

Pacific Journal of Mathematics

CONSTRUCTING NEW *R*-SEQUENCES

MARK BERNARD RAMRAS

CONSTRUCTING NEW R -SEQUENCES

MARK RAMRAS

R -sequences play an important role in modern commutative algebra. The purpose of this paper is to show how new R -sequences may be constructed from a given one. In the first section we give some general results, which are applied in the second section to obtain an explicit method of construction.

Recall that a sequence of elements x_1, \dots, x_n in R is an R -sequence if $(x_1, \dots, x_n)R \neq R$, x_1 is a nonzero divisor on R , and for $2 \leq i \leq n$, x_i is a nonzero divisor on $R/(x_1, \dots, x_{i-1})R$.

Throughout this paper R will be a commutative noetherian ring which contains a field K . Moreover, R will either be local or graded.

I wish to thank Melvin Hochster for showing me Proposition 1.5, which simplified this paper considerably.

1. It is easy to see that if $x_1, \dots, x_n \in R$ and X_1, \dots, X_n are independent indeterminates over K , and if $\varphi: K[X_1, \dots, X_n] \rightarrow R$ by $\varphi(f(X_1, \dots, X_n)) = f(x_1, \dots, x_n)$ is a flat monomorphism, then x_1, \dots, x_n is an R -sequence. The converse, when R is local, is due to Hartshorne [3].

PROPOSITION 1.1 (Hartshorne). *Suppose R is local. If $x_1, \dots, x_n \in R$ form an R -sequence then $\varphi: K[X_1, \dots, X_n] \rightarrow R$ is a flat monomorphism, where φ is the map determined by $\varphi(X_i) = x_i$ for each i and $\varphi(a) = a$ for all $a \in K$.*

REMARK. Saying that φ is a monomorphism is the same as saying that x_1, \dots, x_n are algebraically independent over K .

COROLLARY 1.2. *Assume R is local. Suppose f_1, \dots, f_n is a $K[X_1, \dots, X_n]$ -sequence, and each $f_i \in (X_1, \dots, X_n)K[X_1, \dots, X_n]$. Suppose also that x_1, \dots, x_n is an R -sequence. Then*

$$f_1(x_1, \dots, x_n), \dots, f_n(x_1, \dots, x_n)$$

is an R -sequence.

Proof. By Proposition 1.1 the map φ is a flat monomorphism. By flatness, since f_1, \dots, f_n is a $K[X_1, \dots, X_n]$ -sequence, $\varphi(f_1), \dots, \varphi(f_n)$

is an R -sequence. (The assumption that each $f_i \in (X_1, \dots, X_n)$ guarantees that the $\varphi(f_i)$ generate a *proper* ideal of R .)

REMARK. It is well-known (e.g., [4, Theorem 119]) that for *any* local noetherian ring R , a permutation of an R -sequence is again an R -sequence. However, if R contains a field, the preceding result yields a very simple proof of this fact. For it is clear that for any permutation σ of $\{1, \dots, n\}$, $X_{\sigma(1)}, \dots, X_{\sigma(n)}$ is a $K[X_1, \dots, X_n]$ -sequence. Letting $f_i = X_{\sigma(i)}$, we have $f_i(x_1, \dots, x_n) = x_{\sigma(i)}$, and so by Corollary 1.2, $x_{\sigma(1)}, \dots, x_{\sigma(n)}$ is an R -sequence.

We now give a graded analogue of Proposition 1.1. For in order to use Corollary 1.2 we need $K[X_1, \dots, X_n]$ -sequences.

PROPOSITION 1.3. *Assume R is graded, and let x_1, \dots, x_n be homogeneous elements of R of positive degree. Then x_1, \dots, x_n is an R -sequence \Leftrightarrow (i) x_1, \dots, x_n are algebraically independent over K , and (ii) R is a free $K[x_1, \dots, x_n]$ -module.*

Proof. Let $A = K[x_1, \dots, x_n]$.

(\Leftarrow) Assume (i) and (ii). Hence A is a polynomial ring in n variables and thus x_1, \dots, x_n is an A -sequence. Since R is A -free, any A -sequence is an R -sequence.

(\Rightarrow) (i) follows from [5, p. 199].

