FINITE GROUPS WITH A STANDARD COMPONENT OF TYPE $J_4$

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In this paper, it is shown that if $G$ is a core-free group with a standard component $A$ of type $J_4$, then either $A$ is normal in $G$ or the normal closure of $A$ in $G$ is isomorphic to the direct product of two copies of $J_4$.

1. Introduction. Janko [17] has recently given evidence for the existence of a new finite simple group. In particular, Janko assumes that $G$ is a finite simple group which contains an involution $z$ such that $H = C(z)$ satisfies the following conditions:

(i) The subgroup $E = O_2(H)$ is an extra-special group of order $2^{13}$ and $C_H(E) \subseteq E$.

(ii) $H$ has a subgroup $H_0$ of index 2 such that $H_0/E$ is isomorphic to the triple cover of $M_{22}$.

He then shows that $G$ has order $2^{26} \cdot 3^3 \cdot 5 \cdot 7 \cdot 11 \cdot 23 \cdot 39 \cdot 31 \cdot 37 \cdot 43$ and describes the conjugacy classes and subgroup structure of $G$. In this paper we shall assume that $J_4$ is a finite simple group which satisfies Janko's assumptions and shall prove

**Theorem A.** Let $G$ be finite group with $O(G) = 1$, $A$ a standard component of $G$ isomorphic to $J_4$ and $X = \langle A^0 \rangle$. Then either $X = A$ or $X \cong A \times A$.

Our proof follows the outline given in [6] and makes use of two key facts; namely, that $J_4$ has a 2-local subgroup isomorphic to the split extension of $E_{211}$ by $M_{24}$ and that $J_4$ has one class of elements of order 3 with the centralizer of an element of order 3 isomorphic to the full cover of $M_{22}$. We also make use of the characterization of finite groups with a standard component isomorphic to $M_{24}$ which was recently obtained by Koch [18].

2. Properties of $J_4$. In this section, we shall describe certain properties of $J_4$ and its subgroups which will be required for the proof of Theorem A. Most of these properties are found in [17] and will be listed without proof. $A$ will denote a group isomorphic to $J_4$.

(2.1) $A$ has 2 classes of elements of order 2 denoted by $(2_1)$ and $(2_2)$. If $t \in (2_1)$ and $E = O_2(C(t))$, then $E$ is isomorphic to an extra special group of order $2^{13}$, $C(E) = Z(E)$, $O_{5,3}(C(t))/E$ has order 3 and
\[ C(t)/O_{2a}(C(t)) \cong \text{Aut}(M_{22}). \] Moreover, if \( \langle \beta \rangle \in \text{Syl}_3 \left( O_{2a}(C(t)) \right) \), then \( \langle \beta \rangle \) acts regularly on \( E/Z(E) \). For \( x \in (2_3) \), \( C(x) \) is isomorphic to a split extension of \( E_{211} \) by \( \text{Aut}(M_{22}) \) with \( C(x) \) acting indecomposably on \( O_{2a}(C(x)) \).

(2.2) \( A \) has one class of elements of order 3. If \( \gamma \in A \) has order 3, then \( C(\gamma) \) is isomorphic to the 6-fold cover of \( M_{22} \).

(2.3) \( A \) has two classes of elements of order 7. If \( \delta \in A \) has order 7, then \( C_A(\delta) \cong Z_7 \times S_5 \) and \( \delta \not\sim \delta^{-1} \).

(2.4) Let \( T_0 \in \text{Syl}_3(A) \). Then \( T_0 \) has precisely one \( E_{211} \) subgroup, denoted by \( U \). \( N(U) = UK \) where \( K \cong M_{24} \). The orbits of \( K \) on \( U^* \) are \((2) \cap U \) of order \( 7 \cdot 11 \cdot 23 \) and \((2) \cap U \) of order \( 4 \cdot 3 \cdot 23 \).

In the above, \( U \) is isomorphic to the so-called “Fischer” module for \( M_{24} \). The following is an important property of the Fischer module.

(2.5) Let \( (*) 1 \rightarrow R \rightarrow V \rightarrow U \rightarrow 1 \) be an extension of \( F_2M_{24} \) modules where \( R \) is a trivial module of dimension 1 and \( U \) is isomorphic to the Fischer module. Then the extension splits.

\[ \text{Proof.} \] Let \( \tilde{U} \) and \( \tilde{V} \) be the \( F_2M_{24} \) modules dual to \( U \) and \( V \) respectively. Then we have the extension \( (*) 1 \rightarrow \tilde{U} \rightarrow \tilde{V} \rightarrow R \rightarrow 1 \). It suffices to show that \( (*) \) splits. Since \( U \) is not a self dual module and since there exists precisely 2 nonisomorphic \( F_2M_{24} \) modules of dimension 11 (see James [16]), \( \tilde{U} \) is isomorphic to the so-called Conway module [5]. Thus \( M_{24} \) has 2 orbits on \( (\tilde{U})^* \). If \( u_1 \) and \( u_2 \) are representatives of these 2 orbits, then \( C_{M_{24}}(u_i) \cong \text{Hol}(E_{11}) \) and \( C_{M_{24}}(u_2) \cong \text{Aut}(M_{22}) \).

Since \( |\tilde{V}| = 2^{12} \), there exists a vector \( v \in \tilde{V} - \tilde{U} \) such that \( v \) is fixed by a Sylow 23 subgroup \( S \) of \( M_{24} \). The orbit of \( M_{24} \) on \( (\tilde{V})^* \) which contains \( v \) has order \( [M_{24}: C_{M_{24}}(v)] \) and is not divisible by 23. Therefore, by examining the list of maximal subgroups of \( M_{24} \) [5], together with \( [M_{24}: C_{M_{24}}(v)] \leq 2^{12} \), we see that \( C_{M_{24}}(v) \) contains a subgroup \( L \) isomorphic to \( M_{22} \). Consider the action on \( \tilde{V} \) of an \( M_{22} \) subgroup \( M \) of \( L \). Then \( M \) has no fixed points on \( \tilde{U}^* \), so in fact \( C_M(M) = \langle v \rangle \). Therefore \( N_{M_{24}}(M) \cong \text{Aut}(M_{22}) \) fixes \( \langle v \rangle \) as well. Finally \( \langle L, N_{M_{24}}(M) \rangle = M_{24} \) centralizes \( \langle v \rangle \) and the extension splits.

We shall denote by \( E_{211} \cdot M_{24} \) a split extension of \( E_{211} \) by \( M_{24} \) in which \( E_{211} \) is \( F_2M_{24} \) isomorphic to the Fischer module.

(2.6) Let \( M = UK \) be isomorphic to \( E_{211} \cdot M_{24} \) with \( U = O_4(M) \)
and \( K \cong M_d \). Then the classes of elements of order 2 and 3 of \( M \) and the orders of the centralizers in \( M \) of a representative \( \lambda \) are as follows

| Class | \(|C_U(\lambda)|\) | \(|C_M(\lambda)|\) |
|-------|-------------------|-------------------|
| \((2_1)\) | \(2^{11}\) | \(2^{27} \cdot 3^5 \cdot 5 \cdot 7 \cdot 11 \) |
| \((2_2)\) | \(2^{11}\) | \(2^{27} \cdot 3^5 \cdot 7 \cdot 11 \) |
| \((2_3)\) | \(2^7\) | \(2^{21} \cdot 3^5 \cdot 7 \) |
| \((2_4)\) | \(2^7\) | \(2^{21} \cdot 3^5 \) |
| \((2_5)\) | \(2^5\) | \(2^{21} \cdot 3^5 \cdot 5 \) |
| \((2_6)\) | \(2^5\) | \(2^{21} \cdot 3^5 \) |
| \((3_1)\) | \(2^5\) | \(2^{21} \cdot 3^5 \) |
| \((3_2)\) | \(2^3\) | \(2^{21} \cdot 3^5 \cdot 7 \) |

Moreover, if \( \lambda_i \in (3_i) \cap K \) then \( C_M(\lambda_i) = C_U(\lambda_i)C_K(\lambda_i) \) with \( C_K(\lambda_i) \) isomorphic to the 3-fold cover of \( A_6 \), \( C_K(\lambda_2) \cong Z_3 \times L_4(7) \) and where \( C_K(\lambda_i)/\langle \lambda_i \rangle \) acts faithfully on \( C_U(\lambda_i) \), \( i = 1, 2 \).

