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In this paper, it is shown that if G is a core-free group with a standard component A of type J_4, then either A is normal in G or the normal closure of A in G is isomorphic to the direct product of two copies of J_4.

1. Introduction. Janko [17] has recently given evidence for the existence of a new finite simple group. In particular, Janko assumes that G is a finite simple group which contains an involution z such that $H = C(z)$ satisfies the following conditions:

(i) The subgroup $E = O_2(H)$ is an extra-special group of order 2^{13} and $C_H(E) \leqq E$.

(ii) H has a subgroup H_0 of index 2 such that H_0/E is isomorphic to the triple cover of M_{22}.

He then shows that G has order $2^{21}\cdot 3^3\cdot 5\cdot 7\cdot 11^3\cdot 23\cdot 39\cdot 31\cdot 37\cdot 43$ and describes the conjugacy classes and subgroup structure of G. In this paper we shall assume that J_4 is a finite simple group which satisfies Janko's assumptions and shall prove

THEOREM A. *Let G be finite group with $O(G) = 1$, A a standard component of G isomorphic to J_4 and $X = \langle A^G \rangle$. Then either $X = A$ or $X \cong A \times A$.*

Our proof follows the outline given in [6] and makes use of two key facts; namely, that J_4 has a 2-local subgroup isomorphic to the split extension of $E_{2^{11}}$ by M_{24} and that J_4 has one class of elements of order 3 with the centralizer of an element of order 3 isomorphic to the full cover of M_{22}. We also make use of the characterization of finite groups with a standard component isomorphic to M_{24} which was recently obtained by Koch [18].

2. Properties of J_4. In this section, we shall describe certain properties of J_4 and its subgroups which will be required for the proof of Theorem A. Most of these properties are found in [17] and will be listed without proof. A will denote a group isomorphic to J_4.

(2.1) A has 2 classes of elements of order 2 denoted by (2_1) and (2_2). If $t \in (2_1)$ and $E = O_2(C(t))$, then E is isomorphic to an extra special group of order 2^{13}, $C(E) = Z(E)$, $O_{2,3}(C(t))/E$ has order 3 and

$C(t)/O_{2,3}(C(t)) \cong \mathrm{Aut}(M_{22})$. Moreover, if $\langle\beta\rangle \in \mathrm{Syl}_3(O_{2,3}(C(t)))$, then $\langle\beta\rangle$ acts regularly on $E/Z(E)$. For $x \in (2_2)$, $C(x)$ is isomorphic to a split extension of $E_{2^{11}}$ by $\mathrm{Aut}(M_{22})$ with $C(x)$ acting indecomposably on $O_2(C(x))$.

(2.2) A has one class of elements of order 3. If $\gamma \in A$ has order 3, then $C(\gamma)$ is isomorphic to the 6-fold cover of M_{22}.

(2.3) A has two classes of elements of order 7. If $\delta \in A$ has order 7, then $C_A(\delta) \cong Z_7 \times S_5$ and $\delta \not\sim \delta^{-1}$.

(2.4) Let $T_0 \in \mathrm{Syl}_2(A)$. Then T_0 has precisely one $E_{2^{11}}$ subgroup, denoted by U. $N(U) = UK$ where $K \cong M_{24}$. The orbits of K on $U^\#$ are $(2_1) \cap U$ of order $7\cdot 11\cdot 23$ and $(2_2) \cap U$ of order $4\cdot 3\cdot 23$.

In the above, U is isomorphic to the so-called "Fischer" module for M_{24}. The following is an important property of the Fischer module.

(2.5) Let (*) $1 \to R \to V \to U \to 1$ be an extension of F_2M_{24} modules where R is a trivial module of dimension 1 and U is isomorphic to the Fischer module. Then the extension splits.

Proof. Let $\tilde{U}$ and $\tilde{V}$ be the F_2M_{24} modules dual to U and V respectively. Then we have the extension ($\tilde{*}$) $1 \to \tilde{U} \to \tilde{V} \to R \to 1$. It suffices to show that ($\tilde{*}$) splits. Since U is not a self dual module and since there exists precisely 2 nonisomorphic F_2M_{24} modules of dimension 11 (see James [16]), $\tilde{U}$ is isomorphic to the so-called Conway module [5]. Thus M_{24} has 2 orbits on $(\tilde{U})^\#$. If u_1 and u_2 are representatives of these 2 orbits, then $C_{M_{24}}(u_1) \cong \mathrm{Hol}(E_{16})$ and $C_{M_{24}}(u_2) \cong \mathrm{Aut}(M_{12})$.

Since $|\tilde{V}| = 2^{12}$, there exists a vector $v \in \tilde{V} - \tilde{U}$ such that v is fixed by a Sylow 23 subgroup S of M_{24}. The orbit of M_{24} on $(\tilde{V})^\#$ which contains v has order $[M_{24}: C_{M_{24}}(v)]$ and is not divisible by 23. Therefore, by examining the list of maximal subgroups of M_{24} [5], together with $[M_{24}: C_{M_{24}}(v)] \leq 2^{12}$, we see that $C_{M_{24}}(v)$ contains a subgroup L isomorphic to M_{23}. Consider the action on $\tilde{V}$ of an M_{22} subgroup M of L. Then M has no fixed points on $\tilde{U}^\#$, so in fact $C_{\tilde{V}}(M) = \langle v\rangle$. Therefore $N_{M_{24}}(M) \cong \mathrm{Aut}(M_{22})$ fixes $\langle v\rangle$ as well. Finally $\langle L, N_{M_{24}}(M)\rangle = M_{24}$ centralizes $\langle v\rangle$ and the extension splits.

We shall denote by $E_{2^{11}}\cdot M_{24}$ a split extension of $E_{2^{11}}$ by M_{24} in which $E_{2^{11}}$ is F_2M_{24} isomorphic to the Fischer module.

(2.6) Let $M = UK$ be isomorphic to $E_{2^{11}}\cdot M_{24}$ with $U = O_2(M)$

and $K \cong M_{24}$. Then the classes of elements of order 2 and 3 of M and the orders of the centralizers in M of a representative λ are as follows

Class	$\lvert C_U(\lambda)\rvert$	$\lvert C_M(\lambda)\rvert$
(2_1)	2^{11}	$2^{21}\cdot 3^3\cdot 5$
(2_2)	2^{11}	$2^{19}\cdot 3^2\cdot 5\cdot 7\cdot 11$
(2_3)	2^7	$2^{17}\cdot 3\cdot 7$
(2_4)	2^7	$2^{17}\cdot 3$
(2_5)	2^6	$2^{15}\cdot 3\cdot 5$
(2_6)	2^6	$2^{15}\cdot 3\cdot 5$
(3_1)	2^5	$2^8\cdot 3^3\cdot 5$
(3_2)	2^3	$2^6\cdot 3^2\cdot 7$

Moreover, if $\lambda_i \in (3_i) \cap K$ then $C_M(\lambda_i) = C_U(\lambda_i)C_K(\lambda_i)$ with $C_K(\lambda_1)$ isomorphic to the 3-fold cover of A_6, $C_K(\lambda_2) \cong Z_3 \times L_2(7)$ and where $C_K(\lambda_i)/\langle\lambda_i\rangle$ acts faithfully on $C_U(\lambda_i)$, $i = 1, 2$.