(ii) A is a graded subring of R , with grading induced by that of R . That is, if $R = \bigoplus \Sigma R_k$, let $A_k = A \cap R_k$. Then ΣA_k is a direct sum, which we claim equals A . Since each x_i is homogeneous, $x_i \in A_{m_i}$ for some integer $m_i \geq 1$. Also, $K \subset R$ and R is graded, so $K \subset R_0$, and therefore $K = A_0$. Since every element g of A is a polynomial in the x_i 's with coefficients in K , it follows that $g \in \bigoplus \Sigma A_k$. Hence $A = \bigoplus \Sigma A_k$. Thus, with the grading on A induced by that of R , and with the original grading on R , R is a graded A -module. Now by [2, Ch. VIII, Thm. 6.1] since A_0 is a field and R is a graded A -module, if $\text{Tor}_1^A(R, A_0) = 0$ then R is A -free. Thus to prove (ii) it suffices to show that $\text{Tor}_1^A(R, K) = 0$.

We compute $\text{Tor}_1^A(R, K)$ by taking a projective resolution of K over A and tensoring it with R . Since x_1, \dots, x_n are algebraically independent over K , they form an A -sequence, and so the Koszul complex of the x 's over A is exact and therefore yields a free A -resolution of K . Tensoring it with R gives the Koszul complex of the x 's over R . But since by hypothesis the x 's form an R -sequence, this Koszul complex has zero homology ([1, Cor. 1.2] or

[2, Ch. VIII, 4.3]). In particular, the first homology group, $\text{Tor}_1^4(R, K)$, is 0, and we are done.

We have a graded analogue of Corollary 1.2. Its proof is nearly identical to the latter's and so we omit it.

COROLLARY 1.4. *Suppose R is graded and x_1, \dots, x_n is an R -sequence, where each x_i is homogeneous of positive degree. Suppose f_1, \dots, f_n is a $K[X_1, \dots, X_n]$ -sequence with each $f_i \in (X_1, \dots, X_n)$. Then $f_1(x_1, \dots, x_n), \dots, f_n(x_1, \dots, x_n)$ is an R -sequence.*

We close this section with a proposition due to M. Hochster.

PROPOSITION 1.5. *Let S be a graded Macaulay ring such that S_0 is local. Let x_1, \dots, x_n be homogeneous elements of S . If $\text{rank}(x_1, \dots, x_n) = n$ then x_1, \dots, x_n is an S -sequence.*

Proof. Let $M = M_0 + \sum_{i \geq 1} S_i$, where M_0 is the maximal ideal of S_0 . Then M is maximal in S and contains every proper homogeneous ideal of S . Let $I = (x_1, \dots, x_n)$, and localize at M . Then in the local Macaulay ring S_M , $\text{rank}(f_M) = n$, so x_1, \dots, x_n is an S_M -sequence, by [4, Thms. 129 and 136]. Let \mathcal{K} denote the Koszul complex of the x 's over S . Then $\mathcal{K} \otimes_S S_M$ is acyclic since it is the Koszul complex of the x 's over S_M . Hence for each $i \geq 1$, the i th homology module $H_i(\mathcal{K} \otimes S_M) = 0$. Since S_M is S -flat we have $H_i(\mathcal{K}) \otimes S_M = 0$, so $\text{ann}(H_i(\mathcal{K})) \not\subset M$. Since the x 's are homogeneous, \mathcal{K} is a complex of graded S -modules and hence $H_i(\mathcal{K})$ is also graded. But the annihilator of a graded module is a homogeneous ideal. Thus $\text{ann}(H_i(\mathcal{K})) = S$ and so $H_i(\mathcal{K}) = 0$ for all $i \geq 1$. Therefore \mathcal{K} is acyclic, and so by [1, Prop. 2.8], x_1, \dots, x_n is an S -sequence.

2. Any permutation σ in the symmetric group \mathcal{S}_n acts as an automorphism on the polynomial ring $K[X_1, \dots, X_n]$ by

$$(\sigma f)(X_1, \dots, X_n) = f(X_{\sigma(1)}, \dots, X_{\sigma(n)}).$$

The next lemma is the key to our construction.