**Proof.** Let \( \lambda \) be an involution of \( M - U \), \( \alpha_1, \alpha_2, \ldots, \alpha_n \) the orbits of \( C_M(\lambda U/U) \) on \( \lambda C_U(\lambda) \) and \( \alpha_i \) an element of \( \alpha_i \), \( i = 1, \ldots, n \). Then \( \alpha_i \) is conjugate to \( \alpha_j \) in \( M \) exactly when \( i = j \) and also \( |C_M(\alpha_i)| = |C_M(\lambda U)|/|\alpha_i| \). Now \( K \) has 2 classes of involutions with representatives \( \lambda \) and \( \eta \) having centralizers in \( K \) of order \( 2^{20} \cdot 3 \cdot 7 \) and \( 2^9 \cdot 3 \cdot 5 \) respectively. Noting that the action of \( K \) on \( U \) is dual to the action of \( K \) on the Conway module, it is easy to see that \( |C_U(\lambda)| = 2^7 \) and \( |C_U(\eta)| = 2^5 \). Observe that \( U \) has 8 orbits on \( \lambda C_U(\lambda) \), each of which has length 16. Moreover an element of order 7 of \( C_K(\lambda) \) fixes 2 points of \( C_U(\lambda) \) and therefore must permute 7 of these orbits. Since \( |C_U(\lambda)| = |C_K(\lambda)| = |C_U(\lambda)| = 2^{27} \cdot 3 \cdot 7 \), it then follows that \( C_K(\lambda U/U) \) acting on \( \lambda C_U(\lambda) \) has one orbit of length 16 and one orbit of length 7 \cdot 16 = 112 with \( \lambda \) an element of the orbit of length 16. This accounts for the classes \((2_3)\) and \((2_4)\). Similar reasoning accounts for the classes \((2_5)\) and \((2_6)\). We already know from \((2.4)\) that \( M \) has orbits on \( U^* \) of lengths 4 \cdot 3 \cdot 23 and 7 \cdot 11 \cdot 23 \) and thus the classes of involutions of \( M \) are as described.

Let \( \gamma \) and \( \tau \) be representatives of the classes of element of order 3 of \( K \) with \( C_K(\gamma) \) isomorphic to the 3-fold cover of \( A_6 \) and \( C_K(\tau) \cong Z_3 \times L_4(7) \). Clearly \( \gamma \) and \( \tau \) are representatives of the 2 classes of elements of order 3 of \( M \). It suffices to determine the orders of \( C_U(\gamma) \) and \( C_U(\tau) \). As before, we may appeal to the action of \( K \) on the Conway module to obtain \( |C_U(\gamma)| = 2^5 \) and \( |C_U(\tau)| = 2^9 \) as required.

**NOTATION.** If \( H \) is a simple group, then \( nH \) will denote a proper
n-fold covering of $H$. If the multiplier of $H$ is cyclic, then $nH$ is unique up to isomorphism. Also let $E_{33} \cdot 3A_6$ be the group isomorphic to the centralizer of an element of order 3 of the class (3) of $E_{32} \cdot M_{24}$. Note that $E_{33} \cdot 3A_6$ is isomorphic to a 2-local subgroup of $6M_{22}$.

(2.7) The Schur multiplier of $J_4$ is trivial.

Proof. See Griess [14].

(2.8) $\text{Aut}(J_4) \cong J_4$.

Proof. Let $A \cong J_4$ and suppose that $\alpha \in \text{Aut}(A)$. We may imbed $A$ in $\text{Aut}(A)$ and assume by way of a contradiction that $\alpha \notin A$ but $\alpha^p \in A$ for some prime $p$. Set $G = \langle A, \alpha \rangle$.

By (2.4), we may assume that $\alpha \in N_\alpha(U)$ where $U$ is an $E_{31}$ subgroup of $A$, $N_\alpha(U) = UK = E_{31} \cdot M_{24}$ and $K \cong M_{24}$. Since $\text{Aut}(K) \cong K$, we may further assume that $N_\alpha(U)/U = \langle \alpha \rangle \times K$. It is known [16] that $U$ is an absolutely irreducible $F_2K$ module, hence by a result of Schur, we have $[\alpha, U] = 1$. Two cases now arise; namely $[\alpha, K] = 1$ and $[\alpha, K] \neq 1$.

If $[\alpha, K] \neq 1$, then it is clear that $\alpha$ is a 2-element. Also the fact that $\mathcal{O}'(\langle U, \alpha \rangle)$ is a proper $K$ invariant subgroup of $U$ forces $\mathcal{O}'(\langle U, \alpha \rangle) = 1$. Hence $\langle U, \alpha \rangle \cong E_{32}$ and $K$ acts indecomposably on $\langle U, \alpha \rangle$. Without loss, we may assume that $\alpha$ is centralized by a Sylow 23 subgroup of $K$. By arguing as in (2.5), it then follows that $C_3(\alpha) \cong M_{23}$. Therefore in either case, we have that $C_{UK}(\alpha) \cong UK_0$ where $K_0$ is an $M_{23}$ subgroup of $K$.

Let $\gamma$ be an element of order 3 of $K_0$. Then $C_{K_0}(\gamma) \cong Z_3 \times A_5$ implies that $C_U(\gamma) \cong E_{33}$ by (2.6). Also $C_3(\gamma) \cong 6M_{22}$ and $m_3(C_3(\gamma)) = 5$ [4] gives $O_3(C_3(\gamma)) \leq C_6(\gamma)$. Setting $\overline{C_3(\gamma)} = C_3(\gamma)/Z(C_3(\gamma)) \cong M_{22}$, we conclude that $\alpha$ centralizes a subgroup of $\overline{C_3(\gamma)}$ isomorphic to a split extension of $E_{16}$ by $A_5$. But no nontrivial automorphism of $M_{22}$ centralizes such a subgroup [9] and therefore $[\alpha, C_3(\gamma)] \leq Z(C_3(\gamma))$. By the 3-subgroup lemma, we then have $C_3(\gamma) \leq C_3(\alpha)$. Since $\gamma$ is inverted by an element of $K_0 \leq C_3(\alpha)$, it follows that $N_\alpha(\langle \gamma \rangle) \leq C_3(\alpha)$ as well.

Finally, let $\langle t \rangle = O_3(C_3(\gamma))$ so that $C_3(t) = E \cdot N_3(\langle \gamma \rangle)$ by (2.1), where $E = O_3(C_3(t))$ is extra special of order $2^{13}$. Observe that $C_3(\gamma)$ acts irreducibly on $E/\langle t \rangle$. Combining this with $[C_3(\gamma), \alpha] = 1$ and $C_3(\alpha) \geq U \cap E > \langle t \rangle$, we conclude that $E \leq C_3(\alpha)$. Therefore we are in the position where $C_3(\alpha) \geq C_3(t)$ and $C_{UK}(\alpha) = UK_0$ or $UK$ with $K_0 \cong M_{23}$. But $C_3(t)$ contains a Sylow 2 subgroup of $N_\alpha(U)$ implies that $C_{UK}(\alpha) = UK$ and this gives $C_3(\alpha) \geq \langle UK, C_3(t) \rangle$. An easy argu-
ment shows that $C_A(\alpha)$ is simple with $C_{C_A(\alpha)}(t) = C_A(t)$. Thus by Janko's theorem [17], $|C_A(\alpha)| = |A|$ which of course gives $A = C_A(\alpha)$, a contradiction.

3. Preliminary results. In this section we present certain technical results which are necessary for the proof of Theorem A.

