CONSTRUCTING NEW $R$-SEQUENCES

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R-sequences play an important role in modern commutative algebra. The purpose of this paper is to show how new R-sequences may be constructed from a given one. In the first section we give some general results, which are applied in the second section to obtain an explicit method of construction.

Recall that a sequence of elements \( x_1, \ldots, x_n \) in \( R \) is an \( R \)-sequence if \( (x_1, \ldots, x_n)R \neq R \), \( x_i \) is a nonzero divisor on \( R \), and for \( 2 \leq i \leq n \), \( x_i \) is a nonzero divisor on \( R/(x_1, \ldots, x_{i-1})R \).

Throughout this paper \( R \) will be a commutative noetherian ring which contains a field \( K \). Moreover, \( R \) will either be local or graded.

I wish to thank Melvin Hochster for showing me Proposition 1.5, which simplified this paper considerably.

1. It is easy to see that if \( x_1, \ldots, x_n \in R \) and \( X_1, \ldots, X_n \) are independent indeterminates over \( K \), and if \( \varphi: K[X_1, \ldots, X_n] \rightarrow R \) by \( \varphi(f(X_1, \ldots, X_n)) = f(x_1, \ldots, x_n) \) is a flat monomorphism, then \( x_1, \ldots, x_n \) is an \( R \)-sequence. The converse, when \( R \) is local, is due to Hartshorne [3].

**Proposition 1.1** (Hartshorne). Suppose \( R \) is local. If \( x_1, \ldots, x_n \in R \) form an \( R \)-sequence then \( \varphi: K[X_1, \ldots, X_n] \rightarrow R \) is a flat monomorphism, where \( \varphi \) is the map determined by \( \varphi(X_i) = x_i \) for each \( i \) and \( \varphi(a) = a \) for all \( a \in K \).

**Remark.** Saying that \( \varphi \) is a monomorphism is the same as saying that \( x_1, \ldots, x_n \) are algebraically independent over \( K \).

**Corollary 1.2.** Assume \( R \) is local. Suppose \( f_1, \ldots, f_n \) is a \( K[X_1, \ldots, X_n] \)-sequence, and each \( f_i \in (X_1, \ldots, X_n)K[X_1, \ldots, X_n] \). Suppose also that \( x_1, \ldots, x_n \) is an \( R \)-sequence. Then

\[
f_i(x_1, \ldots, x_n), \ldots, f_n(x_1, \ldots, x_n)
\]

is an \( R \)-sequence.

**Proof.** By Proposition 1.1 the map \( \varphi \) is a flat monomorphism. By flatness, since \( f_1, \ldots, f_n \) is a \( K[X_1, \ldots, X_n] \)-sequence, \( \varphi(f_1), \ldots, \varphi(f_n) \)
is an $R$-sequence. (The assumption that each $f_t \in (X, \ldots, X_n)$ guarantees that the $\varphi(f_t)$ generate a proper ideal of $R$.)

**Remark.** It is well-known (e.g., [4, Theorem 119]) that for any local noetherian ring $R$, a permutation of an $R$-sequence is again an $R$-sequence. However, if $R$ contains a field, the preceding result yields a very simple proof of this fact. For it is clear that for any permutation $\sigma$ of $\{1, \ldots, n\}$, $X_{\sigma(1)}, \ldots, X_{\sigma(n)}$ is a $K[X_1, \ldots, X_n]$-sequence. Letting $f_t = X_{\sigma(i)}$, we have $f_t(x_1, \ldots, x_n) = x_{\sigma(i)}$, and so by Corollary 1.2, $x_{\sigma(1)}, \ldots, x_{\sigma(n)}$ is an $R$-sequence.

We now give a graded analogue of Proposition 1.1. For in order to use Corollary 1.2 we need $K[X_1, \ldots, X_n]$-sequences.

**Proposition 1.3.** Assume $R$ is graded, and let $x_1, \ldots, x_n$ be homogeneous elements of $R$ of positive degree. Then $x_1, \ldots, x_n$ is an $R$-sequence $\iff$ (i) $x_1, \ldots, x_n$ are algebraically independent over $K$, and (ii) $R$ is a free $K[x_1, \ldots, x_n]$-module.

**Proof.** Let $A = K[x_1, \ldots, x_n]$.

(⇒) Assume (i) and (ii). Hence $A$ is a polynomial ring in $n$ variables and thus $x_1, \ldots, x_n$ is an $A$-sequence. Since $R$ is $A$-free, any $A$-sequence is an $R$-sequence.

(⇒) (i) follows from [5, p. 199].