LEMMA 2.1. *Let σ be the cyclic permutation $(1, 2, \dots, n)$, of order n . Let K be a field, with $a \in K$. Define a homogeneous polynomial $f \in K[X_1, \dots, X_n]$ by $f(X_1, \dots, X_n) = X_1^m - ag$, where $g = \prod_{i=1}^k X_{i_t}^{m_t}$, $2 \leq i_1 < i_2 < \dots < i_k \leq n$, each $m_t \geq 1$, and $\sum_{t=1}^k m_t = m$.*

If $a^n \neq 1$, then the only common zero of $f, \sigma f, \dots, \sigma^{n-1} f$ in K^n is $(0, \dots, 0)$.

Proof. We first treat a special case where the basic idea of the proof is not obscured by details. Suppose that $k = n - 1$, i.e., that each X_i , $2 \leq i \leq n$, divides the monomial g . Let $(z_1, \dots, z_n) \in K^n$ be a common zero of $f, \sigma f, \dots, \sigma^{n-1}f$. We have the following system of equations:

$$\begin{aligned} z_1^m &= a z_2^{m_2} \cdots z_{n-1}^{m_{n-1}} z_n^{m_n} \\ z_2^m &= a z_3^{m_2} \cdots z_n^{m_{n-1}} z_1^{m_n} \\ &\vdots \\ z_n^m &= a z_1^{m_2} \cdots z_{n-2}^{m_{n-1}} z_{n-1}^{m_n}. \end{aligned}$$

Equating the product of the left sides with the product of the right sides, and using the fact that $\sum_{i=2}^n m_i = m$, we obtain:

$$\left(\prod_{i=1}^n z_i \right)^m = a^n \left(\prod_{i=1}^n z_i \right)^{m_2} \cdots \left(\prod_{i=1}^n z_i \right)^{m_n} = a^n \left(\prod_{i=1}^n z_i \right)^m.$$

But $a^n \neq 1$, so $\prod_{i=1}^n z_i = 0$ and thus some $z_j = 0$. For all i such that $i \neq j$, z_j appears on the right side of the i th equation of the system above. Hence $z_i = 0$. Thus $(z_1, \dots, z_n) = (0, \dots, 0)$.

In the general case we shall break up the system of n equations into a number of subsystems, for each of which the preceding argument can be used.

Let $H = \langle \sigma^{i_1}, \dots, \sigma^{i_k} \rangle$ be the subgroup of the cyclic group $\langle \sigma \rangle$ generated by $\sigma^{i_1}, \dots, \sigma^{i_k}$. Thus H is cyclic, of order dividing n . In fact, $H = \langle \sigma^b \rangle$ where b is the greatest common divisor of n, i_1, \dots, i_k .

We claim that if X_r divides $\sigma^s(g)$, then $r \equiv s \pmod{b}$. For $r = \sigma^s(i_c)$ for some c , $1 \leq c \leq k$. Thus $r \equiv s + i_c \pmod{n}$. Since b is a common divisor of i_c and n , it follows that $r \equiv s \pmod{b}$.

Now consider $\prod_{s=1}^n \sigma^s(g)$. It is clearly invariant under σ . But if $\sigma(\prod_{i=1}^n X_i^{a_i}) = \prod_{i=1}^n X_i^{a_i}$, then $a_1 = a_2 = \dots = a_n$. Now since $\deg g = m$, $\deg(\prod_{s=1}^n \sigma^s g) = nm$. Thus $\prod_{s=1}^n \sigma^s g = \prod_{i=1}^n X_i^m$. On the other hand, for any r ,

$$\prod_{s=1}^n \sigma^s g = \left(\prod_{s \equiv r \pmod{b}} \sigma^s g \right) \left(\prod_{s \not\equiv r \pmod{b}} \sigma^s g \right),$$

and if $r \not\equiv s \pmod{b}$ then X_r does not divide $\sigma^s g$. Therefore

$$\prod_{s \equiv r \pmod{b}} \sigma^s g = \prod_{s \equiv r \pmod{b}} X_s^m = \left(\prod_{s \equiv r \pmod{b}} X_s \right)^m.$$

Now suppose (z_1, \dots, z_n) is a common zero of $f, \sigma f, \dots, \sigma^{n-1}f$. Then for all $1 \leq s \leq n$, $z_s^m = a(\sigma^s g)(z_1, \dots, z_n)$. Hence

$$\left(\prod_{s \equiv r \pmod{b}} z_s \right)^m = a^{n/b} \prod_{s \equiv r \pmod{b}} (\sigma^s g)(z_1, \dots, z_n) = a^{n/b} \left(\prod_{s \equiv r \pmod{b}} z_s \right)^m.$$

Since $a^n \neq 1$, it follows that $a^{n/b} \neq 1$, and so $z_s = 0$ for some $s \equiv r \pmod{b}$. We shall show that $z_t = 0$ for every $t \equiv r \pmod{b}$.