(3.1) Let $G$ be a group, $A$ a standard component of $G$ with $C(A)$ of 2 rank 1. Let $S \in \text{Syl}_2(N(A))$. Assume that $S \in \text{Syl}_2(G)$ and $Z(S) \leq AC(A)$. Then $[A, O(G)] = 1$.

Proof. See Seitz [19].

(3.2) Let $M$ be a group containing an involution $z$ such that $C(z) = O(C(z)) \times \langle z \rangle \times U K$ where $K \cong M_{24}$ and $U$ is $F_4 K$ isomorphic to the Fischer module. Let $V = \langle z, U \rangle$ and $N = N(V)$. Then either

(i) $z \in Z(N)$ or
(ii) $N = O(N) \times WK$ where $W = \langle z \rangle Y$ is special of order $2^{23}$ with $Z(W) = U$ and where $Y$ is a homocyclic abelian group of order $2^{22}$ invariant under $K$ with $Y/U \cong F_2 K$ isomorphic to $U$.

Proof. Assume that $z \in Z(N)$ and let $\bar{N} = N/O(N)$. By (2.2), the orbits of $K$ on $U^z$ are $t^K$ of order 1771 and $x^K$ of order 276 with $C_K(x) \cong \text{Aut}(M_{24})$. Moreover both $t$ and $x$ are squares in $UK$, hence $z^N \cap U = \emptyset$. Now the orbits of $C(z)$ on $V^z$ are precisely

<table>
<thead>
<tr>
<th>Orbit</th>
<th>${z}$</th>
<th>$t^K$</th>
<th>$x^K$</th>
<th>$(zt)^K$</th>
<th>$(zx)^K$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Length</td>
<td></td>
<td>1771</td>
<td>276</td>
<td>1771</td>
<td>276</td>
</tr>
</tbody>
</table>

Since $z \not\in Z(N)$ and $z^N \cap U = \emptyset$, $z^N$ must be a union of some of the sets $\{z\}, (zt)^K, (zx)^K$. But $|z^N|$ is a divisor of $|L_{13}(2)|$ then gives $z^N = zU$.

Representing $N$ on $z^N = zU$, we have $|N| = 2^{11} |N_{11}(V)|$, hence $|\bar{N}| = 2^{22} |M_{24}|$. Moreover $U$ is generated by those involutions of $V$ not conjugate to $z$ so that $U \lhd N$. Assume that $C_N(U) = O(N)V$. Then $\bar{N}/\bar{V}$ acts faithfully on $\bar{U}$ and is therefore isomorphic to a subgroup of $L_{11}(2)$. Let $S \in \text{Syl}_{11}(K)$ so that $N_K(S)$ is isomorphic to a Frobenius group of order 10·11. Since $S$ fixes 2 points of $zN$, it follows that $|C_N(\bar{S})| = 2 |C_N(\langle z, \bar{S} \rangle)| = 2^{9} \cdot 11$. Hence a Sylow 11 subgroup of $\bar{N}/\bar{V}$ has centralizer of even order which contradicts the fact that a Sylow 11 subgroup of $L_{11}(2)$ has centralizer of odd order. We conclude that $C_N(U)$ properly contains $O(N)V$.

It is easy to see from the action of $K$ on $C_N(U)$ that $C_N(U) = O(N)W$ where $W/U \cong E_{112}$. Furthermore, $C_N(z) = V$ implies that $Z(W) = U$ and $[z, W] = U$. Thus $W$ is a special 2-group of order
with $Z(W) = U$. We will in fact show that $N = O(N) \times WK$. To see this, observe that $V\langle K^\nu \rangle$ covers $\bar{N}$ together with $[VK, O(N)] = 1$ implies that $N = O(N)C_N(O(N))$. A simple argument establishes that $O^\nu(C_N(O(N))) = WK$ and therefore $N = O(N) \times WK$. For the remainder of the proof, we may assume that $O(N) = 1$.

Consider the homomorphism $\varphi: W \rightarrow U$ by $\varphi(w) = [z, w]$. It is easy to see that $\varphi$ induces an $F_2K$ isomorphism between $W/V$ and $U$. But then $W/U$ is an $F_2K$ module which satisfies, the hypotheses of (2.5) and thus $W/U = V/U \times Y/U$ where $Y/U$ is $F_2K$ isomorphic to $U$. It remains for us to show that $Y$ is a homocyclic abelian group. Assume not. Then by the action of $K$ on $Y$, $Z(Y) = U$. Let $L$ be a subgroup of $K$ isomorphic to $\text{Aut}(M_{22})$. It follows from the properties of the Fischer module that $|C_{Y/U}(L)| = |C_U(L)| = 2$ with $C_{Y/U}(L)$ and $C_U(L)$ the unique proper $L$ invariant submodules of $Y/U$ and $U$ respectively. Let $\langle y, U \rangle = C_{Y/U}(L)$ so that $L$ normalizes $\langle y, U \rangle$. Since $y \notin Z(Y)$, $1 \neq [y, Y] < U$ and since $L$ normalizes $\langle y, U \rangle$, $Y = [y, Y]$ we must have $[y, Y] = C_U(L)$. This in turn implies that $[Y: C_Y(y)] = 2$. But $L$ normalizes $C_Y(\langle y, U \rangle) = C_Y(y)$, hence $C_Y(y)/U$ as well and this gives a contradiction.

(3.3) Let $Y \cong E_{222}$ and $M$ a subgroup of $\text{Aut}(Y)$ such that $M = M_i \times M_j$ with $M_i \cong M_j \cong M_{22}$. Then $Y = Y_i \oplus Y_j$ where $[Y_i, M_i] = Y_i$ and $[Y_j, M_j] = 0$, $i \neq j$.

Proof. Let $\gamma$ be an element of order 23 of $\text{Aut}(Y)$. If $\gamma$ acts regularly on $Y$, then $C_{\text{Aut}(Y)}(\gamma)$ is isomorphic to $GL_4(2^{11})$ or is cyclic. Otherwise $\dim(C_{\text{Aut}(Y)}(\gamma)) = 11$ and $C_{\text{Aut}(Y)}(\gamma) \cong Z_{1023} \times L_{11}(2)$. Let $\gamma_i \in M_i$ be an element of order 23. Then it is clear that $\dim(C_{Y_i}(\gamma_i)) = 11$. If we set $Y_i = C_{Y_i}(\gamma_i)$, $i \neq j$, then an easy argument verifies that $Y_i$ and $Y_j$ satisfy $[Y_i, M_j] = 0$, $i \neq j$ and $[Y_i, M_i] = Y_i$, $i = 1, 2$ as required.

In the next result, we list certain properties of $2M_{22}$ which are required for (3.5).

(3.4) Let $D \cong 2M_{22}$, $T \in \text{Syl}_2(D)$. Then
(i) $D$ has 3 classes of involutions.
(ii) $Z(T)$ has order 4 and contains representatives of the classes of involutions of $D$.
(iii) $T$ has precisely 2 $E_{22}$ subgroups, say $F_1$ and $F_2$. Each is normal in $T$ and self-centralizing in $D$. Also $N(F_1)/F_1 \cong A_5$ and $N(F_2)/F_2 \cong S_5$.

Proof. See Burgoyne and Fong [4].

(3.5) Let $\Gamma$ be a group with an involution $z$ such that $C(z) =$
$O(C(z))D(z)$ with $D = E(C(z))$ and $D/O(D) \cong 2M_{22}$. Assume further that $\Gamma$ has a 2-subgroup $R^* = (R_1 \times R_2)z$ where $R_2 = R_1^\alpha$ has type $2M_{22}$ and $R = R_1 \times R_2 \leq O^2(\Gamma)$. Then $\Gamma = O(\Gamma)E(\Gamma)z$ with $E(\Gamma)/O(E(\Gamma)) \cong 2M_{22} \times 2M_{22}$.

