sum_{\substack{1 \le N \le x \\ 2 \mid N}} 2^{\omega(N)} = \frac{2}{3} cx \log x + O(x) ,$$

(3)
$$B(x) = \sum_{\substack{1 \le N \le x \\ 2 \ne N}} 2^{w(N)} = \frac{1}{3} cx \log x + O(x) ,$$

(4)
$$C(x) = \sum_{\substack{1 \le N \le x \\ 4 \mid N}} 2^{w(N)} = \frac{1}{3} cx \log x + O(x) ,$$

(5)
$$D(x) = \sum_{\substack{1 \le N \le x \\ 8 \mid N}} 2^{\omega(N)} = \frac{1}{6} cx \log x + O(x) ,$$

(6)
$$E(x) = \sum_{\substack{1 \le N \le x \\ 16|N}} 2^{\omega(N)} = \frac{1}{12} cx \log x + O(x)$$
, where $c = 6\pi^{-2}$.

Proof. (1) The identity

$$2^{{}^{\omega(N)}}={}_{{}^{k^{2}\mid N}}d\Bigl(rac{N}{k^{2}}\Bigr)\mu(k)$$

is easily proved by induction on the number of distinct prime factors of N. For if N is a prime or a prime power the result is immediate, and then the identity follows on observing that 2^{ω} , d and μ are all multiplicative. Thus

$$\begin{split} F(x) &= \sum_{1 \le N \le x} \sum_{k^2 \mid N} d\left(\frac{N}{k^2}\right) \mu(k) \\ &= \sum_{1 \le k \le x^{1/2}} \sum_{1 \le k_1 \le x^{k-2}} d(k_1) \mu(k) \\ &= \sum_{1 \le k \le x^{1/2}} \mu(k) \sum_{1 \le k_1 \le x^{k-2}} d(k_1) \\ &= \sum_{1 \le k \le x^{1/2}} \mu(k) \left\{\frac{x}{k^2} \log \frac{x}{k^2} + O\left(\frac{x}{k^2}\right)\right\} \\ &= \sum_{1 \le k \le x^{1/2}} \frac{x \mu(k) \log x}{k^2} + O(x) \\ &= \frac{x \log x}{\zeta(2)} + O(x) \\ &= cx \log x + O(x) \;. \end{split}$$

(2) We have

$$\begin{split} A(2x) &= \sum_{\substack{1 < N \leq 2x \\ 2|N}} 2^{\omega(N)} \\ &= \sum_{\substack{1 \leq 1/2N < x \\ 2 \leq 1/2N \leq x}} 2^{\omega(2 \cdot 1/2N)} \\ &= \sum_{\substack{1 < 1/2N \leq x \\ 2|1/2N}} 2^{\omega(2 \cdot 1/2N)} + \sum_{\substack{1 \leq 1/2N \leq x \\ 2|X/2N}} 2^{\omega(2 \cdot 1/2N)} \end{split}$$

$$= \sum_{\substack{1 < 1/2N \leq x \\ 2|1/2N}} 2^{\omega(1/2N)} + \sum_{\substack{1 \leq 1/2N \leq x \\ 2|1/2N}} 2^{1+\omega(1/2N)}$$
$$= A(x) + 2B(x) .$$

Thus A(2x) + A(x) = 2A(x) + 2B(x) = 2F(x). We now prove by induction that

$$A(x) = 2 \sum_{r=1}^{\infty} (-1)^{r-1} F(x \cdot 2^{-r}) .$$

For, if x = 1, the result is clearly true since both sides vanish, and then if true for $x \leq x_0$, we have for $x \leq 2x_0$,

$$A(x) = 2F\left(\frac{1}{2}x\right) - A\left(\frac{1}{2}x\right)$$

which is again of the required form, and this completes the induction. Now F(y) = 0 if y < 1 and so we have

$$A(x) = 2\sum_{r=1}^{k} (-1)^{r-1} F(x \cdot 2^{-r})$$
,

where

$$k = \left[rac{\log x}{\log 2}
ight].$$

Now by (1)

$$|F(y) - cy \log y| < Cy$$
 ,

for some constant C and all y > 1. Thus

$$\left|A(x) - 2c\sum_{r=1}^{k}(-1)^{r-1}rac{x}{2^{r}}\cdot\lograc{x}{2^{r}}
ight| < 2C\sum_{r=1}^{k}rac{x}{2^{r}} < 2Cx\;.$$

Hence

$$ig|A(x) - 2c\sum\limits_{r=1}^k (-1)^{r-1}rac{x}{2^r}\log x ig| < 2Cx + 2cx\log 2{ig}\sum\limits_{r=1}^k r{ig\cdot}2^{-r} < C_1x$$
 .