Proof. Let λ be an involution of $M - U$, $\mathscr{O}_1, \mathscr{O}_2, \cdots, \mathscr{O}_n$ the orbits of $C_M(\lambda U/U)$ on $\lambda C_U(\lambda)$ and α_i an element of $\mathscr{O}_i$, $i = 1, \cdots, n$. Then α_i is conjugate to α_j in M exactly when $i = j$ and also $|C_M(\alpha_i)| = |C_M(\lambda U)|/|\mathscr{O}_i|$. Now K has 2 classes of involutions with representatives λ and η having centralizers in K of order $2^{10}\cdot 3\cdot 7$ and $2^9\cdot 3\cdot 5$ respectively. Noting that the action of K on U is dual to the action of K on the Conway module, it is easy to see that $|C_U(\lambda)| = 2^7$ and $|C_U(\eta)| = 2^6$. Observe that U has 8 orbits on $\lambda C_U(\lambda)$, each of which has length 16. Moreover an element of order 7 of $C_K(\lambda)$ fixes 2 points of $C_U(\lambda)$ and therefore must permute 7 of these orbits. Since $|C_M(\lambda)| = |C_K(\lambda)|\,|C_U(\lambda)| = 2^{17}\cdot 3\cdot 7$, it then follows that $C_M(\lambda U/U)$ acting on $\lambda C_U(\lambda)$ has one orbit of length 16 and one orbit of length $7\cdot 16 = 112$ with λ an element of the orbit of length 16. This accounts for the classes (2_3) and (2_4). Similar reasoning accounts for the classes (2_5) and (2_6). We already know from (2.4) that M has orbits on $U^\#$ of lengths $4\cdot 3\cdot 23$ and $7\cdot 11\cdot 23$ and thus the classes of involutions of M are as described.

Let γ and τ be representatives of the classes of element of order 3 of K with $C_K(\gamma)$ isomorphic to the 3-fold cover of A_6 and $C_K(\tau) \cong Z_3 \times L_2(7)$. Clearly γ and τ are representatives of the 2 classes of elements of order 3 of M. It suffices to determine the orders of $C_U(\gamma)$ and $C_U(\tau)$. As before, we may appeal to the action of K on the Conway module to obtain $|C_U(\gamma)| = 2^5$ and $|C_U(\tau)| = 2^3$ as required.

NOTATION. If H is a simple group, then nH will denote a proper

n-fold covering of H. If the multiplier of H is cyclic, then nH is unique up to isomorphism. Also let $E_{32}\cdot 3A_6$ be the group isomorphic to the centralizer of an element of order 3 of the class (3_1) of $E_{2^{11}}\cdot M_{24}$. Note that $E_{32}\cdot 3A_6$ is isomorphic to a 2-local subgroup of $6M_{22}$.

(2.7) The Schur multiplier of J_4 is trivial.

Proof. See Griess [14].

(2.8) $\operatorname{Aut}(J_4) \cong J_4$.

Proof. Let $A \cong J_4$ and suppose that $\alpha \in \operatorname{Aut}(A)$. We may imbed A in $\operatorname{Aut}(A)$ and assume by way of a contradiction that $\alpha \notin A$ but $\alpha^p \in A$ for some prime p. Set $G = \langle A, \alpha\rangle$.

By (2.4), we may assume that $\alpha \in N_G(U)$ where U is an $E_{2^{11}}$ subgroup of A, $N_A(U) = UK \cong E_{2^{11}}\cdot M_{24}$ and $K \cong M_{24}$. Since $\operatorname{Aut}(K) \cong K$, we may further assume that $\overline{N_G(U)} = N_G(U)/U = \langle\bar\alpha\rangle \times \bar K$. It is known [16] that U is an absolutely irreducible F_2K module, hence by a result of Schur, we have $[\alpha, U] = 1$. Two cases now arise; namely $[\alpha, K] = 1$ and $[\alpha, K] \neq 1$.

If $[\alpha, K] \neq 1$, then it is clear that α is a 2-element. Also the fact that $\mho^1(\langle U, \alpha\rangle)$ is a proper K invariant subgroup of U forces $\mho^1(\langle U, \alpha\rangle) = 1$. Hence $\langle U, \alpha\rangle \cong E_{2^{12}}$ and K acts indecomposably on $\langle U, \alpha\rangle$. Without loss, we may assume that α is centralized by a Sylow 23 subgroup of K. By arguing as in (2.5), it then follows that $C_K(\alpha) \cong M_{23}$. Therefore in either case, we have that $C_{UK}(\alpha) \geqq UK_0$ where K_0 is an M_{23} subgroup of K.

Let γ be an element of order 3 of K_0. Then $C_{K_0}(\gamma) \cong Z_3 \times A_5$ implies that $C_U(\gamma) \cong E_{32}$ by (2.6). Also $C_A(\gamma) \cong 6M_{22}$ and $m_2(C_A(\gamma)) = 5$ [4] gives $O_2(C_A(\gamma)) \leqq C_U(\gamma)$. Setting $\overline{C_A(\gamma)} = C_A(\gamma)/Z(C_A(\gamma)) \cong M_{22}$, we conclude that α centralizes a subgroup of $\overline{C_A(\gamma)}$ isomorphic to a split extension of E_{16} by A_5. But no nontrivial automorphism of M_{22} centralizes such a subgroup [9] and therefore $[\alpha, C_A(\gamma)] \leqq Z(C_A(\gamma))$. Bv the 3-subgroup lemma, we then have $C_A(\gamma) \leqq C_A(\alpha)$. Since γ is inverted by an element of $K_0 \leqq C_A(\alpha)$, it follows that $N_A(\langle\gamma\rangle) \leqq C_A(\alpha)$ as well.

Finally, let $\langle t\rangle = O_2(C_A(\gamma))$ so that $C_A(t) = E\cdot N_A(\langle\gamma\rangle)$ by (2.1), where $E = O_2(C_A(t))$ is extra special of order 2^{13}. Observe that $C_A(\gamma)$ acts irreducibly on $E/\langle t\rangle$. Combining this with $[C_A(\gamma), \alpha] = 1$ and $C_E(\alpha) \geqq U \cap E > \langle t\rangle$, we conclude that $E \leqq C_A(\alpha)$. Therefore we are in the position where $C_A(\alpha) \geqq C_A(t)$ and $C_{UK}(\alpha) = UK_0$ or UK with $K_0 \cong M_{23}$. But $C_A(t)$ contains a Sylow 2 subgroup of $N_A(U)$ implies that $C_{UK}(\alpha) = UK$ and this gives $C_A(\alpha) \geqq \langle UK, C_A(t)\rangle$. An easy argument shows that $C_A(\alpha)$ is simple with $C_{C_A(\alpha)}(t) = C_A(t)$. Thus by Janko's theorem [17], $|C_A(\alpha)| = |A|$ which of course gives $A = C_A(\alpha)$, a contradiction.

3. **Preliminary results.** In this section we present certain technical results which are necessary for the proof of Theorem A.

(3.1) Let G be a group, A a standard component of G with $C(A)$ of 2 rank 1. Let $S \in \mathrm{Syl}_2(N(A))$. Assume that $S \notin \mathrm{Syl}_2(G)$ and $Z(S) \leqq AC(A)$. Then $[A, O(G)] = 1$.

Proof. See Seitz [19].

(3.2) Let M be a group containing an involution z such that $C(z) = O(C(z)) \times \langle z \rangle \times UK$ where $K \cong M_{24}$ and U is F_2K isomorphic to the Fischer module. Let $V = \langle z, U \rangle$ and $N = N(V)$. Then either

(i) $z \in Z(N)$ or

(ii) $N = O(N) \times WK$ where $W = \langle z \rangle Y$ is special of order 2^{23} with $Z(W) = U$ and where Y is a homocyclic abelian group of order 2^{22} invariant under K with Y/U F_2K isomorphic to U.