**Proof.** By assumption and (3.4)(iii), $R$ has a normal subgroup $V = V_1 \times V_2$ where $V_i \vartriangleleft R_i$ and $V_i \cong E_{33}$, $i = 1, 2$. If $\alpha$ is an involution of $R$, then $m_2(C_{R_i}(\alpha)) \geq 3$, $i = 1, 2$, gives $m_2(C_R(\alpha)) \geq 7$. Since $m_2(C(z)) = 6$, it follows that $z^\alpha \cap R = \varnothing$. Also all involutions of $R^* - R$ are conjugate to $z$ which then implies that $z^\alpha \cap R^* = z^{R^*}$. Since $R^e(z) \in \text{Syl}_2(C(z))$, we see that $R^e \in \text{Syl}_2(\Gamma)$. Furthermore by the Thompson transfer lemma and assumption, $z \notin O^2(\Gamma)$ and $R \in \text{Syl}_2(O^2(\Gamma))$. Let $A = O^2(\Gamma)$.

We now examine the structure of $C(D)$. Observe that $C_{C(D)}(z) = O(C(z))z$ where $\langle t \rangle = O_d(D)$. By a result of Suzuki, $O(D)$ has dihedral or semidihedral Sylow 2 subgroups. Let $Z \in \text{Syl}_2(C(D))$ so that $\langle Z, z \rangle \in \text{Syl}_2(C(D))$. Since $C_R(z) \in \text{Syl}_2(D)$ and $Z(R) = C_R(C_R(z)) \in \text{Syl}_2(C_R(C_R(z)))$, we may assume that $Z \subseteq Z(R)$. Therefore $Z$ is elementary abelian by (3.4)(ii) and we have either $\langle Z, z \rangle \cong D_8$ and $Z \cong E_4$, or $Z = \langle t \rangle$. Let $N = N(Z)$ and $\bar{N} = N/Z$. In either case, $\langle z \rangle \in \text{Syl}_2(C_{\bar{N}}(\bar{D}))$ and $C_{\bar{N}}(\bar{z}) \leq N_{\bar{N}}(\bar{D})$ together imply that $\bar{D}$ is a standard component of $\bar{N}$. By Theorem A [8] and (3.1), $E(\bar{N}) = \langle \bar{D}^2 \rangle$, $Z(E(\bar{N}))$ has odd order and $E(\bar{N})/Z(E(\bar{N})) \cong M_{22} \times M_{22}$. Let $K = E(N)$ have components $K_1$ and $K_2$ with $K^e_1 = K_2$ and $K_1/K_2 \subseteq M_{22}$. Then $D = C_K(D)$ and $D/O(D) \cong 2M_{22}$ implies that $K/O(K) \cong 2M_{22} \times 2M_{22}$. Thus $|Z| = 4$ and $K = O^2(C_d(Z))$.

Note that $R \leq K$. Without loss, we may assume that $R_i \leq K_i$, $i = 1, 2$. By (3.4)(iii), let $V_i$ and $W_i$ be the 2 $E_{33}$ subgroups of $R_i$ with $C_{R_1}(V_i) = O(K_1)V_i$, $C_{R_1}(w_i) = O(K_2)W_i$, $N_{R_1}(V_i)/C_{R_1}(V_i) \cong S_6$ and $N_{R_2}(W_i)/C_{R_2}(W_i) \cong A_8$, $i = 1, 2$. Set $W = W_1 \times W_2$, $M = N(W)$ and $\bar{M} = M/W$. Then $M \cap K = E(M \cap K)O(M \cap K)$ with $E(M \cap K)/O(E(M \cap K)) \cong A_8 \times A_8$. Since $W_1 = W_2$, $C_M(zW) = N(\langle z, W \rangle) = WC_M(z)$. Also $K = K_1K_2$ with $K^e_1 = K_2$ implies that $C_{M \cap K}(z)$ involves $A_8$. Hence by (3.4)(iii), $C_M(\bar{z}) = \langle \bar{z} \rangle \times O(C_d(\bar{z}))(D \cap \bar{M})$ where $D \cap \bar{M} = E(C_{\bar{M}}(\bar{z}))$ and $D \cap \bar{M}/O(D \cap \bar{M}) \cong A_8$. It now follows that $D \cap \bar{M}$ is a standard component of $\bar{M}$ and we have from Proposition 2.3 [7] and (3.1) that $\bar{M} = O(\bar{M})E(\bar{M})\langle \bar{z} \rangle$ with $E(\bar{M})/O(E(\bar{M})) \cong A_8 \times A_8$. Furthermore $E(D \cap \bar{M}) = E(\bar{M})$ then implies that $Z = C_M(E(\bar{M}))$ and this yields $Z \vartriangleleft M$.

Our next goal is to show that $ZO(\Gamma) \vartriangleleft \Gamma$. Towards this end, observe that $W$, $W_1 \times V_2$, $V_1 \times W_2$ and $V_1 \times V_2$ are the only $E_{30}$ subgroups of $R$ and that $S_5$ is not involved in $N_4(W)$ whereas $S_5$ is involved in $N_4(W_1 \times V_2)$, $N_4(V_1 \times W_2)$ and $N_4(V_1 \times V_2)$. This prevents $W$ from fusing in $\Gamma$ to $W_1 \times V_2$, $V_1 \times W_2$ or $V_1 \times V_2$ and
yields \( W \triangleleft N_\lambda(R) \). Now \( Z(R) \) contains representatives of the classes of involutions of \( K \) by (3.4i), hence of \( A \) as well. Since \( Z \leq Z(R) \), \( Z \) fails to be strongly closed in \( R \) with respect to \( A \) only when \( Z^x \cap Z(R) \not\subseteq Z \) for some \( \lambda \in A \). If in fact this happens, then we may choose \( \lambda \in N_\lambda(R) \). But \( W \triangleleft N_\lambda(W) \) and \( Z \triangleleft N_\lambda(W) \) then gives \( Z^x = Z \), a contradiction. Applying Goldschmidt's theorem [11], we conclude that \( ZO(\Gamma) \triangleleft \Gamma \). This in turn yields \( \Gamma = O(\Gamma)N \).

Since \( K = E(N) = O^\prime(N) \), it suffices to show that \( [K, O(\Gamma)] = 1 \). Recall that \( E(C(z)) = D = C_K(z) \). Let \( T = C_K(z) \in \text{Syl}_2(D) \) and \( Z(T) = \langle t, t_x \rangle = Z(T) \leq Z(R) \). Then for \( X = O(\Gamma) \), we have \( X = C_X(z)C_x(zt_x)C_x(t_x) \). Now \( C_X(z) \leq O(C(z)) \) and \( [O(C(z)), D] = 1 \) gives \( C_X(z) \leq C_X(t_x) \). Also \( z^x = zt_x \) for some \( \lambda \in Z(R) \), hence \( t_x = t_x^\lambda \in D^\lambda = E(C(zt_x)) \). By the same reasoning, \( C_X(zt_x) \leq C_X(t_x) \) and so \( [t_x, X] = 1 \). But \( \langle t_x^\lambda \rangle = K \) and therefore \( [K, X] = 1 \) as required.

The next result will be used in conjunction with (3.5).

(3.6) Let \( \Gamma_0 = \Gamma_0 \times \Gamma \) with \( \Gamma_0 \leq \Gamma_0 \leq 6M_{22} \) and suppose \( H = H_1 \times H_2 \) is a perfect subgroup of \( \Gamma_0 \). Then by reindexing if necessary \( H_1 \leq \Gamma_1 \) and \( H_2 \leq \Gamma_2 \).

**Proof.** Let \( \tilde{\Gamma}_0 = \Gamma_0 / \Gamma \) and observe that \( \tilde{H} = \tilde{H}_1 \tilde{H}_2 \) where \( \tilde{H}_1 \) is perfect and \( [\tilde{H}_1, \tilde{H}_2] = 1 \). Since \( \tilde{\Gamma}_0 \cong 6M_{22} \) and \( 6M_{22} \) contains no subgroup which is the central product of two proper perfect subgroups (see Conway [5], p. 235), \( \tilde{H} \neq 1 \) and either \( H_1 \leq \Gamma_1 \) or \( H_2 \leq \Gamma_2 \). Assume that \( H_1 \leq \Gamma_1 \). Then by the same reasoning applied to \( \Gamma_0 / \Gamma_2 \), we have \( H_2 \leq \Gamma_2 \).