Finally,

$$\begin{split} \sum_{r=1}^{k} (-1)^{r-1} \frac{x}{2^{r}} \log x &= \frac{1}{2} x \log x \cdot \frac{1 - \left(-\frac{1}{2}\right)^{k}}{1 - \left(-\frac{1}{2}\right)} \\ &= \frac{1}{3} x \log x \{1 + O(x^{-1})\} \\ &= \frac{1}{3} x \log x + O(\log x) \;, \end{split}$$

and so (2) follows.

- (3) now follows since B(x) = F(x) A(x).
- (4) follows since

$$C(x) = \sum_{\substack{1 < 1/2N \le 1/2x \\ 21/12N}} 2^{w(2 \cdot 1/2N)} = A\left(\frac{1}{2}x\right).$$

(5) and (6) now follow similarly since D(x) = C(1/2x) and E(x) = D(1/2x).

Proof of Theorem 1. The idea of the proof is to combine the results of Lemmas 3-6. We have immediately that

$$p(d) \leq \sum_{1 \leq N \leq 2d^{1/2}} 2^{\omega^{(N)}} = cd^{1/2} \log d + O(d^{1/2})$$

and the remainder of the proof deals with reducing the constant in the above. There are two ways of doing this; in the first place if 2||N, then the upper bound $2^{\omega(N)}$ appearing above can immediately be halved in view of Lemmas 3 and 4; secondly depending upon the value of d, there are certain residue classes modulo 16 such that for any N belonging to one of them, the equation $x^2 - dy^2 = N$ cannot have any primitive solutions at all. In each case, it is not possible to dispose of all the odd values of N in this way, and corresponding to these we always obtain a term

$$\sum_{\substack{1 \leq N \leq 2d^{1-2} \\ 2
ext{/}N}} 2^{\omega(N)} = B(2d^{1/2})$$
 .

There are various cases to consider.

(a) $d \equiv 1 \pmod{8}$. In this case, since x and y cannot both be even, we find that $x^2 - dy^2 = N$ is either odd or divisible by 8. Thus we find that $p(d) \leq B(2d^{1/2}) + D(2d^{1/2}) = 1/2cd^{1/2} \log d + O(d^{1/2})$, as required.

(b) $d \equiv 5 \pmod{8}$. In this case, we find that if N is even, then $2^2 || N$, and accordingly

$$p(d) \leq B(2d^{1/2}) + C(2d^{1/2}) - D(2d^{1/2}) = rac{1}{2}cd^{1/2}\log d + O(d^{1/2})$$
 .

(c) If $d \equiv 2$ or $3 \pmod{4}$ then N can be even only if 2 || N and we obtain

$$egin{aligned} p(d) &\leq B(2d^{1/2}) + \sum \limits_{\substack{1 < N \leq 2d^{1/2} \ 2 \mid \mid N}} 2^{\omega(N)-1} \ &= B(2d^{1/2}) + rac{1}{2} \{A(2d^{1/2}) - C(2d^{1/2})\} \ &= rac{1}{2} c d^{1/2} \log d \, + O(d^{1/2}) \;. \end{aligned}$$

It is to be noted for future reference that if $4 \nmid d$, then the 7c/12 of the theorem can be improved to 1/2c.

(d) If $d \equiv 0 \pmod{4}$, then for a primitive solution of $x^2 - dy^2 = N$ we must have either that x is odd, in which case N is also odd, or else x is even, y odd and 4|N. In the latter case we find that $(1/2x)^2 - (1/4d)y^2 = 1/4N$ and so we obtain a primitive solution of the equation $X^2 - (1/4d)Y^2 = 1/4N$, in which moreover y is odd. Thus we have

either $1/4d \equiv 0$ or $1 \pmod{4}$ in which case 1/4N is odd or divisible by 4,

or $1/4d \equiv 2$ or $3 \pmod{4}$ in which case 1/4N is odd or 2 || 1/4N.

