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Given an algebra C over a commutative ring k and an element (called a C-two-cocycle) $\sigma = \sum_i a_i \otimes b_i \otimes c_i$ in $C \otimes_k C \otimes_k C$ satisfying certain relations, Sweedler defined a new multiplication $*$ on C by $x*y = \sum_i a_i x b_i y c_i$ for all x, y in C and denoted C with this new multiplication by C^σ. This paper studies three rigidity properties which arise by asking whether:

(i) $C^\sigma \simeq C$ as algebras;

(ii) a certain functor from the category of C-bimodules to the category of C^σ-bimodules is an equivalence;

(iii) a certain functor from the category of algebras over C to the category of algebras over C^σ is an equivalence. For certain algebras over a field k (including finite dimensional algebras possessing a Wedderburn factor), these rigidity properties are shown to be equivalent to (respectively): (i) all k-separable subalgebras B of C are commutative and for a separability idempotent $\sum_i x_i \otimes y_i$ of B, $\{c \in C \mid \sum_i x_i c y_i = 0\}$ is an ideal with square $\{0\}$; (ii) all k-separable subalgebras of C are central; (iii) k is the only k-separable subalgebra of C.

We recall Sweedler's basic definitions [7] and determine some elementary properties of multiplication alteration in §§1 and 2. The behavior of an algebra under alteration by Waterhouse's C-two-cocycle $\sigma_e = e \otimes 1 + 1 \otimes e - (e \otimes 1)(1 \otimes e)$ associated with a k-separable subalgebra B of C having separability idempotent e is studied in §3.

Section 4 introduces the notion of dominance: the k-algebra C is said to dominate the k-algebra D (written $C > D$) if there is a C-two-cocycle σ with $D \simeq C^\sigma$. C is called rigid if $C > D$ implies $D \simeq C$. Dominance is a partial order on the class of k-algebras. In the course of proving this an alternate characterization of a C-two-cocycle σ in terms of the existence of a certain functor $F^\sigma: A(C) \rightarrow A(C^\sigma)$ is given. (For any k-algebra D, $A(D)$ is the category of k-algebras over D.) We provide a dominance description of the central simple algebras over a field k as the "highly nonrigid" algebras and characterize those algebras over a perfect field k with nilpotent Jacobson radical $J(C)$ and k-dim $C/J(C)$ finite which are rigid. The main step in our study of rigidity is a theorem which states that if the kernel of an idempotent algebra endomorphism p of C satisfies a certain nilpotency condition every C-two-cocycle σ is "equivalent" to the $p(C)$-two cocycle $p(\sigma)$ (cf. Theorem 4.7).

Section 5 deals with a notion of rigidity on the bimodule level. If $\sigma = \sum_i a_i \otimes b_i \otimes c_i$ is a C-two-cocycle and M is an object of the category $M(C)$ of C-bimodules, we define actions of C^σ on M by $x^\sigma * m = \sum_i a_i x b_i m c_i$ and $m * x^\sigma = \sum_i a_i m b_i x c_i$ for all x in C, m in M. Denoting the resulting C^σ-bimodule by M^σ, we obtain a functor $(\ )^\sigma: M(C) \to M(C^\sigma)$ taking M to M^σ which we show can also be described as the change of rings functor associated with a certain algebra map $\varphi_\sigma: C^\sigma \otimes_k C^0 \to C \otimes_k C^0$. C is called modularly rigid (modularly semi-rigid) if $(\ )^\sigma$ is an equivalence (dense) for all C-two-cocycles σ. If k is a field, we find $(\ )^{\sigma_e}$ dense for some separability idempotent e of $B \subseteq C$ implies B is central in C. We use this to prove: If k is a field, and C is a k-algebra with nilpotent Jacobson radical $J(C)$ and $C/J(C)$ locally finite, then C is modularly rigid iff C is modularly semi-rigid iff all k-separable subalgebras of C are central.

As mentioned above, σ being a C-two-cocycle is equivalent to the existence of a certain functor $F^\sigma: A(C) \to A(C^\sigma)$. In §6 we study these functors. We show that if C is commutative and σ is an Amitsur (i.e., invertible) C-two-cocycle, then F^σ is an equivalence of categories. C is called categorically rigid (categorically semi-rigid) if F^σ is an equivalence (dense) for all C-two-cocycles σ. The paper concludes with a theorem relating categorically rigid algebras and algebras with all two-cocycles invertible. This theorem includes: If k is a field, a k-algebra C with nilpotent Jacobson radical $J(C)$ and $C/J(C)$ locally finite is categorically rigid iff C is categorically semi-rigid iff C has no nontrivial k-separable subalgebras iff all C-two-cocycles are invertible.