4. Proof of Theorem A. Let \( G \) be a group with \( O(G) = 1 \), \( A \) a standard component of \( G \) with \( A/\text{Z}(A) \cong J_4 \) and \( X = \langle A^\theta \rangle \). Furthermore, let \( K = C(A) \) and \( R \in \text{Syl}_2(K) \). It follows from (2.7) that \( Z(A) = 1 \) and from (2.8) that \( N(A) = KA \). We shall assume that \( G \) is a minimal counterexample to Theorem A. Thus \( X \neq A \) whereupon \( X \) is simple and \( G \leq \text{Aut}(X) \) by Lemma 2.5 [1].

(4.1) \(|R| = 2\). Consequently \( G = \langle X, R \rangle \).

**Proof.** Let \( g \in G - N(A) \) be chosen so that \( Q = K^g \cap N(A) \) has a Sylow 2 subgroup \( T \) of maximal order. If \( m(R) > 1 \), then by ([13], (3.2) and (3.3)), \( R \) is elementary abelian and we may choose \( g \) so that \( T = R^g \). On the other hand, if \( m(R) = 1 \) and \( T \) is trivial, then \( \Omega_1(R) \) is isolated in \( C(\Omega_1(R)) \), hence \( \Omega_1(R) \) is contained in \( Z^*(G) \) by [10] contradicting \( F^*(G) \) is simple. Thus in either case, we may assume that \( T \) is nontrivial.
Now $Q = N(A) = K \times A$ implies that $T$ is isomorphic to a subgroup of $A$ under the projection map $\pi: N(A) \rightarrow A$. An easy argument shows that $Q$ is tightly embedded in $QA$. Moreover, $\pi(Q)^a = \pi(Q^a)$ for $a \in A$ then implies that $\pi(Q)$ is normalized by $\langle C_A(a); a \in \pi(T)^a \rangle$. Assume first that $m(R) > 1$ so that $R$ is elementary abelian and $T = R^a$. Let $a \in \pi(T)^a$. Then $\pi(Q) \cap C_A(a)$ is a normal subgroup of $C_A(a)$ with Sylow 2 subgroup $\pi(T) \cong T$. The structure of $C_A(a)$ is given in (2.1) and from this we conclude that $a$ belongs to the class $(2,2)$ of $A$ and $\pi(Q) \cap C_A(a) = \pi(T) \cong E_{23}$. But $\pi(T)$ also contains involutions of the class $(2,2)$ and this gives a contradiction.

Assume finally that $m(T) = 1$ and let $\langle a \rangle = O_1(\pi(T))$. Arguing as before, $\pi(Q) \cap C_A(a)$ is a normal subgroup of $C_A(a)$ with Sylow 2 subgroup $\pi(T)$, hence by (2.1), $\pi(T)$ has order 2. Since $\pi(T) \cong T$, we may set $T = \langle ra \rangle$ with $1 \neq a \in A$ and $r \in R$. Now $[A, R] = 1$ gives $N_R(T) = C_R(r)$ and since $N_R(T) \cong T$ by [2, Theorem 2], we conclude that $R$ has order 2 proving the result.

Since $G$ is a minimal counterexample to Theorem A and $A$ is a standard component of $\langle R, X \rangle$, with $X = \langle A^X \rangle$, it follows that $\langle R, X \rangle$ is also a counterexample to Theorem A. Hence $G = \langle X, R \rangle$.

**Notation.** By (4.1), we may set $\langle z \rangle = R$ so that $G = \langle X, z \rangle$. Also $C(z) = O(C(z)) \times \langle z \rangle \times A$ by (2.7) and (2.8). Let $T_0 \in Syl_2(A)$, $T = \langle z \rangle \times T_0 \in Syl_2(C(z))$ and $\{V\} = \langle z \rangle \times U = \mathcal{S}_{12}(T)$ where $U = \mathcal{S}_{12}(T_0)$. Recall from (2.4) that $N_{C(z)}(V) = O(C(z)) \times \langle z \rangle \times UK$ where $UK = N_A(U)$, $K \cong M_2$, and $U$ is $F_2K$ isomorphic to the Fischer module.

(4.2) $z^g \cap A = \emptyset$.

**Proof.** Note that $z$ is not a square in $G$ whereas every involution of $A$ is a square by (2.1).

(4.3) Let $N = N(V)$. Then $z^g \cap V = zU$. $N = O(N) \times WK$ where $W = \langle z \rangle Y$ is special of order $2^{23}$ with $Z(W) = U$, $Y$ is a homocyclic abelian group of order $2^{23}$ invariant under $K$ and $Y/U$ is $F_2K$ isomorphic to $U$.

**Proof.** Since $C_N(z) = O(C(z)) \times \langle z \rangle \times UK$, it suffices, in light of (3.1), to show that $z \in Z(N)$. Assume in fact that $z \in Z(N)$. Then $V = J(T)$ and $T \in Syl_2(N)$ together imply that $T \in Syl_2(G)$. Furthermore $V$ is weakly closed in $N$ with respect to $G$ and so $N$ controls fusion of $C(V) = O(N) \times V$. But $V$ contains representatives of the classes of involutions of $C(z)$ and therefore $z$ is isolated in $C(z)$. Applying the $Z^*$ theorem [10], we then have $z \in Z^*(G)$ which is incompatible with $G \leq \text{Aut}(X)$. 

We continue our analysis using the structure and notation for $N$ set up in (4.3). In order to eliminate the ambiguity in the structure of $Y$ we need the following result.

(4.4) Let $\langle \delta \rangle \in \text{Syl}_7(A)$, $\Delta = C(\delta)$ and $\bar{\Delta} = \Delta/O(\Delta)$. Then either $
abla \cong S_5 \triangleleft Z_2$ or $\bar{\Delta} = E(\bar{\Delta})\langle \bar{z} \rangle$ where $E(\bar{\Delta}) \cong U_3(5), L_3(5)$ or $L_2(25)$.

Proof. According to (2.3), $C_A(\delta) = \langle \delta \rangle \times D$ where $D \cong S_5$. Moreover if $e$ and $d$ are involutions in $D'$ and $D - D'$ respectively, then by (2.1), $e \in (2_2)$ and $d \in (2_2)$. We shall first show that $z$ fuses to $zd$ and $ze$ in $\Delta$. We know from (4.3) that $z$ fuses to both $zd$ and $ze$ in $G$. Set $\Sigma = C(z)$ and assume that $(zd)^g = z$, $g \in G$. Now $C_B(zd)^g = C(\langle z, zd \rangle)^g = C(\langle z^g, z \rangle) = C_H(z^g)$. Since $z^g \cap H = \{ z \} \cup (zd)^h \cup (ze)^h$ and $C_H(zd) \neq C_H(ze)$, we may replace $g$ by $gh$, $h \in H$, if necessary, to insure that $z^g = zd$. Thus $C_B(zd)^g = C_B(zd)$. Let $B = O^*(C_H(zd)) = \langle z \rangle \times C_A(d)$ and $B = B/O_2(B) \cong \text{Aut}(M_{22})$. Since $B^g = B$ and $\langle \delta \rangle \in \text{Syl}_7(B)$, we may assume that $\langle \delta \rangle^g = \langle \delta \rangle$. If $\delta^g \sim \delta^{-1}$, then $g$ induces an automorphism of $O^*(B) \cong M_{22}$ in which an element of order 7 is inverted, a contradiction. Therefore $\delta^g \sim \delta$ in $U$ and again we may replace $g$ by $gb$, $b \in B$, if necessary to obtain $\delta^g = \delta$ as required.

We may prove that $z$ fuses to $ze$ in $\Delta$ in the exact same way making use of the fact that $O^*(C_H(zd))/O_2(C_H(zd)) \cong \text{Aut}(M_{22})$ by (2.1).