In the first case we obtain

$$egin{aligned} p(d) &\leq B(2d^{1/2}) + C(2d^{1/2}) - D(2d^{1/2}) + E(2d^{1/2}) \ &= rac{7}{12} c d^{1/2} \log d \, + \, O(d^{1/2}) \; , \end{aligned}$$

and in the second case we obtain similarly

$$egin{aligned} p(d) &\leq B(2d^{1/2}) + C(2d^{1/2}) - E(2d^{1/2}) \ &= rac{7}{12} c d^{1/2} \log d + O(d^{1/2}) \;, \end{aligned}$$

which concludes the proof.

LEMMA 7. As $x \to \infty$,

$$F_1(x) = \sum_{1 \leq N \leq x} 2^{\omega(N)} \log \frac{x}{N} = cx \log x + O(x) .$$

Proof. Let $1 < \rho < x$; then

$$\begin{split} F_1(x) - F_1(x\rho^{-1}) &= \sum_{1 \le N \le x} 2^{\omega(N)} \log \frac{x}{N} - \sum_{1 \le N \le x\rho^{-1}} 2^{\omega(N)} \log \frac{x}{\rho N} \\ &= \sum_{1 \le N \le x\rho^{-1}} 2^{\omega(N)} \log \rho + \sum_{x\rho^{-1} < N \le x} 2^{\omega(N)} \log \frac{x}{N} \end{split}$$

and so

$$\log
ho \cdot F(x
ho^{-1}) \leq F_1(x) - F_1(x
ho^{-1}) \leq \log
ho \cdot F(x)$$
,

since $x/N < \rho$ for $N > x\rho^{-1}$.

Thus if $1 <
ho^n \leq x <
ho^{n+1}$, we find that

$$\log \rho \cdot \sum_{r=1}^{n} F(x \rho^{-r}) \leq F_1(x) - F_1(x \rho^{-n}) \leq \log \rho \cdot \sum_{r=0}^{n-1} F(x \rho^{-r}),$$

and so to complete the proof it suffices to show that

$$\log \rho \cdot \sum_{0}^{n-1} F(x \rho^{-r}) \longrightarrow cx \log x + O(x) \text{ as } \rho \longrightarrow 1+$$
,

where $n = [(\log x / \log \rho)]$.

Now for all y > 1, we have for some constant A,

$$cy \log y - Ay < F(y) < cy \log y + Ay$$
.

Thus

$$egin{aligned} \log
ho \sum\limits_{0}^{n-1} F(x
ho^{-r}) &< \log
ho \sum\limits_{0}^{n-1} (cx \log x + Ax)
ho^{-1} \ &<
ho rac{\log
ho}{
ho - 1} (cx \log x + Ax) \longrightarrow cx \log x + Ax \,, \ & ext{ as }
ho \longrightarrow 1 + \ . \end{aligned}$$

On the other hand

Now

$$X = \frac{\rho(cx\log x - Ax)\log\rho}{\rho - 1} \left\{ 1 - \frac{1}{\rho^n} \right\} \longrightarrow (cx\log x - Ax)(1 - x^{-1})$$

as $\rho \rightarrow 1$, since x lies between ρ^n and ρ^{n+1} . Also

$$Y < cx(\log \rho)^2 \sum_{0}^{\infty} r \rho^{-r} = \rho^2 cx \left\{ \frac{\log \rho}{\rho - 1} \right\}^2 \longrightarrow cx \quad \text{as} \quad \rho \longrightarrow 1 +$$

and so the result follows.

LEMMA 8. Let

$$A_{1}(x) = \sum_{\substack{1 < N \leq x \\ 2 \mid N}} 2^{\omega(N)} \log rac{x}{N}$$

with analogous definitions for B_1 , C_1 , and D_1 . Then the results of Lemma 6, (2)-(5) hold also for the functions A_1 etc.

Proof. These results follow from Lemma 7 in exactly the same way as the corresponding results follow from Lemma 6(1).

Proof of Theorem 2. We have for each convergent

$$\left| \, d^{\scriptscriptstyle 1/2} - rac{P_r}{Q_r}
ight| < rac{1}{Q_r Q_{r+1}}$$
 ,

whence

$$rac{Q_{r+1}}{Q_r}\!<\!rac{1}{Q_r|P_r\!-\!Q_r\!d^{1/2}|}=\!rac{d^{1/2}+rac{P_r}{Q_r}}{|P_r^2-dQ_r^2|}\!<\!rac{2d^{1/2}+1}{N_r}$$
 ,

where

$$|P_{r}^{2} - dQ_{r}^{2}| = N_{r}$$
 .