Proof. Assume that $z \notin Z(N)$ and let $\bar{N} = N/O(N)$. By (2.2), the orbits of K on $U^\#$ are t^K of order 1771 and x^K of order 276 with $C_K(x) \cong \mathrm{Aut}(M_{22})$. Moreover both t and x are squares in UK, hence $z^N \cap U = \varnothing$. Now the orbits of $C(z)$ on $V^\#$ are precisely

Orbit	$\{z\}$	t^K	x^K	$(zt)^K$	$(zx)^K$
Length	1	1771	276	1771	276

Since $z \notin Z(N)$ and $z^N \cap U = \varnothing$, z^N must be a union of some of the sets $\{z\}$, $(zt)^K$, $(zx)^K$. But $|z^N|$ is a divisor of $|L_{12}(2)|$ then gives $z^N = zU$.

Representing N on $z^N = zU$, we have $|N| = 2^{11}|N_H(V)|$, hence $|\bar{N}| = 2^{23}|M_{24}|$. Moreover U is generated by those involutions of V not conjugate to z so that $U \lhd N$. Assume that $C_N(U) = O(N)V$. Then $\bar{N}/\bar{V}$ acts faithfully on $\bar{U}$ and is therefore isomorphic to a subgroup of $L_{11}(2)$. Let $S \in \mathrm{Syl}_{11}(K)$ so that $N_K(S)$ is isomorphic to a Frobenius group of order $10 \cdot 11$. Since S fixes 2 points of zN, it follows that $|C_{\bar{N}}(\bar{S})| = 2|C_{\bar{N}}(\langle \bar{z}, \bar{S} \rangle)| = 2^3 \cdot 11$. Hence a Sylow 11 subgroup of $\bar{N}/\bar{V}$ has centralizer of even order which contradicts the fact that a Sylow 11 subgroup of $L_{11}(2)$ has centralizer of odd order. We conclude that $C_N(U)$ properly contains $O(N)V$.

It is easy to see from the action of K on $\overline{C_N(U)}$ that $C_N(U) = O(N)W$ where $W/U \cong E_{2^{12}}$. Furthermore, $C_W(z) = V$ implies that $Z(W) = U$ and $[z, W] = U$. Thus W is a special 2-group of order

2^{23} with $Z(W) = U$. We will in fact show that $N = O(N) \times WK$. To see this, observe that $V\langle K^N\rangle$ covers $\bar{N}$ together with $[VK, O(N)] = 1$ implies that $N = O(N)C_N(O(N))$. A simple argument establishes that $O^{2'}(C_N(O(N))) = WK$ and therefore $N = O(N) \times WK$. For the remainder of the proof, we may assume that $O(N) = 1$.

Consider the homomorphism $\varphi: W \rightarrow U$ by $\varphi(w) = [z, w]$. It is easy to see that φ induces an F_2K isomorphism between W/V and U. But then W/U is an F_2K module which satisfies, the hypotheses of (2.5) and thus $W/U = V/U \times Y/U$ where Y/U is F_2K isomorphic to U. It remains for us to show that Y is a homocyclic abelian group. Assume not. Then by the action of K on Y, $Z(Y) = U$. Let L be a subgroup of K isomorphic to $\mathrm{Aut}(M_{22})$. It follows from the properties of the Fischer module that $|C_{Y/U}(L)| = |C_U(L)| = 2$ with $C_{Y/U}(L)$ and $C_U(L)$ the unique proper L invariant submodules of Y/U and U respectively. Let $\langle yU\rangle = C_{Y/U}(L)$ so that L normalizes $\langle y, U\rangle$. Since $y \notin Z(Y)$, $1 \neq [y, Y] < U$ and since L normalizes $[\langle y, U\rangle, Y] = [y, Y]$ we must have $[y, Y] = C_U(L)$. This in turn implies that $[Y: C_Y(y)] = 2$. But L normalizes $C_Y(\langle y, U\rangle) = C_Y(y)$, hence $C_Y(y)/U$ as well and this gives a contradiction.

(3.3) Let $Y \cong E_{2^{22}}$ and M a subgroup of $\mathrm{Aut}(Y)$ such that $M = M_1 \times M_2$ with $M_1 \cong M_2 \cong M_{24}$. Then $Y = Y_1 \oplus Y_2$ where $[Y_i, M_i] = Y_i$ and $[Y_i, M_j] = 0$, $i \neq j$.

Proof. Let γ be an element of order 23 of $\mathrm{Aut}(Y)$. If γ acts regularly on Y, then $C_{\mathrm{Aut}(Y)}(\gamma)$ is isomorphic to $GL_2(2^{11})$ or is cyclic. Otherwise $\dim(C_Y(\gamma)) = 11$ and $C_{\mathrm{Aut}(Y)}(\gamma) \cong Z_{1023} \times L_{11}(2)$. Let $\gamma_i \in M_i$ be an element of order 23. Then it is clear that $\dim(C_Y(\gamma_i)) = 11$. If we set $Y_i = C_Y(\gamma_j)$, $i \neq j$, then an easy argument verifies that Y_1 and Y_2 satisfy $[Y_i, M_j] = 0$, $i \neq j$ and $[Y_i, M_i] = Y_i$, $i = 1, 2$ as required.

In the next result, we list certain properties of $2M_{22}$ which are required for (3.5).

(3.4) Let $D \cong 2M_{22}$, $T \in \mathrm{Syl}_2(D)$. Then

(i) D has 3 classes of involutions.

(ii) $Z(T)$ has order 4 and contains representatives of the classes of involutions of D.

(iii) T has precisely 2 E_{32} subgroups, say F_1 and F_2. Each is normal in T and self-centralizing in D. Also $N(F_1)/F_1 \cong A_6$ and $N(F_2)/F_2 \cong S_5$.

Proof. See Burgoyne and Fong [4].

(3.5) Let Γ be a group with an involution z such that $C(z) =$

$O(C(z))D\langle z\rangle$ with $D = E(C(z))$ and $D/O(D) \cong 2M_{22}$. Assume further that Γ has a 2-subgroup $R^* = (R_1 \times R_2)\langle z\rangle$ where $R_2 = R_1^z$ has type $2M_{22}$ and $R = R_1 \times R_2 \leqq O^2(\Gamma)$. Then $\Gamma = O(\Gamma)E(\Gamma)\langle z\rangle$ with $E(\Gamma)/O(E(\Gamma)) \cong 2M_{22} \times 2M_{22}$.

Proof. By assumption and (3.4)(iii), R has a normal subgroup $V = V_1 \times V_2$ where $V_i \lhd R_i$ and $V_i \cong E_{32}$, $i = 1, 2$. If α is an involution of R, then $m_2(C_{V_i}(\alpha)) \geqq 3$, $i = 1, 2$, gives $m_2(C_R(\alpha)) \geqq 7$. Since $m_2(C(z)) = 6$, it follows that $z^\Gamma \cap R = \varnothing$. Also all involutions of $R^* - R$ are conjugate to z which then implies that $z^\Gamma \cap R^* = z^{R^*}$. Since $C_{R^*}(z) \in \mathrm{Syl}_2(C(z))$, we see that $R^* \in \mathrm{Syl}_2(\Gamma)$. Furthermore by the Thompson transfer lemma and assumption, $z \notin O^2(\Gamma)$ and $R \in \mathrm{Syl}_2(O^2(\Gamma))$. Let $\Lambda = O^2(\Gamma)$.