Returning to the structure of $\bar{\Delta} = \Delta/O(\Delta)$, we have $C_{\bar{\Delta}}(\bar{z}) = O(\bar{\Delta}) \times \langle \bar{z} \rangle \times \bar{D}$ so that $\bar{D}'$ is standard in $\bar{\Delta}$. Since $\bar{\Delta}$ has sectional 2 rank at most 4 by a result of Harada [14], we may apply the main theorem of [13] to conclude that $E(\bar{\Delta})$ is isomorphic (i) $A_5$, (ii) $A_5 \times A_5$, (iii) $L_3(4)$, (iv) $M_{12}$, (v) $U_3(5)$, (vi) $L_3(5)$, (vii) $L_4(25)$, or (viii) $A_7$. Furthermore except in case (i), $\bar{\Delta} \leq \text{Aut}(E(\bar{\Delta}))$. Since $\bar{zd} \sim \bar{z} \sim \bar{ze}$ in $\bar{\Delta}$, and $\bar{d} \sim \bar{z} \sim \bar{e}$ by (4.2), we may easily eliminate cases (i), (iii), (iv) and (viii) and show that in case (ii), $\bar{\Delta} \cong S_5 \triangleleft Z_2$.

REMARK. If $E(\bar{\Delta})$ is simple then both $O_{z',e}(\Delta)$ and $\Delta - O_{z',e}(\Delta)$ contain one class of involutions. In particular, $z \in O_{z',e}(\Delta)$ and $\bar{d} \not\sim z \not\sim e$ together imply that the classes $(2_2)$ and $(2_2)$ of $A$ fuse in $G$.

(4.5) $Y \cong E_{z^222}$.

Proof. It follows from (4.3) that either the result is true or $Y$ is homocyclic of exponent 4. Assume the latter for purpose of a contradiction. We know that $N = O(N) \times WK$. Thus if $\langle \delta \rangle \in \text{Syl}_7(K)$, and $\Delta = C(\delta)$, then the structure of $\bar{\Delta} = \Delta/O(\Delta)$ is given by (4.4). Now $C_T(\delta) \cong Z_i \times Z_i$ and $C_k(\delta)$ contains an element of order 3 which acts regularly on $C_T(\delta)$. This implies that $O^*(\Delta)$ contains a $Z_i \times Z_i$ subgroup and we conclude from (4.4) that $\bar{\Delta} = E(\bar{\Delta})\langle \bar{z} \rangle$ with
$E(\bar{J}) \cong L_6(5)$. Since $E(\bar{J})$ has wreathed Sylow 2 subgroups of order $2^5$ and $\bar{z}$ acts as the graph automorphism, $z$ must invert $C_{r}(\delta)$. But the set of all elements of $Y$ inverted by $z$ forms a subgroup of $Y$ properly containing $U$ and invariant under $K$ which forces $z$ to invert $Y$.

We claim that $Y$ is the unique $(Z_4)^n$ subgroup of $N$. In fact let $Y'$ be another such subgroup of $N$. Then $WK = \bar{W}K/V \cong E_{24} \cdot M_{24}$ together with $m_4(\bar{Y}') = 11$ gives $\bar{Y}' = \bar{W}$. Therefore $Y \subseteq W = \langle z \rangle Y$ and since $z$ inverts $Y$, we must have $Y = Y'$. This in turn implies that $W$ must be the unique subgroup of $N$ of its isomorphism type as well. In particular, if $N = N(W)$, then $W$ is weakly closed in its normalizer with respect to $G$. Hence $N$ contains a Sylow 2 subgroup of $G$ and this in turn forces $N$ to control fusion of $C(W) = O(N)U$. Now the 2 $N$ classes of involutions of $U$ are the sets $(2_1) \cap U$ and $(2_2) \cap U$ of $A$. Also in the remark following (4.4), we observed that the classes $(2_1)$ and $(2_2)$ of $A$ fuse in $G$ if $E(\bar{J}) \cong L_6(5)$. Thus $N$ must act transitively on $U$ which is clearly not the case and we conclude that $N < N(W)$.

We now investigate the structure of $N(W)$. First observe that $C(W) \leq C(V)$ gives $C(W) = UO(N)$. Set $\bar{N}(W) = N(W)/U$ and consider the action of $\bar{N}(W)$ on $\bar{W}$. Since $Y$ is characteristic in $W$, $\bar{Y}$ is normal in $\bar{N}(W)$. Also $C_{\bar{N}(W)}(\bar{z}) = \bar{N} = \langle \bar{z} \rangle \times O(\bar{N}) \times \bar{Y}K$. Therefore we may apply (3.1) to conclude that $N(W) = O(N) \times W^*K$ where $W^*$ is a 2-group containing $W$ invariant under $K$, $W = \langle z \rangle Y^*$ where $Y^*$ contains $Y$ and is invariant under $K$ with $\bar{Y}^*/\bar{Y}$ $F_2K$ isomorphic to $\bar{Y}$.

But $Y^*/Y$, $Y/U$ and $U$ are all $F_2K$ isomorphic, hence $|C_{r}(\delta)| = 2^s$ and this in turn gives $|C_{r}(\delta)| = 2^r$ which contradicts $|\Delta| = 2^s$.

(4.6) $W \in \text{Syl}_4(C(U))$. Hence $Y \in \text{Syl}_4(C(Y))$.

Proof. The second statement follows easily from the first. Now $z^g \cap Y = \emptyset$ together with $z^w = zU$ by (4.3) gives $\langle z^g \cap W \rangle = V$. Thus $V$ is weakly closed in $W$ with respect to $G$. This implies that $N_{\text{cent}}(W) = N \cap C(U) = O(N) \times W$ by (4.3), hence $W \in \text{Syl}_4(C(U))$ as required.

(4.7) Let $M = N(Y)$ and $\bar{M} = M/Y$. Then

(i) $C_M(\bar{z}) = \bar{N} = O(N) \times \langle \bar{z} \rangle \times K$.

(ii) $\bar{z} \in Z^*(\bar{M})$.

Proof. Suppose $z^\alpha \in zY$, $\alpha \in M$. Since $z^g \cap W = z^w = zU$ by (4.3), $\alpha w$ normalizes $V$, hence $\alpha w \in N$. This in turn implies that $\alpha \in N$ and we see that $\bar{N} = C_M(z) = O(N) \times \langle \bar{z} \rangle \times K$, proving (i).
To prove (ii), let $b$ be an involution of $UK - U$. Since $z$ fuses to $za$ for any involution $a \in A$ by (4.3), there exists $g \in G$ such that $z^g = zb$. By (2.4), we see that $m_2(C(zb)) = 12$ and all $E_{q_2}$ subgroups of $C(zb)$ are conjugate. Therefore $\langle zb, C_T(zb) \rangle = V^{e_h}$ for some $h \in C(zb)$. Observe that $C_T(zb)$ is generated by those involutions of $\langle zb, C_T(zb) \rangle$ which are not conjugate to $zb$. Hence $U^{e_h} = C_T(zb)$.

Also $W \in \text{Syl}_2(C(U))$ by (4.6) implies that $W^{e_h} \in \text{Syl}_2(C(C_T(zb)))$. Since $\langle Y, zb \rangle \in \text{Syl}_2(C(C_T(zb)))$ as well, there exists $k \in G$ such that $W^{e_h} = \langle Y, zb \rangle$. Finally, $z^{e_hk} \in z^g \cap \langle Y, zb \rangle = (zb)^g$ implies that $z^{e_hkl} = zb$ for $l \in \langle Y, zb \rangle$. Setting $g' = ghk$, we have $z^{g'} = zb$ and $W^{g'} = \langle Y, zb \rangle$. Therefore $Y^{g'} = Y$ and $z \sim zb$ in $M$. We have shown that $\bar{z} \sim zb$ in $\bar{M}$ and thus $\bar{z} \in Z^*(\bar{M})$.

(4.8) $M = O(M)(M_1 \times M_2)\langle z \rangle$ where $M_1 = M_2 \cong E_{q_1} \cdot M_{q_2}$.