The author wishes to thank Moss Sweedler for suggesting the study of these rigidity notions as a thesis problem and for directing this research.

1. Review of basic notions. Throughout this paper k will always denote at least a commutative ring with unit 1. By an algebra over k or a k-algebra we mean an associative, unitary algebra over k. Unadorned $\otimes$, Hom represent $\otimes_k$, Hom_k respectively. For any k-algebra C, we denote the n-fold tensor product $C \otimes \overset{n}{\cdots} \otimes C$ by $C^{\otimes n}$. Given a map $C \xrightarrow{f} D$ of k-algebras, we have an induced algebra map $C^{\otimes n} \to D^{\otimes n}$ for each n given by $x_1 \otimes \cdots \otimes x_n \mapsto f(x_1) \otimes \cdots \otimes f(x_n)$ for x_i in C which we denote by $f^{\otimes n}$ or by f if no confusion seems likely. If C is a k-algebra, we denote its opposite k-algebra by C^0 and we call a left $C \otimes C^0$-module a C-bimodule. By an ideal of the k-algebra C we mean a two-sided ideal of C. $J(C)$ denotes the Jacobson radical of C and $Z(C)$ denotes the center of C. By a central simple algebra over the field k we mean a finite k-dimensional

k-algebra C with no proper ideals and $Z(C) = k$. Semi-simple means that the Jacobson radical is trivial and the descending chain condition on left ideals holds.

In this section we give a brief review of the theory of multiplication alteration by two-cocycles introduced by Sweedler [7]. Given an algebra C over the commutative ring k, let $\sigma = \sum_i a_i \otimes b_i \otimes c_i$ be in $C \otimes C \otimes C$. We form a new k-algebra C^σ as follows. As an abelian group, C^σ is equal to C. For any x in C we use the notation x^σ to indicate that we are considering x as an element of C^σ. We define the product $*$ of any two elements x^σ and y^σ in C^σ by

$$x^\sigma * y^\sigma = (\sum_i a_i x b_i y c_i)^\sigma .$$

DEFINITION 1.1. σ is called a C-two-cocycle if

$$\text{(1.1a)} \qquad \sum_{i,j} a_i a_j \otimes b_j \otimes c_j b_i \otimes c_i = \sum_{i,j} a_i \otimes b_i a_j \otimes b_j \otimes c_j c_i$$

and there is an element e_σ in C with

$$\text{(1.1b)} \qquad \sum_i a_i e_\sigma b_i \otimes c_i = 1 \otimes 1 = \sum_i a_i \otimes b_i e_\sigma c_i .$$

If σ is a C-two-cocycle C^σ is an associative k-algebra with unit element e_σ^σ. This paper may be briefly described as follows: Given a k-algebra C and an arbitrary C-two-cocycle σ, we "compare" C^σ with C. In §§4 through 6 we investigate three ways of "comparing" C^σ with C, including whether $C^\sigma \simeq C$ as k-algebras.

EXAMPLE 1.2. Let C be a commutative k-algebra and $\sigma = \sum_i a_i \otimes b_i \otimes c_i$ be a C-two-cocycle. From (1.1b) $(\sum_i a_i b_i c_i) e_\sigma = 1$ and hence e_σ is invertible in C with $e_\sigma^{-1} = \sum_i a_i b_i c_i$. Since $x^\sigma * y^\sigma = (xye_\sigma^{-1})^\sigma$ for any x, y in C, the k-linear map $C \to C^\sigma$ given by $x \mapsto (xe_\sigma)^\sigma$ is a k-algebra map and is bijective since e_σ is invertible. Thus $C^\sigma \simeq C$.

Let $\sigma = \sum_i a_i \otimes b_i \otimes c_i$ and $\tau = \sum_i r_i \otimes s_i \otimes t_i$ be C-two-cocycles. Associated with any element $\delta = \sum_i u_i \otimes v_i$ in $C \otimes C$ we have a linear map $R^\delta: C^\sigma \to C^\tau$ given by $x^\sigma \mapsto (\sum_i u_i x v_i)^\tau$.