We now examine the structure of $C(D)$. Observe that $C_{C(D)}(z) = O(C(z))\langle z, t\rangle$ where $\langle t\rangle = O_2(D)$. By a result of Suzuki, $C(D)$ has dihedral or semidihedral Sylow 2 subgroups. Let $Z \in \mathrm{Syl}_2(C_\Lambda(D))$ so that $\langle Z, z\rangle \in \mathrm{Syl}_2(C(D))$. Since $C_R(z) \in \mathrm{Syl}_2(D)$ and $Z(R) = C_R(C_R(z)) \in \mathrm{Syl}_2(C_\Lambda(C_R(z)))$, we may assume that $Z \leqq Z(R)$. Therefore Z is elementary abelian by (3.4)(ii) and we have either $\langle Z, z\rangle \cong D_8$ and $Z \cong E_4$, or $Z = \langle t\rangle$. Let $N = N(Z)$ and $\bar{N} = N/Z$. In either case, $\langle \bar{z}\rangle \in \mathrm{Syl}_2(C_{\bar{N}}(\bar{D}))$ and $C_{\bar{N}}(\bar{z}) \leqq N_{\bar{N}}(\bar{D})$ together imply that $\bar{D}$ is a standard component of $\bar{N}$. By Theorem A [8] and (3.1), $E(\bar{N}) = \langle \bar{D}^{\bar{N}}\rangle$, $Z(E(\bar{N}))$ has odd order and $E(\bar{N})/Z(E(\bar{N})) \cong M_{22} \times M_{22}$. Let $K = E(N)$ have components K_1 and K_2 with $K_1^z = K_2$ and $K_1/Z(K_1) \cong M_{22}$. Then $D = C_K(D)$ and $D/O(D) \cong 2M_{22}$ implies that $K/O(K) \cong 2M_{22} \times 2M_{22}$. Thus $|Z| = 4$ and $K = O^{2'}(C_\Lambda(Z))$.

Note that $R \leqq K$. Without loss, we may assume that $R_i \leqq K_i$, $i = 1, 2$. By (3.4iii), let V_i and W_i be the 2 E_{32} subgroups of R_i with $C_{K_i}(V_i) = O(K_i)V_i$, $C_{K_i}(W_i) = O(K_i)W_i$, $N_{K_i}(V_i)/C_{K_i}(V_i) \cong S_5$ and $N_{K_i}(W_i)/C_{K_i}(W_i) \cong A_6$, $i = 1, 2$. Set $W = W_1 \times W_2$, $M = N(W)$ and $\bar{M} = M/W$. Then $\overline{M \cap K} = E(\overline{M \cap K})O(\overline{M \cap K})$ with $E(\overline{M \cap K})/O(E(\overline{M \cap K})) \cong A_6 \times A_6$. Since $W_1^z = W_2$, $C_M(zW) = N(\langle z, W\rangle) = WC_M(z)$. Also $K = K_1K_2$ with $K_1^z = K_2$ implies that $C_{M \cap K}(z)$ involves A_6. Hence by (3.4iii), $C_{\bar{M}}(\bar{z}) = \langle \bar{z}\rangle \times O(C_{\bar{M}}(\bar{z}))(\overline{D \cap M})$ where $\overline{D \cap M} = E(C_{\bar{M}}(\bar{z}))$ and $\overline{D \cap M}/O(\overline{D \cap M}) \cong A_6$. It now follows that $\overline{D \cap M}$ is a standard component of $\bar{M}$ and we have from Proposition 2.3 [7] and (3.1) that $\bar{M} = O(\bar{M})E(\bar{M})\langle \bar{z}\rangle$ with $E(\bar{M})/O(E(\bar{M})) \cong A_6 \times A_6$. Furthermore $E(\overline{M \cap K}) = E(\bar{M})$ then implies that $Z = C_W(E(\bar{M}))$ and this yields $Z \lhd M$.

Our next goal is to show that $ZO(\Gamma) \lhd \Gamma$. Towards this end, observe that W, $W_1 \times V_2$, $V_1 \times W_2$ and $V_1 \times V_2$ are the only $E_{2^{10}}$ subgroups of R and that S_5 is not involved in $N_\Lambda(W)$ whereas S_5 is involved in $N_\Lambda(W_1 \times V_2)$, $N_\Lambda(V_1 \times W_2)$ and $N_\Lambda(V_1 \times V_2)$. This prevents W from fusing in Λ to $W_1 \times V_2$, $V_1 \times W_2$ or $V_1 \times V_2$ and yields $W \lhd N_A(R)$. Now $Z(R)$ contains representatives of the classes of involutions of K by (3.4i), hence of A as well. Since $Z \leqq Z(R)$, Z fails to be strongly closed in R with respect to A only when $Z^\lambda \cap Z(R) \not\leqq Z$ for some $\lambda \in A$. If in fact this happens, then we may choose $\lambda \in N_A(R)$. But $W \lhd N_A(R)$ implies that $\lambda \in N_A(W)$ and $Z \lhd N_A(W)$ then gives $Z^\lambda = Z$, a contradiction. Applying Goldschmidt's theorem [11], we conclude that $ZO(\Gamma) \lhd \Gamma$. This in turn yields $\Gamma = O(\Gamma)N$.

Since $K = E(N) = O^{2'}(N)$, it suffices to show that $[K, O(\Gamma)] = 1$. Recall that $E(C(z)) = D = C_K(z)$. Let $T = C_R(z) \in \mathrm{Syl}_2(D)$ and $Z(T) = \langle t, t_1\rangle = Z(T) \leqq Z(R)$. Then for $X = O(\Gamma)$, we have $X = C_X(z)C_X(zt_1)C_X(t_1)$. Now $C_X(z) \leqq O(C(z))$ and $[O(C(z)), D] = 1$ gives $C_X(z) \leqq C_X(t_1)$. Also $z^\lambda = zt_1$ for some $\lambda \in Z(R)$, hence $t_1 = t_1^\lambda \in D^\lambda = E(C(zt_1))$. By the same reasoning, $C_X(zt_1) \leqq C_X(t_1)$ and so $[t_1, X] = 1$. But $\langle t_1^K\rangle = K$ and therefore $[K, X] = 1$ as required.

The next result will be used in conjunction with (3.5).

(3.6) Let $\Gamma_0 = \Gamma_1 \times \Gamma_2$ with $\Gamma_1 \cong \Gamma_2 \cong 6M_{22}$ and suppose $H = H_1 \times H_2$ is a perfect subgroup of Γ_0. Then by reindexing if necessary $H_1 \leqq \Gamma_1$ and $H_2 \leqq \Gamma_2$.

Proof. Let $\tilde{\Gamma}_0 = \Gamma_0/\Gamma_1$ and observe that $\tilde{H} = \tilde{H}_1\tilde{H}_2$ where $\tilde{H}_i$ is perfect and $[\tilde{H}_1, \tilde{H}_2] = 1$. Since $\tilde{\Gamma}_0 \cong 6M_{22}$ and $6M_{22}$ contains no subgroup which is the central product of two proper perfect subgroups (see Conway [5], *p.* 235), $\tilde{H} \neq 1$ and either $H_1 \leqq \Gamma_1$ or $H_2 \leqq \Gamma_1$. Assume that $H_1 \leqq \Gamma_1$. Then by the same reasoning applied to Γ_0/Γ_2, we have $H_2 \leqq \Gamma_2$.

4. **Proof of Theorem A.** Let G be a group with $O(G) = 1$, A a standard component of G with $A/Z(A) \cong J_4$ and $X = \langle A^G\rangle$. Furthermore, let $K = C(A)$ and $R \in \mathrm{Syl}_2(K)$. It follows from (2.7) that $Z(A) = 1$ and from (2.8) that $N(A) = KA$. We shall assume that G is a minimal counterexample to Theorem A. Thus $X \neq A$ whereupon X is simple and $G \leqq \mathrm{Aut}(X)$ by Lemma 2.5 [1].