Consider first the case $N(\varepsilon) = -1$. Then

$$egin{aligned} arepsilon_1 &= arepsilon^2 &= P_{2p(d)-1} + Q_{2p(d)-1} d^{1/2} \ &< (2d^{1/2}+1)Q_{2p(d)-1} \ &= (2d^{1/2}+1)^{2p(d)-2} rac{Q_{r+1}}{Q_r} \ &< (2d^{1/2}+1)^{2p(d)-2} rac{2d^{1/2}+1}{N_r} \ &= \prod_{0}^{2p(d)-1} rac{2d^{1/2}+1}{N_r} \ . \end{aligned}$$

Thus

$$egin{aligned} &2\logarepsilon < \sum\limits_{0}^{2p(d)^{-1}}\lograc{2d^{1/2}+1}{N_r} \ &\leq \sum\limits_{0< N< 2d^{1/2}} \{f(N;\,d)+f(-N;\,d)\}\lograc{2d^{1/2}+1}{N} \ &= \sum\limits_{0< N< 2d^{1/2}} \{f(N;\,d)+f(-N;\,d)\}\lograc{2d^{1/2}}{N}+O\{d^{-1/2}F(2d^{1/2})\} \ &= 2\sum\limits_{0< N< 2d^{1/2}} f(N;\,d)\lograc{2d^{1/2}}{N}+O(\log d) \;, \end{aligned}$$

since in this case f(N; d) = f(-N; d). Thus

$$egin{aligned} \log arepsilon &< \sum\limits_{^{0 < N < 2d^{1/2}}} &f(N;\,d)\lograc{2d^{^{1/2}}}{N} + O(\log d) \ &< rac{1}{2}cd^{^{1/2}}\log d + O(d^{^{1/2}}) ext{ ,} \end{aligned}$$

as before, using Lemmas 7 and 8 in place of Lemma 6, since in this case $4 \nmid d$. In the case $N(\varepsilon) = +1$, we have

$$egin{aligned} arepsilon &= {P}_{p(d)-1} + {Q}_{p(d)-1} d^{1/2} \ &< (2d^{1/2}+1) Q_{p(d)-1} \ &< \prod_{0}^{p(d)-1} rac{2d^{1/2}+1}{N_r} \ , \end{aligned}$$

as before.

Thus

$$egin{aligned} \logarepsilon &< \sum\limits_{0}^{p(d)-1}\lograc{2d^{1/2}\,+\,1}{N_r} \ &\leq \sum\limits_{0< N<2d^{1/2}}\{f(N;\,d)\,+\,f(-N;\,d)\}\lograc{2d^{1/2}\,+\,1}{N} \ &= \sum\limits_{0< N<2d^{1/2}}\{f(N;\,d)\,+\,f(-N;\,d)\}\lograc{2d^{1/2}}{N}\,+\,O(\log d) \ &\leq rac{1}{2}cd^{1/2}\log d\,+\,O(d^{1/2})$$
 ,

as before, provided $4 \nmid d$.

Finally if 4|d we observe that $\varepsilon = \eta$ or η^2 where η is the fundamental unit of the ring $Z[((1/4)d)^{1/2}]$. Then the result for this case follows by descent since now $\log \varepsilon \leq 2 \log \eta$.

This concludes the proof of Theorem 2.

Proof of Theorem 3. We have as before

$$\logarepsilon < \sum_{r=0}^{p(d)-1} \log rac{2d^{1/2}+1}{N_r}$$

and so for any K satisfying $1 < K < 2d^{\scriptscriptstyle 1/2}$

$$egin{aligned} \log arepsilon &< \sum\limits_{r=0}^{p(d)-1} \log rac{2d^{1/2}}{N_r} + O(\log d) \ &= \sum\limits_{\substack{N_r \leq K \ 0 \leq r < p(d)}} \log rac{2d^{1/2}}{N_r} + \sum\limits_{\substack{0 \leq r < p(d) \ 0 \leq r < p(d)}} \log rac{2d^{1/2}}{N_r} + O(\log d) \ &< \sum\limits_{1 \leq N \leq K} \{f(N; \, d) + f(-N; \, d)\} \log 2d^{1/2} \ &+ p(d) \log rac{2d^{1/2}}{K} + O(\log d) \ &< A \log d \cdot K \log K + rac{1}{2} p(d) \log (4dK^{-2}) + O(K \log d) \end{aligned}$$

In particular taking $K = 2d^{1/2}(\log d)^{-3}$ we obtain

$$\logarepsilon < 3p(d)\log\log d + o(d^{\scriptscriptstyle 1/2})$$
 .