For $1 \leq j \leq k$, X_{i_j} divides g : Thus $X_t = \sigma^{t-i_j}(X_{i_j})$ divides $\sigma^{t-i_j}(g)$, say $x_t h = \sigma^{t-i_j}(g)$. Now $\sigma^{t-i_j}(f) = \sigma^{t-i_j}(X_1^m) - a\sigma^{t-i_j}(g) = X_{t-i_j}^m - ax_t h$. If $z_t = 0$, then $z_{t-i_j}^m = 0$ since (z_1, \dots, z_n) is a zero of $\sigma^{t-i_j}(f)$, and so $z_{t-i_j} = 0$. Thus for all j and for all q with $q \equiv s \pmod{i_j}$, we have $z_q = 0$. This implies $z_t = 0$ for all $t \equiv r \pmod{b}$. Since r was arbitrary, $(z_1, \dots, z_n) = (0, \dots, 0)$.

THEOREM 2.2. *Let K , σ , a , and f be as in the preceding lemma. Then $f, \sigma f, \dots, \sigma^{n-1}f$ is a $K[X_1, \dots, X_n]$ -sequence.*

Proof. Let $I = (f, \sigma f, \dots, \sigma^{n-1}f)$ and let $R = K[X_1, \dots, X_n]$. Let $S = \bar{K}[X_1, \dots, X_n]$, where \bar{K} is the algebraic closure of K . By Lemma 2.1 the variety of IS in \bar{K}^n contains only the origin. Hence by the Nullstellensatz, the radical of IS is the maximal ideal $(X_1, \dots, X_n)S$. Therefore $\text{rank}(IS) = n$, and so by Proposition 1.5 $f, \sigma f, \dots, \sigma^{n-1}f$ is an S -sequence. Now $S = R \otimes_K \bar{K}$, so S is R -free. Hence S is faithfully R -flat, and thus $f, \sigma f, \dots, \sigma^{n-1}f$ is also an R -sequence.

Combining Theorem 2.2 with Corollaries 1.2 and 1.4, we have:

COROLLARY 2.3. *Suppose R contains a field K , and x_1, \dots, x_n is an R -sequence. Define $f \in K[X_1, \dots, X_n]$ as in Lemma 2.1, and assume $a^n \neq 1$. If R is local, or if R is graded and each x_i is homogeneous of positive degree, then*

$$f(x_1, \dots, x_n), (\sigma f)(x_1, \dots, x_n), \dots, (\sigma^{n-1}f)(x_1, \dots, x_n)$$

is an R -sequence.

REMARK. Since f is a homogeneous polynomial of positive degree, when the original R -sequence consists of homogeneous elements of positive degree, the same is true for the resulting R -sequence. Thus in the graded case as well as in the local case, the procedure may be iterated.

EXAMPLE. Let $R = K[X, Y, Z]$, where X, Y, Z are independent indeterminates. By Theorem 2.2, if $a^2 \neq 1$, then $X^2 - aYZ, Y^2 - aXZ, Z^2 - aXY$ is an R -sequence, and if $b \in K$ and $b^3 \neq 1$, then $X^3 - bY^3, Y^3 - bZ^3, Z^3 - bX^3$ is another. Hence by Corollary 2.3, $(X^2 - aYZ)^3 - b(Y^2 - aXZ)^3, (Y^2 - aXZ)^3 - b(Z^2 - aXY)^3, (Z^2 - aXY)^3 - b(X^2 - aYZ)^3$ is again an R -sequence, as is $(X^3 - bY^3)^2 - a(Y^3 - bZ^3)(Z^3 - bX^3), (Y^3 - bZ^3)^2 - a(Z^3 - bX^3)(X^3 - bY^3), (Z^3 -$

$$bX^3)^2 - a(X^3 - bY^3)(Y^3 - bZ^3).$$

REFERENCES

1. M. Auslander and D. Buchsbaum, *Codimension and multiplicity*, Ann. Math., **68**, no. 3 (1958), 625-657.
2. H. Cartan and S. Eilenberg, *Homological Algebra*, Princeton University Press, 1956.
3. R. Hartshorne, *A property of A-sequences*, Bull. Soc. Math. France, **94** (1966), 61-65.
4. I. Kaplansky, *Commutative Rings*, Allyn and Bacon, 1970.
5. ———, *R-sequences and homological dimension*, Nagoya Math. J., **20** (1962), 195-199.