(4.1) $|R| = 2$. Consequently $G = \langle X, R\rangle$.

Proof. Let $g \in G - N(A)$ be chosen so that $Q = K^g \cap N(A)$ has a Sylow 2 subgroup T of maximal order. If $m(R) > 1$, then by ([3], (3.2) and (3.3)), R is elementary abelian and we may choose g so that $T = R^g$. On the other hand, if $m(R) = 1$ and T is trivial, then $\Omega_1(R)$ is isolated in $C(\Omega_1(R))$, hence $\Omega_1(R)$ is contained in $Z^*(G)$ by [10] contradicting $F^*(G)$ is simple. Thus in either case, we may assume that T is nontrivial.

Now $Q = N(A) = K \times A$ implies that T is isomorphic to a subgroup of A under the projection map $\pi: N(A) \to A$. An easy argument shows that Q is tightly embedded in QA. Moreover, $\pi(Q)^a = \pi(Q^a)$ for $a \in A$ then implies that $\pi(Q)$ is normalized by $\langle C_A(a): a \in \pi(T)^{\#}\rangle$. Assume first that $m(R) > 1$ so that R is elementary abelian and $T = R^g$. Let $a \in \pi(T)^{\#}$. Then $\pi(Q) \cap C_A(a)$ is a normal subgroup of $C_A(a)$ with Sylow 2 subgroup $\pi(T) \cong T$. The structure of $C_A(a)$ is given in (2.1) and from this we conclude that a belongs to the class (2_2) of A and $\pi(Q) \cap C_A(a) = \pi(T) \cong E_{2^{11}}$. But $\pi(T)$ also contains involutions of the class (2_1) and this gives a contradiction.

Assume finally that $m(T) = 1$ and let $\langle a\rangle = \Omega_1(\pi(T))$. Arguing as before, $\pi(Q) \cap C_A(a)$ is a normal subgroup of $C_A(a)$ with Sylow 2 subgroup $\pi(T)$, hence by (2.1), $\pi(T)$ has order 2. Since $\pi(T) \cong T$, we may set $T = \langle ra\rangle$ with $1 \neq a \in A$ and $r \in R$. Now $[A, R] = 1$ gives $N_R(T) = C_R(r)$ and since $N_R(T) \cong T$ by [2, Theorem 2], we conclude that R has order 2 proving the result.

Since G is a minimal counterexample to Theorem A and A is a standard component of $\langle R, X\rangle$, with $X = \langle A^X\rangle$, it follows that $\langle R, X\rangle$ is also a counterexample to Theorem A. Hence $G = \langle X, R\rangle$.

NOTATION. By (4.1), we may set $\langle z\rangle = R$ so that $G = \langle X, z\rangle$. Also $C(z) = O(C(z)) \times \langle z\rangle \times A$ by (2.7) and (2.8). Let $T_0 \in \mathrm{Syl}_2(A)$, $T = \langle z\rangle \times T_0 \in \mathrm{Syl}_2(C(z))$ and $\{V\} = \{\langle z\rangle \times U\} = \mathscr{E}_{12}(T)$ where $U = \mathscr{E}_{11}(T_0)$. Recall from (2.4) that $N_{C(z)}(V) = O(C(z)) \times \langle z\rangle \times UK$ where $UK = N_A(U)$, $K \cong M_{24}$ and U is F_2K isomorphic to the Fischer module.

(4.2) $z^G \cap A = \varnothing$.

Proof. Note that z is not a square in G whereas every involution of A is a square by (2.1).

(4.3) Let $N = N(V)$. Then $z^G \cap V = zU$. $N = O(N) \times WK$ where $W = \langle z\rangle Y$ is special of order 2^{23} with $Z(W) = U$, Y is a homocyclic abelian group of order 2^{22} invariant under K and Y/U is F_2K isomorphic to U.

Proof. Since $C_N(z) = O(C(z)) \times \langle z\rangle \times UK$, it suffices, in light of (3.1), to show that $z \notin Z(N)$. Assume in fact that $z \in Z(N)$. Then $V = J(T)$ and $T \in \mathrm{Syl}_2(N)$ together imply that $T \in \mathrm{Syl}_2(G)$. Furthermore V is weakly closed in N with respect to G and so N controls fusion of $C(V) = O(N) \times V$. But V contains reprssentatives of the classes of involutions of $C(z)$ and therefore z is isolated in $C(z)$. Applying the Z^* theorem [10], we then have $z \in Z^*(G)$ which is incompatible with $G \leqq \mathrm{Aut}(X)$.

We continue our analysis using the structure and notation for N set up in (4.3). In order to eliminate the ambiguity in the structure of Y we need the following result.

(4.4) Let $\langle\delta\rangle \in \mathrm{Syl}_7(A)$, $\varDelta = C(\delta)$ and $\bar{\varDelta} = \varDelta/O(\varDelta)$. Then either $\bar{\varDelta} \cong S_5 \wr Z_2$ or $\bar{\varDelta} = E(\bar{\varDelta})\langle\bar{z}\rangle$ where $E(\bar{\varDelta}) \cong U_3(5)$, $L_3(5)$ or $L_2(25)$.

Proof. According to (2.3), $C_A(\delta) = \langle\delta\rangle \times D$ where $D \cong S_5$. Moreover if e and d are involutions in D' and $D - D'$ respectively, then by (2.1), $e \in (2_2)$ and $d \in (2_1)$. We shall first show that z fuses to zd and ze in $\varDelta$. We know from (4.3) that z fuses to both zd and ze in G. Set $H = C(z)$ and assume that $(zd)^g = z$, $g \in G$. Now $C_H(zd)^g = C(\langle z, zd\rangle)^g = C(\langle z^g, z\rangle) = C_H(z^g)$. Since $z^G \cap H = \{z\} \cup (zd)^H \cup (ze)^H$ and $C_H(zd) \not\cong C_H(ze)$, we may replace g by gh, $h \in H$, if necessary, to insure that $z^g = zd$. Thus $C_H(zd)^g = C_H(zd)$. Let $B = O^{2'}(C_H(zd)) = \langle z\rangle \times C_A(d)$ and $B = B/O_{2,3}(B) \cong \mathrm{Aut}\,(M_{22})$. Since $B^g = B$ and $\langle\delta\rangle \in \mathrm{Syl}_7(B)$, we may assume that $\langle\delta\rangle^g = \langle\delta\rangle$. If $\delta^g \sim \delta^{-1}$, then g induces an automorphism of $O^2(\bar{B}) \cong M_{22}$ in which an element of order 7 is inverted, a contradiction. Therefore $\delta^g \sim \delta$ in U and again we may replace g by gb, $b \in B$, if necessary to obtain $\delta^g = \delta$ as required. We may prove that z fuses to ze in $\varDelta$ in the exact same way making use of the fact that $O^{2'}(C_H(zd))/O_2(C_H(zd)) \cong \mathrm{Aut}\,(M_{22})$ by (2.1).

Returning to the structure of $\bar{\varDelta} = \varDelta/O(\varDelta)$, we have $C_{\bar{\varDelta}}(\bar{z}) = \overline{O(H)} \times \langle\bar{z}\rangle \times \bar{D}$ so that $\bar{D}'$ is standard in $\bar{\varDelta}$. Since $\bar{\varDelta}$ has sectional 2 rank at most 4 by a result of Harada [14], we may apply the main theorem of [13] to conclude that $E(\bar{\varDelta})$ is isomorphic (i) A_5, (ii) $A_5 \times A_5$, (iii) $L_3(4)$, (iv) M_{12}, (v) $U_3(5)$, (vi) $L_3(5)$, (vii) $L_2(25)$, or (viii) A_7. Furthermore except in case (i), $\bar{\varDelta} \leqq \mathrm{Aut}\,(E(\bar{\varDelta}))$. Since $\overline{zd} \sim \bar{z} \sim \overline{ze}$ in $\bar{\varDelta}$, and $\bar{d} \not\sim \bar{z} \not\sim \bar{e}$ by (4.2), we may easily eliminate cases (i), (iii), (iv) and (viii) and show that in case (ii), $\bar{\varDelta} \cong S_5 \wr Z_2$.