Now for $d = 2^{2k+1}$ we have $\varepsilon = (1 + \sqrt{2})^{2^k}$, i.e., $\log \varepsilon > Ad^{1/2}$ where A > 0 and so $p(d) \neq o(d^{1/2}/\log \log d)$, as required.

Proof of Theorem 4. If $\log \varepsilon \neq o(d^{1/2} \log d)$, then there exists a positive constant $c_1 < c$ so that for infinitely many values of d, $\log \varepsilon > c_1 d^{1/2} \log d$. Let g(N; d) denote the number of distinct primitive classes of solutions of $x^2 - dy^2 = N$ for which x/y occurs as a convergent to the continued fraction for $d^{1/2}$. Then

$$2p(d) \ge \sum_{-2d^{1/2} < N < 2d^{1/2}} g(N; d)$$

and

$$\log arepsilon < \sum_{-2d^{1/2} < N < 2d^{1/2}} g(N; d) \log rac{2d^{1/2}}{|N|} + O(\log d)$$
 .

Thus if $k \geq 1$,

$$\begin{split} \log \varepsilon - 2p(d) \log k <& \sum_{-2d^{1/2} < N < 2d^{1/2}} g(N; d) \log \frac{2d^{1/2}}{k |N|} + O(\log d) \\ & \leq \sum_{0 < |N| < 2d^{1/2}k^{-1}} g(N; d) \log \frac{2d^{1/2}}{k |N|} + O(\log d) \\ & \leq \sum_{0 < N < 2d^{1/2}k^{-1}} 2^{\omega(N)} \log \frac{2d^{1/2}k^{-1}}{N} + O(\log d) \end{split}$$

since $g(N; d) \leq f(N; d)$. Thus

$$egin{aligned} \log arepsilon &- 2p(d) \log k < F_1(2d^{1/2}k^{-1}) + O(\log d) \ &< cd^{1/2}k^{-1}\log d \,+\, O(d^{1/2}) \,. \end{aligned}$$

Thus if $k > c/c_1$, we have for infinitely many values of d,

$$p(d) > rac{kc_1-c}{2k\log k} d^{1/2}\log d + O(d^{1/2})$$
 ,

as required.

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Charalambos D. Aliprantis and Owen Sidney Burkinshaw, <i>On universally</i> <i>complete Riesz spaces</i>	1
Stephen Richard Bernfeld and Jagdish Chandra, <i>Minimal and maximal</i> solutions of nonlinear boundary value problems	13
John H. E. Cohn, <i>The length of the period of the simple continued fraction of</i> $d^{1/2}$	21
Earl Vern Dudley, <i>Sidon sets associated with a closed subset of a compact abelian group</i>	33
Larry Finkelstein, <i>Finite groups with a standard component of type J</i> ₄	41
Louise Hay, Alfred Berry Manaster and Joseph Goeffrey Rosenstein, Concerning partial recursive similarity transformations of linearly ordered sets	57
Richard Michael Kane, <i>On loop spaces without p torsion. II</i>	71
William A. Kirk and Rainald Schoneberg, <i>Some results on</i>	, 1
pseudo-contractive mappings	89
Philip A. Leonard and Kenneth S. Williams, <i>The quadratic and quartic</i>	
character of certain quadratic units. I	101
Lawrence Carlton Moore, A comparison of the relative uniform topology	
and the norm topology in a normed Riesz space	107
Mario Petrich, Maximal submonoids of the translational hull	119
Mark Bernard Ramras, <i>Constructing new R-sequences</i>	133
Dave Riffelmacher, <i>Multiplication alteration and related rigidity properties</i> of algebras	139
Jan Rosiński and Wojbor Woyczynski, Weakly orthogonally additive	
functionals, white noise integrals and linear Gaussian stochastic	
processes	159
Ryōtarō Satō, Invariant measures for ergodic semigroups of operators	173
Peter John Slater and William Yslas Vélez, <i>Permutations of the positive</i>	
integers with restrictions on the sequence of differences	193
Edith Twining Stevenson, <i>Integral representations of algebraic cohomology</i>	
classes on hypersurfaces	197
Laif Swanson, Generators of factors of Bernoulli shifts	213
Nicholas Th. Varopoulos, <i>BMO functions and the</i> $\overline{\partial}$ <i>-equation</i>	221