(ii) $A$ is a graded subring of $R$, with grading induced by that of $R$. That is, if $R = \bigoplus \Sigma R_k$, let $A_k = A \cap R_k$. Then $\Sigma A_k$ is a direct sum, which we claim equals $A$. Since each $x_i$ is homogeneous, $x_i \in A_{m_i}$ for some integer $m_i \geq 1$. Also, $K \subseteq R$ and $R$ is graded, so $K \subseteq R_0$, and therefore $K = A_0$. Since every element $g$ of $A$ is a polynomial in the $x_i$'s with coefficients in $K$, it follows that $g \in \Sigma A_k$. Hence $A = \bigoplus \Sigma A_k$. Thus, with the grading on $A$ induced by that of $R$, and with the original grading on $R$, $R$ is a graded $A$-module. Now by [2, Ch. VIII, Thm. 6.1] since $A_0$ is a field and $R$ is a graded $A$-module, if $\text{Tor}_1^A(R, A_0) = 0$ then $R$ is $A$-free. Thus to prove (ii) it suffices to show that $\text{Tor}_1^A(R, K) = 0$.

We compute $\text{Tor}_1^R(R, K)$ by taking a projective resolution of $K$ over $A$ and tensoring it with $R$. Since $x_1, \ldots, x_n$ are algebraically independent over $K$, they form an $A$-sequence, and so the Koszul complex of the $x$'s over $A$ is exact and therefore yields a free $A$-resolution of $K$. Tensoring it with $R$ gives the Koszul complex of the $x$'s over $R$. But since by hypothesis the $x$'s form an $R$-sequence, this Koszul complex has zero homology ([1, Cor. 1.2] or
In particular, the first homology group, \( \text{Tor}_1(R, K) \), is 0, and we are done.

We have a graded analogue of Corollary 1.2. Its proof is nearly identical to the latter's and so we omit it.

**Corollary 1.4.** Suppose \( R \) is graded and \( x_1, \ldots, x_n \) is an \( R \)-sequence, where each \( x_i \) is homogeneous of positive degree. Suppose \( f_1, \ldots, f_n \) is a \( K[X_1, \ldots, X_n] \)-sequence with each \( f_i \in (X_1, \ldots, X_n) \). Then \( f_1(x_1, \ldots, x_n), \ldots, f_n(x_1, \ldots, x_n) \) is an \( R \)-sequence.

We close this section with a proposition due to M. Hochster.

**Proposition 1.5.** Let \( S \) be a graded Macaulay ring such that \( S_0 \) is local. Let \( x_1, \ldots, x_n \) be homogeneous elements of \( S \). If \( \text{rank}(x_1, \ldots, x_n) = n \) then \( x_1, \ldots, x_n \) is an \( S \)-sequence.

**Proof.** Let \( M = M_0 + \sum_{i=1}^n S_i \), where \( M_0 \) is the maximal ideal of \( S_0 \). Then \( M \) is maximal in \( S \) and contains every proper homogeneous ideal of \( S \). Let \( I = (x_1, \ldots, x_n) \), and localize at \( M \). Then in the local Macaulay ring \( S_M \), \( \text{rank}(f_M) = n \), so \( x_1, \ldots, x_n \) is an \( S_M \)-sequence, by [4, Thms. 129 and 136]. Let \( \mathcal{K} \) denote the Koszul complex of the \( x \)'s over \( S \). Then \( \mathcal{K} \otimes S_M \) is acyclic since it is the Koszul complex of the \( x \)'s over \( S_M \). Hence for each \( i \geq 1 \), the \( i \)th homology module \( H_i(\mathcal{K} \otimes S_M) = 0 \). Since \( S_M \) is \( S \)-flat we have \( H_i(\mathcal{K}) \otimes S_M = 0 \), so \( \text{ann}(H_i(\mathcal{K})) \not\subset M \). Since the \( x \)'s are homogeneous, \( \mathcal{K} \) is a complex of graded \( S \)-modules and hence \( H_i(\mathcal{K}) \) is also graded. But the annihilator of a graded module is a homogeneous ideal. Thus \( \text{ann}(H_i(\mathcal{K})) = S \) and so \( H_i(\mathcal{K}) = 0 \) for all \( i \geq 1 \). Therefore \( \mathcal{K} \) is acyclic, and so by [1, Prop. 2.8], \( x_1, \ldots, x_n \) is an \( S \)-sequence.

2. Any permutation \( \sigma \) in the symmetric group \( S_n \) acts as an automorphism on the polynomial ring \( K[X_1, \ldots, X_n] \) by

\[
(\sigma f)(X_1, \ldots, X_n) = f(X_{\sigma(1)}, \ldots, X_{\sigma(n)}).
\]

The next lemma is the key to our construction.