REMARK. If $E(\bar{\varDelta})$ is simple then both $O_{2',E}(\varDelta)$ and $\varDelta - O_{2',E}(\varDelta)$ contain one class of involutions. In particular, $z \notin O_{2',E}(\varDelta)$ and $d \not\sim z \not\sim e$ together imply that the classes (2_1) and (2_2) of A fuse in G.

(4.5) $Y \cong E_{2^{22}}$.

Proof. It follows from (4.3) that either the result is true or Y is homocyclic of exponent 4. Assume the latter for purpose of a contradiction. We know that $N = O(N) \times WK$. Thus if $\langle\delta\rangle \in \mathrm{Syl}_7(K)$, and $\varDelta = C(\delta)$, then the structure of $\bar{\varDelta} = \varDelta/O(\varDelta)$ is given by (4.4). Now $C_Y(\delta) \cong Z_4 \times Z_4$ and $C_K(\delta)$ contains an element of order 3 which acts regularly on $C_Y(\delta)$. This implies that $O^2(\varDelta)$ contains a $Z_4 \times Z_4$ subgroup and we conclude from (4.4) that $\bar{\varDelta} = E(\bar{\varDelta})\langle\bar{z}\rangle$ with

$E(\bar{\Delta}) \cong L_3(5)$. Since $E(\bar{\Delta})$ has wreathed Sylow 2 subgroups of order 2^5 and $\bar{z}$ acts as the graph automorphism, z must invert $C_Y(\delta)$. But the set of all elements of Y inverted by z forms a subgroup of Y properly containing U and invariant under K which forces z to invert Y.

We claim that Y is the unique $(Z_4)^{11}$ subgroup of N. In fact let Y_1 be another such subgroup of N. Then $WK = \widetilde{WK/V} \cong E_{2^{11}} \cdot M_{24}$ together with $m_2(\tilde{Y}_1) = 11$ gives $\tilde{Y}_1 = \tilde{W}$. Therefore $Y_1 \leqq W = \langle z \rangle Y$ and since z inverts Y, we must have $Y = Y_1$. This in turn implies that W must be the unique subgroup of N of its isomorphism type as well. In particular, if $N = N(W)$, then W is weakly closed in its normalizer with respect to G. Hence N contains a Sylow 2 subgroup of G and this in turn forces N to control fusion of $C(W) = O(N)U$. Now the 2 N classes of involutions of U are the sets $(2_1) \cap U$ and $(2_2) \cap U$ of A. Also in the remark following (4.4), we observed that the classes (2_1) and (2_2) of A fuse in G if $E(\bar{\Delta}) \cong L_3(5)$. Thus N must act transitively on U which is clearly not the case and we conclude that $N < N(W)$.

We now investigate the structure of $N(W)$. First observe that $C(W) \leqq C(V)$ gives $C(W) = UO(N)$. Set $\overline{N(W)} = N(W)/U$ and consider the action of $\overline{N(W)}$ on $\bar{W}$. Since Y is characteristic in W, $\bar{Y}$ is normal in $\overline{N(W)}$. Also $C_{\overline{N(W)}}(\bar{z}) = \bar{N} = \langle \bar{z} \rangle \times O(\bar{N}) \times \bar{Y}\bar{K}$. Therefore we may apply (3.1) to conclude that $N(W) = O(N) \times W^*K$ where W^* is a 2-group containing W invariant under K, $W = \langle z \rangle Y^*$ where Y^* contains Y and is invariant under K with $\bar{Y}^*/\bar{Y}$ F_2K isomorphic to $\bar{Y}$.

But Y^*/Y, Y/U and U are all F_2K isomorphic, hence $|C_Y(\delta)| = 2^6$ and this in turn gives $|C_{W^*}(\delta)| = 2^7$ which contradicts $|\Delta|_2 = 2^6$.

(4.6) $W \in \mathrm{Syl}_2(C(U))$. Hence $Y \in \mathrm{Syl}_2(C(Y))$.

Proof. The second statement follows easily from the first. Now $z^G \cap Y = \varnothing$ together with $z^N = zU$ by (4.3) gives $\langle z^G \cap W \rangle = V$. Thus V is weakly closed in W with respect to G. This implies that $N_{C(U)}(W) = N \cap C(U) = O(N) \times W$ by (4.3), hence $W \in \mathrm{Syl}_2(C(U))$ as required.

(4.7) Let $M = N(Y)$ and $\bar{M} = M/Y$. Then
(i) $C_{\bar{M}}(\bar{z}) = \bar{N} = \overline{O(N)} \times \langle \bar{z} \rangle \times \bar{K}$.
(ii) $\bar{z} \notin Z^*(\bar{M})$.

Proof. Suppose $z^\alpha \in zY$, $\alpha \in M$. Since $z^G \cap W = z^W = zU$ by (4.3), αw normalizes V, hence $\alpha w \in N$. This in turn implies that $\alpha \in N$ and we see that $\bar{N} = \overline{C_M(z)} = O(\bar{N}) \times \langle \bar{z} \rangle \times \bar{K}$, proving (i).

To prove (ii), let b be an involution of $UK - U$. Since z fuses to za for any involution $a \in A$ by (4.3), there exists $g \in G$ such that $z^g = zb$. By (2.4), we see that $m_2(C(zb)) = 12$ and all $E_{2^{12}}$ subgroups of $C(zb)$ are conjugate. Therefore $\langle zb, C_Y(zb)\rangle = V^{gh}$ for some $h \in C(zb)$. Observe that $C_Y(zb)$ is generated by those involutions of $\langle zb, C_Y(zb)\rangle$ which are not conjugate to zb. Hence $U^{gh} = C_Y(zb)$. Also $W \in \mathrm{Syl}_2(C(U))$ by (4.6) implies that $W^{gh} \in \mathrm{Syl}_2(C(C_Y(zb)))$. Since $\langle Y, zb\rangle \in \mathrm{Syl}_2(C(C_Y(zb)))$ as well, there exists $k \in G$ such that $W^{ghk} = \langle Y, zb\rangle$. Finally, $z^{ghk} \in z^G \cap \langle Y, zb\rangle = (zb)^Y$ implies that $z^{ghkl} = zb$ for $l \in \langle Y, zb\rangle$. Setting $g' = ghkl$, we have $z^{g'} = zb$ and $W^g = \langle Y, zb\rangle$. Therefore $Y^{g'} = Y$ and $z \sim zb$ in M. We have shown that $\bar{z} \sim \overline{zb}$ in $\bar{M}$ and thus $\bar{z} \notin Z^*(\bar{M})$.

(4.8) $M = O(M)(M_1 \times M_2)\langle z\rangle$ where $M_1^z = M_2 \cong E_{2^{11}} \cdot M_{24}$.