DEFINITION 1.3. σ is cohomologous to τ via δ, denoted $\sigma \sim^\delta \tau$, if

$$\sum_{i,j} u_i a_j \otimes b_j \otimes c_j v_i = \sum_{i,j,l} r_i u_j \otimes v_j s_i u_l \otimes v_l t_i ,$$
$$\sum_i u_i e_\sigma v_i = e_\tau .$$

Thus if $\sigma \sim^\delta \tau$, $R^\delta: C^\sigma \to C^\tau$ is a k-algebra map.

DEFINITION 1.4. $\delta = \sum_i u_i \otimes v_i$ is called vertible if there is an element $\bar{\delta} = \sum_i \bar{u}_i \otimes \bar{v}_i$ in $C \otimes C$ with

$$\text{(1.4a)} \qquad \sum_{i,j} u_i \bar{u}_j \otimes \bar{v}_j v_i = 1 \otimes 1 = \sum_{i,j} \bar{u}_i u_j \otimes v_j \bar{v}_i .$$

$\bar{\delta}$ is called the verse of δ.

Hence if $\sigma \sim^{\delta} \tau$ with δ vertible the map $C^{\delta} \xrightarrow{R^{\delta}} C^{\tau}$ is an isomorphism of k-algebras with inverse $R^{\bar{\delta}}$. Because of the existence of this nice isomorphism, we say that σ and τ are equivalent if $\sigma \sim^{\delta} \tau$ with δ vertible.

EXAMPLES 1.5.

(a) Let $C = k \oplus k$ and $f = (1, 0)$. Then

$$\sigma_f = 1 \otimes 1 \otimes 1 + f \otimes f \otimes 1 + 1 \otimes f \otimes f - f \otimes 1 \otimes f - 1 \otimes f \otimes 1$$

is a C-two-cocycle with $e_{\sigma_f} = 1$.

(b) Let $C = k[x]$ with $x^2 = 0$. Then

$$\sigma_x = 1 \otimes 1 \otimes 1 + x \otimes x \otimes 1 + 1 \otimes x \otimes x - x \otimes 1 \otimes x$$

is a C-two-cocycle with $e_{\sigma_x} = 1$. In addition, $\sigma_x \sim^{\delta} 1 \otimes 1 \otimes 1$ with $\delta = 1 \otimes 1 - x \otimes x$ vertible.

2. Structure of C inherited by C^{σ}. Let σ be a C-two-cocycle. If I is an ideal of C, we have an injective map {ideals of C} $\longrightarrow$ {ideals of C^{σ}} given by $I \mapsto I^{\sigma}$. Also, $(I^{\sigma})^2 = I^{\sigma} * I^{\sigma} \subseteqq (I \cdot I)^{\sigma} = (I^2)^{\sigma}$ and by induction $(I^{\sigma})^n \subseteqq (I^n)^{\sigma}$ for all n. Hence, if $J(C)$ is nilpotent $J(C)^{\sigma} \subseteqq J(C^{\sigma})$. If $C \xrightarrow{f} D$ is an algebra map, $f^{\otimes 3}(\sigma)$ is a D-two-cocycle with $e_{f(\sigma)} = f(e_{\sigma})$. In particular, if I is any ideal of C we may take $D = C/I$ and f the canonical projection $C \longrightarrow C/I$. If $C \subseteqq D$ we may take f to be the inclusion and in this way view a C-two-cocycle as a D-two-cocycle. If $C/J(C)$ is commutative and $C \xrightarrow{\pi} C/J(C)$ is the canonical projection, $\{C/J(C)\}^{\pi(\sigma)} \simeq C^{\sigma}/J(C)^{\sigma}$, and $C/J(C)$ and $C^{\sigma}/J(C)^{\sigma}$ are isomorphic by Example 1.2. Thus $J\{C^{\sigma}/J(C)^{\sigma}\} = \{0\}^{\sigma}$ and $J(C)^{\sigma} \supseteqq J(C^{\sigma})$.