**Lemma 2.1.** Let \( \sigma \) be the cyclic permutation \((1, 2, \ldots, n)\) of order \( n \). Let \( K \) be a field, with \( a \in K \). Define a homogeneous polynomial \( f \in K[X_1, \ldots, X_n] \) by

\[
f(X_1, \ldots, X_n) = X_1^n - ag,
\]

where

\[
g = \prod_{i=1}^k X_{i_1}^{m_1} \cdots X_{i_k}^{m_k}, \quad 2 \leq i_1 < i_2 < \cdots < i_k \leq n, \quad \text{each } m_i \geq 1, \quad \text{and } \sum_{i=1}^k m_i = m.
\]

If \( a^n \neq 1 \), then the only common zero of \( f, \sigma f, \ldots, \sigma^{n-1}f \) in \( K^n \) is \((0, \ldots, 0)\).
Proof. We first treat a special case where the basic idea of the proof is not obscured by details. Suppose that \( k = n - 1 \), i.e., that each \( X_i, \ 2 \leq i \leq n \), divides the monomial \( g \). Let \((z_1, \ldots, z_n) \in K^n \) be a common zero of \( f, \sigma f, \ldots, \sigma^{n-1} f \). We have the following system of equations:

\[
\begin{align*}
    z_1^m &= a z_2^{m_2} \cdots z_{n-1}^{m_{n-1}} z_n^m \\
    z_2^m &= a z_3^{m_3} \cdots z_{n-1}^{m_{n-1}} z_1^m \\
    &\vdots \\
    z_n^m &= a z_1^{m_1} \cdots z_{n-2}^{m_{n-2}} z_{n-1}^{m_{n-1}}.
\end{align*}
\]

Equating the product of the left sides with the product of the right sides, and using the fact that \( \sum_{i=2}^{n-1} m_i = m \), we obtain:

\[
\left( \prod_{i=1}^{n} z_i \right)^m = a^n \left( \prod_{i=1}^{n} z_i \right)^{m_2} \cdots \left( \prod_{i=1}^{n} z_i \right)^{m_n} = a^n \left( \prod_{i=1}^{n} z_i \right)^m.
\]

But \( a^n \neq 1 \), so \( \prod_{i=1}^{n} z_i = 0 \) and thus some \( z_i = 0 \). For all \( i \) such that \( i \neq j \), \( z_j \) appears on the right side of the \( i \)th equation of the system above. Hence \( z_i = 0 \). Thus \((z_1, \ldots, z_n) = (0, \ldots, 0)\).

In the general case we shall break up the system of \( n \) equations into a number of subsystems, for each of which the preceding argument can be used.

Let \( H = \langle \sigma^i, \ldots, \sigma^k \rangle \) be the subgroup of the cyclic group \( \langle \sigma \rangle \) generated by \( \sigma^i, \ldots, \sigma^k \). Thus \( H \) is cyclic, of order dividing \( n \). In fact, \( H = \langle \sigma^b \rangle \) where \( b \) is the greatest common divisor of \( n, i_1, \ldots, i_k \).

We claim that if \( X_r \) divides \( \sigma^i(g) \), then \( r \equiv s \) \( \pmod{b} \). For \( r = \sigma^i(i_c) \) for some \( c, 1 \leq c \leq k \). Thus \( r \equiv s + i_c \) \( \pmod{n} \). Since \( b \) is a common divisor of \( i_c \) and \( n \), it follows that \( r \equiv s \) \( \pmod{b} \).

Now consider \( \prod_{s=1}^{n} \sigma^i(g) \). It is clearly invariant under \( \sigma \). But if \( \sigma(\prod_{s=1}^{n} X_s^{i_s}) = \prod_{s=1}^{n} X_s^{i_s} \), then \( a_1 = a_2 = \cdots = a_n \). Now since \( \deg g = m, \ \deg (\prod_{s=1}^{n} \sigma^i g) = nm \). Thus \( \prod_{s=1}^{n} \sigma^i g = \prod_{s=1}^{n} X_s^m \). On the other hand, for any \( r \),

\[
\prod_{s=r \pmod{b}}^{n} \sigma^i g = \left( \prod_{s=r \pmod{b}}^{n} \sigma^i g \right) \left( \prod_{s \neq r \pmod{b}}^{n} \sigma^i g \right),
\]

and if \( r \neq s \pmod{b} \) then \( X_s \) does not divide \( \sigma^i g \). Therefore

\[
\prod_{s=r \pmod{b}}^{n} \sigma^i g = \prod_{s=r \pmod{b}}^{n} X_s^m = \left( \prod_{s=r \pmod{b}}^{n} X_s \right)^m.
\]

Now suppose \((z_1, \ldots, z_n) \) is a common zero of \( f, \sigma f, \ldots, \sigma^{n-1} f \). Then for all \( 1 \leq s \leq n \), \( z_s^m = a(\sigma^i g)(z_1, \ldots, z_n) \). Hence

\[
\left( \prod_{s=r \pmod{b}}^{n} z_s \right)^m = a^n \left( \prod_{s=r \pmod{b}}^{n} (\sigma^i g)(z_1, \ldots, z_n) \right) = a^n \left( \prod_{s=r \pmod{b}}^{n} z_s \right)^m.
\]
Since \( a^n \neq 1 \), it follows that \( a^{n/b} \neq 1 \), and so \( z_s = 0 \) for some \( s \equiv r \) (mod \( b \)). We shall show that \( z_t = 0 \) for every \( t \equiv r \) (mod \( b \)).