Proof. If follows from (4.7) that $C_{\bar{M}}(\bar{z}) = \langle \bar{z}\rangle \times \bar{K}$ and $\bar{z} \notin Z^*(\bar{K})$. Therefore, by a result of Koch [18] and (3.1), $\bar{M} = O(\bar{M})E(\bar{M})\langle \bar{z}\rangle$ where $E(\bar{M}) \cong M_{24} \times M_{24}$. Let M_1 and M_2 be the minimal normal subgroups of M which map onto the direct factors of $E(\bar{M})$. By (3.2), $Y = U_1 \times U_2$ where $[M_i, U_i] = U_i$ and $[M_i, U_j] = 1$, $i \neq j$. It is clear that either $O_2(M_i) = U_i$ or $O_2(M_i) = Y$, $i = 1, 2$. Assume the latter happens and set $\tilde{M}_1 = M_1/U_1$. Since M_1 is perfect and U_2 is central in M_1, $\tilde{M}_1$ is a perfect central extension of $E_{2^{11}}$ by M_{24}. But this contradicts the fact that M_{24} has trivial multiplier [4]. Therefore $O_2(M_i) = U_i$, $i = 1, 2$. Now $M_1 \cap M_2 \leqq O_2(M_1) \cap O_2(M_2) = U_1 \cap U_2 = 1$ gives $M_1M_2 = M_1 \times M_2$. Finally $M_1^z = M_2 \cong C_{M_1M_2}(z) \cong E_{2^{11}} \cdot M_{24}$ proving the result.

NOTATION. From (4.8), let $M_0 = (M_1 \times M_2)\langle z\rangle$ with $M_2 = M_1^z \cong E_{2^{11}} \cdot M_{24}$. Set $M_1 = U_1K_1$ with $U_1 = O_2(M_1)$, $K_1 \cong M_{24}$ and set $M_2 = U_2K_2$ with $U_2 = U_1^z$, $K_2 = K_1^z$. Furthermore, let $UK = C_{M_1M_2}(z)$ with $U = C_{U_1U_2}(z)$ and $K = C_{K_1K_2}(z)$. Finally, let $S_1 \in \mathrm{Syl}_2(M_1)$, $S_2 = S_1^z \in \mathrm{Syl}_2(M_2)$, $S = S_1 \times S_2$ and $S^* = \langle S, z\rangle \in \mathrm{Syl}_2(M_0)$.

(4.9) $S^* \in \mathrm{Syl}_2(G)$, $S = S^* \cap X \in \mathrm{Syl}_2(X)$ and $z \notin X$.

Proof. First observe that all involutions of $S^* - S$ are conjugate in S^* to z and $C_{S^*}(z) \in \mathrm{Syl}_2(C(z))$. Furthermore, it is easy to see that $z^G \cap S = \varnothing$. In fact, if s is an involution of S, then $C_Y(s) = C_{Y_1}(s) \times C_{Y_2}(s)$ has order at least 2^{12} gives $m_2(C_Y(s)) \geqq 13$ whereas $m_2(C(z)) = 12$ by (2.4). Therefore $z^{S^*} = z^G \cap S$ and we have at once that $S^* \in \mathrm{Syl}_2(G)$. It is clear from the Thompson transfer lemma that $z \notin O^2(G)$. Since $G = \langle X, z\rangle$, we have $X = O^2(G)$. Thus $z \notin X$. Also $S \leqq O^2(M_0) \leqq X$ gives $S = S^* \cap X \in \mathrm{Syl}_2(X)$.

(4.10) Let γ be an element of order 3 of A and $\Gamma = C(\gamma)$. Then $\Gamma = O(\Gamma)E(\Gamma)\langle z\rangle$ where $E(\Gamma) = \Gamma_1 \times \Gamma_2$ and $\Gamma_1^z = \Gamma_2 \cong 6M_{22}$.

Proof. First observe from (2.2) that $C_\Gamma(z) = O(C(z)) \times \langle z\rangle \times C_A(\gamma)$ where $C_A(\gamma) \cong 6M_{22}$. Also by (2.2) we may assume that γ belongs to the class (3_1) of UK. Thus we may write $\gamma = \gamma_1\gamma_2$ where $\gamma_2 = \gamma_1^z$ and γ_i belongs to the class (3_1) of M_i, $i = 1, 2$. Applying (2.6) gives $C_{M_0}(\gamma) = (C_{M_1}(\gamma_1) \times C_{M_2}(\gamma_2))\langle z\rangle$ where $C_{M_1}(\gamma_1)^z = C_{M_2}(\gamma_2) \cong E_{32}\cdot 3A_6$. Since $C_{M_1}(\gamma_1)$ is isomorphic to a 2-local subgroup of $6M_{22}$ which contains a Sylow 2 subgroup of $6M_{22}$, we may set $R^* \in \mathrm{Syl}_2(C_{M_0}(\gamma))$ where $R^* = (R_1 \times R_2)\langle z\rangle$, $R_2 \in \mathrm{Syl}(C_{M_1}(\gamma_1))$ and $R_2 = R_2^z$ has type $2M_{22}$. Also $R_1 \times R_2 \leqq O^2(\Gamma)$. Thus by (3.5), $\Gamma = O(\Gamma)E(\Gamma)\langle z\rangle$ where $E(\Gamma)/O(E(\Gamma)) \cong 2M_{22} \times 2M_{22}$. But $(C_{M_0}(\gamma))^{(\infty)} = C_{M_1}(\gamma) \times C_{M_2}(\gamma) \leqq E(\Gamma)$ then gives $E(\Gamma) = \Gamma_1 \times \Gamma_2$ where $\Gamma_2 = \Gamma_1^z \cong 6M_{22}$.

(4.11) Let γ_i and τ_i be representatives of the classes (3_1) and (3_2) respectively of M_i with $\gamma_1^z = \gamma_2$ and $\tau_1^z = \tau_2$. Let $\gamma = \gamma_1\gamma_2$ and $\tau = \tau_1\tau_2$. Then $\gamma_1\tau_2$, $\tau_1\gamma_2$, τ and γ are conjugate in X.

Proof. We know that τ is conjugate to γ in A by (2.2). Since z leaves γ^X invariant under conjugation and $(\tau_1\gamma_2)^z = \gamma_1\tau_2$, it suffices to show that $\tau_1\gamma_2$ fuses to γ in X. This in turn may be proved by verifying that τ_1 fuses to γ_1 in $C_X(\gamma_2)$. Let $P_i \in \mathrm{Syl}_3(M_i)$ with $P_1^z = P_2$, $Z(P_i) = \langle\gamma_i\rangle$ and assume that $\tau_i \in P_i$, $i = 1, 2$. Since $C_{M_0}(\gamma)^{(\infty)} = C_{M_1}(\gamma_1) \times C_{M_2}(\gamma_2)$ is contained in $E(\Gamma) = \Gamma_1 \times \Gamma_2$, it follows from (3.6), that subject to reindexing, if necessary, $C_{M_i}(\gamma_i) \leqq \Gamma_i$, $i = 1, 2$. In particular, $P_i \in \mathrm{Syl}_3(\Gamma_i)$ and $\langle\gamma_i\rangle = O_3(\Gamma_i)$, $i = 1, 2$. Now P_1 contains an E_9 subgroup $\langle\gamma_1, \gamma_1^*\rangle$ all of whose elements of order 3 are conjugate in M_1 to γ_1. On the other hand, M_{22} contains one class of elements of order 3, hence τ_1 is conjugate in Γ_1 to an element of $\langle\gamma_1, \gamma_1^*\rangle$. Therefore, γ_1 is conjugate to τ_1 in $\langle M_1, \Gamma_1\rangle \leqq C_X(\gamma_2)$ as required.

(4.12) $I(S_i) = U_i^X \cap I(S)$.

Proof. Since S has type $J_4 \times J_4$, $Y = J(S)$ by (2.4). Therefore $N_X(Y)$ controls fusion of Y and we have that $U_i^X \cap Y = U_i$, $i = 1, 2$.