**Proof.** If follows from (4.7) that $C_T(\bar{z}) = \langle \bar{z} \rangle \times \bar{K}$ and $\bar{z} \in Z^*(\bar{K})$. Therefore, by a result of Koch [18] and (3.1), $\bar{M} = O(\bar{M})E(\bar{M})\langle \bar{z} \rangle$ where $E(\bar{M}) \cong M_1 \times M_2$. Let $M_1$ and $M_2$ be the minimal normal subgroups of $M$ which map onto the direct factors of $E(\bar{M})$. By (3.2), $Y = U_i \times U_2$ where $[M_i, U_i] = U_i$ and $[M_i, U_j] = 1$, $i \neq j$. It is clear that $O_2(M_i) \neq U_i$ or $O_2(M_i) = Y$, $i = 1, 2$. Assume the latter happens and set $\bar{M}_i = M_i/U_i$. Since $M_i$ is perfect and $U_2$ is central in $M_i$, $\bar{M}_i$ is a perfect central extension of $E_{q_1}$ by $M_{q_2}$. But this contradicts the fact that $M_{q_2}$ has trivial multiplier [4]. Therefore $O_2(M_i) = U_i$, $i = 1, 2$. Now $M_i \cap M_2 \leq O_2(M_i) \cap O_2(M_2) = U_i \cap U_2 = 1$ gives $M_iM_2 = M_i \times M_2$. Finally $M_1 = M_2 \cong C_{M_1M_2}(z) \cong E_{q_1} \cdot M_{q_2}$ proving the result.

**Notation.** From (4.8), let $M_0 = (M_1 \times M_2)\langle z \rangle$ with $M_1 = M_2 \cong E_{q_1} \cdot M_{q_2}$. Set $M_i = U_iK_i$ with $U_i = O_2(M_i)$, $K_i \cong M_{q_2}$ and set $M_i = U_2K_2$ with $U_2 = U_i$, $K_2 = K_i$. Furthermore, let $UK = C_{M_1M_2}(z)$ with $U = U_1U_2(z)$ and $K = C_{K_1K_2}(z)$. Finally, let $S_1 \in \text{Syl}_2(M_1)$, $S_2 = S_1 \in \text{Syl}_2(M_2)$, $S = S_1 \times S_2$ and $S^* = \langle S, z \rangle \in \text{Syl}_2(M_0)$.

(4.9) $S^* \in \text{Syl}_2(G)$, $S = S^* \cap X \in \text{Syl}_2(X)$ and $z \in X$.

**Proof.** First observe that all involutions of $S^* - S$ are conjugate in $S^*$ to $z$ and $C_{S^*}(z) \in \text{Syl}_2(C(z))$. Furthermore, it is easy to see that $z^g \cap S = \varnothing$. In fact, if $s$ is an involution of $S$, then $C_T(s) = C_T(s) \times C_T(s)$ has order at least $2^a$ gives $m_2(C_T(s)) \geq 13$ whereas $m_2(C(z)) = 12$ by (2.4). Therefore $z^{S^*} = m_2 \cap S$ and we have at once that $S^* \in \text{Syl}_2(G)$. It is clear from the Thompson transfer lemma that $z \in O^2(G)$. Since $G = \langle X, z \rangle$, we have $X = O^2(G)$. Thus $z \in X$. Also $S \leq O^2(M_0) \leq X$ gives $S = S^* \cap X \in \text{Syl}_2(X)$. 


(4.10) Let \( \gamma \) be an element of order 3 of \( A \) and \( \Gamma = C(\gamma) \). Then
\[
\Gamma = O(\Gamma)E(\Gamma)\langle z \rangle \text{ where } E(\Gamma) = \Gamma_1 \times \Gamma_2 \text{ and } \Gamma_i = \Gamma_2 \cong 6M_{22}.
\]

**Proof.** First observe from (2.2) that \( C_i(z) = O(C(z)) \times \langle z \rangle \times C_A(\gamma) \) where \( C_A(\gamma) \cong 6M_{22} \). Also by (2.2) we may assume that \( \gamma \) belongs to the class (3) of \( UK \). Thus we may write \( \gamma = \gamma _1 \gamma _2 \) where \( \gamma _1 = \gamma _i \) and \( \gamma _i \) belongs to the class (2) of \( M_i, i = 1, 2 \). Applying (2.6) gives
\[
C_{M_0}(\gamma) = \langle C_{M_1}(\gamma), C_{M_2}(\gamma) \rangle \langle z \rangle \text{ where } C_{M_1}(\gamma)^i = C_{M_2}(\gamma)^i \cong E_{29} \rtimes 3A_4.
\]
Since \( C_{M_1}(\gamma) \) is isomorphic to a 2-local subgroup of \( 6M_{22} \) which contains a Sylow 2 subgroup of \( 6M_{22} \), we may set \( R^* \in Syl_2(C_{M_0}(\gamma)) \) where \( R = \langle R_1 \times R_2 \rangle \langle z \rangle, R_2 \in Syl(C_{M_1}(\gamma)) \) and \( R_2 = R_3^* \) has type \( 2M_{22} \). Also \( R_1 \times R_2 \leq O^2(\Gamma) \). Thus by (3.5), \( \Gamma = O(\Gamma)E(\Gamma)\langle z \rangle \) where \( E(\Gamma)/O(\Gamma(\Gamma)) \cong 2M_{22} \times 2M_{22} \). But \( (C_{M_0}(\gamma))^{(\circ \circ)} = C_{M_1}(\gamma) \times C_{M_2}(\gamma) \leq E(\Gamma) \) then gives \( E(\Gamma) = \Gamma_1 \times \Gamma_2 \) where \( \Gamma_2 = \Gamma_1 \cong 6M_{22} \).

(4.11) Let \( \gamma_i \) and \( \tau_i \) be representatives of the classes (3) and (3) respectively of \( M_i \) with \( \gamma_i = \gamma_i \) and \( \tau_i = \tau_i \). Let \( \gamma = \gamma \gamma_i \tau_i \gamma_i \) and \( \tau = \tau \tau_i \tau_i \). Then \( \gamma, \tau_i, \tau_i \gamma_i \) and \( \gamma \) are conjugate in \( X \).

**Proof.** We know that \( \tau \) is conjugate to \( \gamma \) in \( A \) by (2.2). Since \( z \) leaves \( \gamma^2 \) invariant under conjugation and \( (\tau_i \gamma_i)^2 = \gamma_i \tau_i \), it suffices to show that \( \tau_i \gamma_i \) fuses to \( \gamma \) in \( X \). This in turn may be proved by verifying that \( \tau_i \) fuses to \( \gamma_i \) in \( C_X(\gamma_i) \). Let \( P_i \in Syl_3(M_i) \) with \( P_i = P_2, Z(P_i) = \langle \gamma_i \rangle \) and assume that \( \tau_i \in P_i, i = 1, 2 \). Since \( C_{M_0}(\gamma)^{(\circ \circ)} = C_{M_1}(\gamma) \times C_{M_2}(\gamma) \) is contained in \( E(\Gamma) = \Gamma_1 \times \Gamma_2 \), it follows from (3.6), that subject to reindexing, if necessary, \( C_{M_i}(\gamma_i) \leq \Gamma_i, i = 1, 2 \). In particular, \( P_i \in Syl_3(\Gamma_i) \) and \( \langle \gamma_i \rangle = O_3(\Gamma_i), i = 1, 2 \). Now \( P_1 \) contains an \( E_9 \) subgroup \( \langle \gamma_i, \gamma_i^* \rangle \) all of whose elements of order 3 are conjugate in \( M_i \) to \( \gamma_i \). On the other hand, \( M_{22} \) contains one class of elements of order 3, hence \( \tau_i \) is conjugate in \( \Gamma_i \) to an element of \( \langle \gamma_i, \gamma_i^* \rangle \). Therefore, \( \gamma_i \) is conjugate to \( \tau_i \) in \( \langle M_i, \Gamma_i \rangle \leq C_X(\gamma_i) \) as required.

(4.12) \( I(S_i) = U_i^X \cap I(S) \).