Received August 2, 1976.

NORTHEASTERN UNIVERSITY
BOSTON, MA 02115

PACIFIC JOURNAL OF MATHEMATICS

EDITORS

RICHARD ARENS (Managing Editor)

University of California
Los Angeles, California 90024

C. W. CURTIS

University of Oregon
Eugene, OR 97403

C. C. MOORE

University of California
Berkeley, CA 94720

J. DUGUNDJI

Department of Mathematics
University of Southern California
Los Angeles, California 90007

R. FINN AND J. MILGRAM

Stanford University
Stanford, California 94305

ASSOCIATE EDITORS

E. F. BECKENBACH

B. H. NEUMANN

F. WOLF

K. YOSHIDA

SUPPORTING INSTITUTIONS

UNIVERSITY OF BRITISH COLUMBIA
CALIFORNIA INSTITUTE OF TECHNOLOGY
UNIVERSITY OF CALIFORNIA
MONTANA STATE UNIVERSITY
UNIVERSITY OF NEVADA, RENO
NEW MEXICO STATE UNIVERSITY
OREGON STATE UNIVERSITY
UNIVERSITY OF OREGON
OSAKA UNIVERSITY

UNIVERSITY OF SOUTHERN CALIFORNIA
STANFORD UNIVERSITY
UNIVERSITY OF TOKYO
UNIVERSITY OF UTAH
WASHINGTON STATE UNIVERSITY
UNIVERSITY OF WASHINGTON
* * *
AMERICAN MATHEMATICAL SOCIETY
NAVAL WEAPONS CENTER

Charalambos D. Aliprantis and Owen Sidney Burkinshaw, <i>On universally complete Riesz spaces</i>	1
Stephen Richard Bernfeld and Jagdish Chandra, <i>Minimal and maximal solutions of nonlinear boundary value problems</i>	13
John H. E. Cohn, <i>The length of the period of the simple continued fraction of $d^{1/2}$</i>	21
Earl Vern Dudley, <i>Sidon sets associated with a closed subset of a compact abelian group</i>	33
Larry Finkelstein, <i>Finite groups with a standard component of type J_4</i>	41
Louise Hay, Alfred Berry Manaster and Joseph Goeffrey Rosenstein, <i>Concerning partial recursive similarity transformations of linearly ordered sets</i>	57
Richard Michael Kane, <i>On loop spaces without p torsion. II</i>	71
William A. Kirk and Rainald Schoneberg, <i>Some results on pseudo-contractive mappings</i>	89
Philip A. Leonard and Kenneth S. Williams, <i>The quadratic and quartic character of certain quadratic units. I</i>	101
Lawrence Carlton Moore, <i>A comparison of the relative uniform topology and the norm topology in a normed Riesz space</i>	107
Mario Petrich, <i>Maximal submonoids of the translational hull</i>	119
Mark Bernard Ramras, <i>Constructing new R-sequences</i>	133
Dave Riffelmacher, <i>Multiplication alteration and related rigidity properties of algebras</i>	139
Jan Rosiński and Wojbor Woyczynski, <i>Weakly orthogonally additive functionals, white noise integrals and linear Gaussian stochastic processes</i>	159
Ryōtarō Satō, <i>Invariant measures for ergodic semigroups of operators</i>	173
Peter John Slater and William Yslas Vélez, <i>Permutations of the positive integers with restrictions on the sequence of differences</i>	193
Edith Twining Stevenson, <i>Integral representations of algebraic cohomology classes on hypersurfaces</i>	197
Laif Swanson, <i>Generators of factors of Bernoulli shifts</i>	213
Nicholas Th. Varopoulos, <i>BMO functions and the $\bar{\partial}$-equation</i>	221