For \( 1 \leq j \leq k \), \( X_{ij} \) divides \( g \): Thus \( X_i = \sigma^{t-i_j}(X_{ij}) \) divides \( \sigma^{t-i_j}(g) \), say \( x_h = \sigma^{t-i_j}(g) \). Now \( \sigma^{t-i_j}(f) = \sigma^{t-i_j}(X_i^n) - a\sigma^{t-i_j}(g) = X_i^n - ax_h \). If \( z_t = 0 \), then \( z_{s-i_j} = 0 \) since \((z_1, \ldots, z_n)\) is a zero of \( \sigma^{t-i_j}(f) \), and so \( z_{t-i_j} = 0 \). Thus for all \( j \) and for all \( q \) with \( q \equiv s \) (mod \( i_j \)), we have \( z_q = 0 \). This implies \( z_t = 0 \) for all \( t \equiv r \) (mod \( b \)). Since \( r \) was arbitrary, \((z_s, \ldots, z_n) = (0, \ldots, 0)\).

**Theorem 2.2.** Let \( K, \sigma, a, \) and \( f \) be as in the preceding lemma. Then \( f, \sigma f, \ldots, \sigma^{n-1}f \) is a \( K[X_1, \ldots, X_n] \)-sequence.

**Proof.** Let \( I = (f, \sigma f, \ldots, \sigma^{n-1}f) \) and let \( R = K[X_1, \ldots, X_n] \). Let \( S = \overline{K}[X_1, \ldots, X_n] \), where \( \overline{K} \) is the algebraic closure of \( K \). By Lemma 2.1 the variety of \( IS \) in \( \overline{K}^n \) contains only the origin. Hence by the Nullstellensatz, the radical of \( IS \) is the maximal ideal \((X_1, \ldots, X_n)S \). Therefore \( \operatorname{rank}(IS) = n \), and so by Proposition 1.5 \( f, \sigma f, \ldots, \sigma^{n-1}f \) is an \( S \)-sequence. Now \( S = R \otimes_K \overline{K} \), so \( S \) is \( R \)-free. Hence \( S \) is faithfully \( R \)-flat, and thus \( f, \sigma f, \ldots, \sigma^{n-1}f \) is also an \( R \)-sequence.

Combining Theorem 2.2 with Corollaries 1.2 and 1.4, we have:

**Corollary 2.3.** Suppose \( R \) contains a field \( K \), and \( x_1, \ldots, x_n \) is an \( R \)-sequence. Define \( f \in K[X_1, \ldots, X_n] \) as in Lemma 2.1, and assume \( a^n \neq 1 \). If \( R \) is local, or if \( R \) is graded and each \( x_i \) is homogeneous of positive degree, then

\[
f(x_1, \ldots, x_n), (\sigma f)(x_1, \ldots, x_n), \ldots, (\sigma^{n-1} f)(x_1, \ldots, x_n)
\]

is an \( R \)-sequence.

**Remark.** Since \( f \) is a homogeneous polynomial of positive degree, when the original \( R \)-sequence consists of homogeneous elements of positive degree, the same is true for the resulting \( R \)-sequence. Thus in the graded case as well as in the local case, the procedure may be iterated.

**Example.** Let \( R = K[X, Y, Z] \), where \( X, Y, Z \) are independent indeterminates. By Theorem 2.2, if \( a^2 \neq 1 \), then \( X^2 - aYZ, Y^2 - aXZ, Z^2 - aXY \) is an \( R \)-sequence, and if \( b \in K \) and \( b^2 \neq 1 \), then \( X^3 - bY^3, Y^3 - bZ^3, Z^3 - bX^3 \) is another. Hence by Corollary 2.3, \( (X^2 - aYZ)^2 - b(Y^2 - aXZ)^2, (Y^2 - aXZ)^2 - b(Z^2 - aXY)^2, (Z^2 - aXY)^2 - b(X^2 - aYZ)^2 \) is again an \( R \)-sequence, as is \( (X^3 - bY^3)^2 - a(Y^3 - bZ^3)(Z^3 - bX^3), (Y^3 - bZ^3)^2 - a(Z^3 - bX^3)(X^3 - bY^3), (Z^3 - bX^3)^2 - a(X^3 - bY^3)(Y^3 - bZ^3) \).
\[ bX^2 + a(X^3 - bY^3)(Y^3 - bZ^3). \]

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