**Proof.** Since \( S \) has type \( J_s \times J_s = Y = J(S) \) by (2.4). Therefore \( N_X(Y) \) controls fusion of \( Y \) and we have that \( U_i^X \cap Y = U_i, i = 1, 2 \).

We now observe from (2.6) that every involution of \( M_i M_2 - Y \) centralizes an element of order 3 of \( M_i M_2 \), which is conjugate to \( \tau_i \tau_2 = \tau, \gamma_i \gamma_2 = \gamma, \gamma_i \gamma_2 \) or \( \gamma_i \gamma_2 \). Also \( C_{M_1}(\gamma_i) = C_{U_i}(\gamma_i)C_{K_1}(\gamma_i) \cong E_{29} \rtimes 3A_4 \) and \( C_{M_1}(\gamma_i) \cong C_{U_i}(\gamma_i)C_{K_1}(\gamma_i) \cong E_9(L_3(2) \times Z_3) \). In the course of proving (4.11), we showed that up to reindexing, it may be assumed that \( C_{M_1}(\gamma_i) \leq \Gamma_i, i = 1, 2 \). Let \( R = R_1 \times R_2 \in Syl_2(\Gamma_i \Gamma_i) \) where \( R_1 \in Syl_2(\Gamma_i) \) and \( R_2 \leq C_{M_1}(\gamma_i), i = 1, 2 \). By (3.4), \( Z(R_i) \) has order 4 and contains representatives of the 3 classes of involutions of \( \Gamma_i, i = 1, 2 \). But
then every involution of \( R_i \) is conjugate to an element of \( Z(R_i) \) whereas every involution of \( R - R_i \) is conjugate to an element of \( Z(R) - Z(R_i) \). Since \( Y \cap R = (U_i \cap R_i) \times (U_i \cap R_i) \) with \( U_i \cap R_i \cong E_2 \), we have \( Z(R_i) \leq U_i \) and \( Z(R) - Z(R_i) \leq U - U_i \). Therefore \( U_i^n \cap Y = U_i \) then yields \( Z(R_i)^x \cap Z(R) = Z(R_i) \). We now conclude that \( I(R_i) = U_i^n \cap I(R) \), \( i = 1,2 \) and this in turn gives \( I(\Gamma_i) = U_i^n \cap I(\Gamma) \), \( i = 1,2 \).

Our next objective is to show that \( I(C_{M_i}(\tau_i)) = U_i^n \cap I(C_{M_i,M_2}(\tau_i)) \), \( i = 1,2 \). By (4.11) there exists \( g \in X \) such that \( \tau^g = \gamma \), hence \( (C_{M_i,M_2}(\gamma))^g \leq C_X(\gamma) \). Since \( O^2(\tau^g) = (C_{M_i,M_2}(\tau))^g \), \( (C_{M_i}(\tau_i))^g \times (C_{M_2}(\tau_2))^g \leq O^2(\tau^g) = O^2(C_X(\gamma)) = \Gamma_i \Gamma_i \) by (3.5). Furthermore by (3.6), \( C_{M_i}(\tau_i) \leq \Gamma_j \) with \( j \neq j_2 \). But \( O_d(C_{M_i}(\tau_i)) = C_{U_i}(\tau_i) \cong E_8 \) combined with \( U_i^n \cap \Gamma_i = I(\Gamma_i) \) yields \( (C_{M_i}(\tau_i))^g \leq \Gamma_i \). Therefore \( I(C_{M_i}(\tau_i))^g = U_i^n \cap I(C_{M_i,M_2}(\tau_i))^g \) and this implies that \( I(C_{M_i}(\tau_i)) = U_i^n \cap I(C_{M_i,M_2}(\tau_i)) \), \( i = 1,2 \). The same argument then gives \( I(C_{M_i}(\tau_i)) = U_i^n \cap I(C_{M_i,M_2}(\tau_i)) \) and \( I(C_{M_i}(\tau_i)) = U_i^n \cap I(C_{M_i,M_2}(\tau_i)) \), \( i \neq j \). Since a conjugate of every involution of \( M_i,M_2 \) centralizes \( \gamma, \tau, \gamma_2 \), or \( \tau_1 \gamma_2 \), we see at once that \( I(M_i) = U_i^n \cap I(M,M_2) \), \( i = 1,2 \). Therefore \( I(S_i) = U_i^n \cap I(S) \), \( i = 1,2 \) proving the result.

(4.13) The following holds:

(i) \( S_i \) is a Sylow 2 subgroup of \( O^2(C_X(S_i)) \) and \( O^2(C_X(U_i)) \), \( i \neq j \).

(ii) Every involution of \( S_i \) is conjugate in \( C_X(S_j) \) to an element of \( U_i \), \( i \neq j \).

Proof. Since \( U_j \triangleleft S \), \( S_i \times U_j \in \text{Syl}_2(C_X(U_j)) \), \( i \neq j \). By Gaschutz's theorem we may write \( C_{\Omega}(U_j) = C_j U_j \) where \( C_j \) is a complement to \( U_j \) in \( C_X(U_j) \). Also \( U_j \) is central in \( C_X(U_j) \) gives \( C_X(U_j) = C_j \times U_j \). Clearly \( O^2(C_X(U_j)) \leq C_j \). Also \( S_i \leq M_i \) and \( [M_i, S_i] = 1 \) yields \( S_i \leq C_j \). It now follows directly that \( S_i \in \text{Syl}_2(O^2(C_X(U_j))) \). The same proof may be used to verify that \( S_i \in \text{Syl}_2(O^2(C_X(S_i))) \) and this completes the proof of (i).

In order to prove (ii), first observe that \( S_j = \Omega_i(S_j) \), hence by (4.12), \( S_j \) is weakly closed in \( S \) with respect to \( X \). Therefore \( N_X(S_j) \) controls fusion of \( C_X(S_j) \). Since \( S_i \in \text{Syl}_2(O^2(C_X(S_j))) \) by (i), the Frattini argument gives \( N_X(S_j) = C_X(S_j)N_X(S) \). Now \( N_X(S) \leq N_X(Y) \) where \( N_X(Y) = M \cap X = O(M)(M_i \times M_2) \). Clearly \( \tilde{S} \) is self normalizing in \( \tilde{M} = M \cap X/O(M) \) and this yields \( N_X(S) = O(N_X(S)) \). Consequently \( N_X(S_j) = C_X(S_j)S_j \). But \( [S_i, S_j] = 1 \) implies that \( C_X(S_j) \) controls fusion of \( S_i \times Z(S_j) \in \text{Syl}_2(C_X(S_j)) \) and the result now follows from (4.12).

(4.14) \( S_i \) is strongly closed in \( S \) with respect to \( X \), \( i = 1,2 \).
Proof. By symmetry, we need only prove the result for $S_\lambda$. Assume in fact that $S_\lambda$ is not strongly closed in $S$ with respect to $X$. Let $s_i \in S_i$ be an element of minimal order of $S_i$ such that $s_i^g \not\in S_i$. Then $s_i = s_is_i^g$ for some $g \in X$, $s_i \in S_i$, $i = 1, 2$, and $s_i^g \neq 1$. By (4.12), we may assume that $|s_i| > 2$. Also $(s_i^g)^a = (s_i^g)^b(s_i^g)^a$ together with the minimality of $|s_i|$ implies that $s_i'$ is an involution. By (4.13ii), $s_i'$ is conjugate in $C_X(S_i)$ to an element of $U_2$, so we may further assume that $s_i' \in U_2$. But $U_2$ is weakly closed in $S$ with respect to $X$ by (2.4) and (4.12), therefore $N_X(U_2)$ controls fusion of $C_X(U_2)$. A contradiction may now be established by observing that

$$s_i \in S_i \in \text{Syl}_2(O^2(C_X(U_2)))$$

whereas $s_is_i' \in O^2(C_X(U_2))$ by (4.13i).

We are now in the position to complete the proof of Theorem A. By (4.14) and the Aschbacher-Goldschmidt theorem [12], $X$ is not simple. This of course contradicts our condition that $X$ is simple and $G \leq \text{Aut } X$.

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