We now observe from (2.6) that every involution of $M_1M_2 - Y$ centralizes an element of order 3 of M_1M_2 which is conjugate to $\tau_1\tau_2 = \tau$, $\gamma_1\gamma_2 = \gamma$, $\tau_1\gamma_2$ or $\gamma_1\tau_2$. Also $C_{M_i}(\gamma_i) = C_{U_i}(\gamma_i)C_{K_i}(\gamma_i) \cong E_{32}\cdot 3A_6$ and $C_{M_i}(\tau_i) \cong C_{U_i}(\tau_i)C_{K_i}(\tau_i) \cong E_8(L_3(2) \times Z_3)$. In the course of proving (4.11), we showed that up to reindexing, it may be assumed that $C_{M_i}(\gamma_i) \leqq \Gamma_i$, $i = 1, 2$. Let $R = R_1 \times R_2 \in \mathrm{Syl}_2(\Gamma_1\Gamma_2)$ where $R_i \in \mathrm{Syl}_2(\Gamma_i)$ and $R_i \leqq C_{M_i}(\gamma_i)$, $i = 1, 2$. By (3.4), $Z(R_i)$ has order 4 and contains representatives of the 3 classes of involutions of Γ_i, $i = 1, 2$. But then every involution of R_i is conjugate to an element of $Z(R_i)$ whereas every involution of $R - R_i$ is conjugate to an element of $Z(R) - Z(R_i)$. Since $Y \cap R = (U_1 \cap R_1) \times (U_2 \cap R_2)$ with $U_i \cap R_i \cong E_{32}$, we have $Z(R_i) \leqq U_i$ and $Z(R) - Z(R_i) \subseteq U - U_i$. Therefore $U_i^X \cap Y = U_i$ then yields $Z(R_i)^X \cap Z(R) = Z(R_i)$. We now conclude that $I(R_i) = U_i^X \cap I(R)$, $i = 1, 2$ and this in turn gives $I(\Gamma_i) = U_i^X \cap I(\Gamma)$, $i = 1, 2$.

Our next objective is to show that $I(C_{M_i}(\tau_i)) = U_i^X \cap I(C_{M_1M_2}(\tau))$, $i = 1, 2$. By (4.11) there exists $g \in X$ such that $\tau^g = \gamma$, hence $(C_{M_1M_2}(\gamma))^g \leqq C_X(\gamma)$. Since $O^{2'}(C_{M_1M_2}(\tau)) = C_{M_1}(\tau_1)' \times C_{M_2}(\tau_2)'$, we have $(C_{M_1}(\tau_1)')^g \times (C_{M_2}(\tau_2)')^g = O^{2'}(C_{M_1M_2}(\tau))^g \leqq O^{2'}(C_X(\gamma)) = \Gamma_1\Gamma_2$ by (3.5). Furthermore by (3.6), $C_{M_i}(\tau_i)' \leqq \Gamma_{j_i}$ with $j_1 \neq j_2$. But $O_2(C_{M_i}(\tau_i)') = C_{U_i}(\tau_i) \cong E_8$ combined with $U_i^X \cap \Gamma_i = I(\Gamma_i)$ yields $(C_{M_i}(\tau_i)')^g \leqq \Gamma_i$. Therefore $I(C_{M_i}(\tau_i)^g) = U_i^X \cap I(C_{M_1M_2}(\tau)^g)$ and this implies that $I(C_{M_i}(\tau_i)) = U_i^X \cap I(C_{M_1M_2}(\tau))$, $i = 1, 2$. The same argument then gives $I(C_{M_i}(\tau_i)) = U_i^X \cap I(C_{M_1M_2}(\tau_i\delta_j))$ and $I(C_{M_i}(\gamma_i)) = U_i^X \cap I(C_{M_1M_2}(\gamma_i\delta_j))$, $i \neq j$, $\delta_j = \tau_j$ or γ_j. Since a conjugate of every involution of M_1M_2 centralizes $\gamma, \tau, \gamma_1\tau_2$ or $\tau_1\gamma_2$, we see at once that $I(M_i) = U_i^X \cap I(M_1M_2)$, $i = 1, 2$. Therefore $I(S_i) = U_i^X \cap I(S)$, $i = 1, 2$ proving the result.

(4.13) The following holds:

(i) S_i is a Sylow 2 subgroup of $O^2(C_X(S_j))$ and $O^2(C_X(U_j))$, $i \neq j$.
(ii) Every involution of S_i is conjugate in $C_X(S_j)$ to an element of U_i, $i \neq j$.

Proof. Since $U_j \lhd S$, $S_i \times U_j \in \mathrm{Syl}_2(C_X(U_j))$, $i \neq j$. By Gaschutz's theorem we may write $C_X(U_j) = C_jU_j$ where C_j is a complement to U_j in $C_X(U_j)$. Also U_j is central in $C_X(U_j)$ gives $C_X(U_j) = C_j \times U_j$. Clearly $O^2(C_X(U_j)) \leqq C_j$. Also $S_i \leqq M_i$ and $[M_i, S_j] = 1$ yields $S_i \leqq C_j$. It now follows directly that $S_i \in \mathrm{Syl}_2(O^2(C_X(U_j)))$. The same proof may be used to verify that $S_i \in \mathrm{Syl}_2(O^2(C_X(S_j))$ and this completes the proof of (i).

In order to prove (ii), first observe that $S_j = \Omega_1(S_j)$, hence by (4.12), S_j is weakly closed in S with respect to X. Therefore $N_X(S_j)$ controls fusion of $C_X(S_j)$. Since $S_i \in \mathrm{Syl}_2(O^2(C_X(S_j)))$ by (i), the Frattini argument gives $N_X(S_j) = C_X(S_j)N_X(S)$. Now $N_X(S) \leqq N_X(Y)$ where $N_X(Y) = M \cap X = O(M)(M_1 \times M_2)$. Clearly $\bar{S}$ is self normalizing in $\overline{M \cap X} = M \cap X/O(M)$ and this yields $N_X(S) = O(N_X(S))S$. Consequently $N_X(S_j) = C_X(S_j)S_j$. But $[S_i, S_j] = 1$ implies that $C_X(S_j)$ controls fusion of $S_i \times Z(S_j) \in \mathrm{Syl}_2(C_X(S_j))$ and the result now follows from (4.12).

(4.14) S_i is strongly closed in S with respect to X, $i = 1, 2$.

Proof. By symmetry, we need only prove the result for S_1. Assume in fact that S_1 is not strongly closed in S with respect to X. Let $s_1 \in S_1$ be an element of minimal order of S_1 such that $s_1^X \cap S \nsubseteq S_1$. Then $s_1^g = s_1' s_2'$ for some $g \in X$, $s_i' \in S$, i $= 1, 2$, and $s_2' \neq 1$. By (4.12), we may assume that $|s_1| > 2$. Also $(s_1^2)^g = (s_1')^2(s_2')^2$ together with the minimality of $|s_1|$ implies that s_2' is an involution. By (4.13ii), s_2' is conjugate in $C_X(S_1)$ to an element of U_2, so we may further assume that $s_2' \in U_2$. But U_2 is weakly closed in S with respect to X by (2.4) and (4.12), therefore $N_X(U_2)$ controls fusion of $C_X(U_2)$. A contradiction may now be established by observing that $s_1 \in S_1 \in \mathrm{Syl}_2(O^2(C_X(U_2)))$ whereas $s_1' s_2' \notin O^2(C_X(u_2))$ by (4.13i).

We are now in the position to complete the proof of Theorem A. By (4.14) and the Aschbacher-Goldschmidt theorem [12], X is not simple. This of course contradicts our condition that X is simple and $G \leqq \mathrm{Aut}\, X$.

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