For any x in $Z(C)$ and y in C, (1.1b) implies $(xe_{\sigma})^{\sigma} * y^{\sigma} = y^{\sigma} * (xe_{\sigma})^{\sigma}$. Therefore $(Z(C)e_{\sigma})^{\sigma} \subseteqq Z(C^{\sigma})$. The map $Z(C) \xrightarrow{i} Z(C^{\sigma})$ given by $x \mapsto (xe_{\sigma})^{\sigma}$ is an injective algebra map by (1.1b). Suppose $C/J(C)$ is commutative and let $\sigma = \sum_i a_i \otimes b_i \otimes c_i$ be a C-two-cocycle. Then

$$\{e_{\sigma} + J(C)\}\{\sum_i a_i b_i c_i + J(C)\} = 1 + J(C)$$

by (1.1b). Therefore $e_{\sigma} + J(C)$ is invertible in $C/J(C)$ which implies that e_{σ} is invertible in C. If we let $\tau = \sum_i a_i \otimes e_{\sigma} b_i \otimes e_{\sigma} c_i e_{\sigma}^{-1}$, τ is a C-two-cocycle with $e_{\tau} = 1$ and $\tau \sim^{\delta} \sigma$, where $\delta = 1 \otimes e_{\sigma}$ is vertible.

For convenience, we assemble our preceding comments and two easy consequences in the following lemma.

LEMMA 2.1. *Let σ be a C-two-cocycle.*
(i) *If I is an ideal of C, I^σ is an ideal of C^σ.*
(ii) *If $J(C)$ is nilpotent, $J(C)^\sigma \subseteq J(C^\sigma)$.*
(iii) *If $C/J(C)$ is commutative, $J(C^\sigma) \subseteq J(C)^\sigma$.*
(iv) *There is a k-algebra injection $Z(C) \hookrightarrow Z(C^\sigma)$.*
(v) *If C^σ is simple* (i.e., has no proper ideals), *C is simple.*
(vi) *If C^σ has center k, C has center k.*

3. **Waterhouse two-cocycles.** In this section, C is a fixed k-algebra and B is a k-separable subalgebra of C. We investigate some properties of a B-two-cocycle discovered by Waterhouse. Recall that the k-algebra B is separable over k iff there is an element $e = \sum_i a_i \otimes b_i$ in $B \otimes B$ (called a separability idempotent for B over k) with

$$(3.1) \qquad \sum_i a_i b_i = 1$$
$$\sum_i x a_i \otimes b_i = \sum_i a_i \otimes b_i x \qquad \text{for all } x \text{ in } B\,.$$

The reader may verify that $\sigma_e = e \otimes 1 + 1 \otimes e - (e \otimes 1)(1 \otimes e)$ is a B-two-cocycle with $e_{\sigma_e} = 1$.

DEFINITION 3.2. σ_e is called the Waterhouse two-cocycle associated to the separable k-algebra B with separability idempotent e.

The Waterhouse two-cocycle σ_e figures prominently in our work. In fact, Example 1.5(a) is the Waterhouse two-cocycle for $B = k \oplus k$ and separability idempotent $f \otimes f + (1-f) \otimes (1-f)$, $f = (1, 0)$. Using (3.1) it can be shown that $\sigma_e^2 = \sigma_e$ in $B^{\otimes 3}$. Since B is a subalgebra of C, we may view σ_e as a C-two-cocycle as mentioned in §2. We examine the algebra C^{σ_e} in detail.

Define $\Gamma_e\colon C \to C$ by $\Gamma_e(x) = \sum_i a_i x b_i$ for any x in C. Γ_e is a $Z_C(B)$-module endomorphism of C, where $Z_C(B) = \{x \text{ in } c \mid xb = bx \text{ for all } b \text{ in } B\}$. We have a $Z_C(B)$-module decomposition $C = Z_C(B) \oplus \operatorname{Ker} \Gamma_e$. Let a, b be in $Z_C(B)$, x, y be in $\operatorname{Ker} \Gamma_e$. Then it is easily seen from the definition of σ_e that

$$(3.3) \qquad \begin{aligned} a^{\sigma_e} * b^{\sigma_e} &= (ab)^{\sigma_e} \\ a^{\sigma_e} * y^{\sigma_e} &= (ay)^{\sigma_e} \\ x^{\sigma_e} * b^{\sigma_e} &= (xb)^{\sigma_e} \\ x^{\sigma_e} * y^{\sigma_e} &= 0^{\sigma_e}\,. \end{aligned}$$

Thus $(\operatorname{Ker} \Gamma_e)^{\sigma_e}$ is an ideal of C^{σ_e} with $(\operatorname{Ker} \Gamma_e)^{\sigma_e} * (\operatorname{Ker} \Gamma_e)^{\sigma_e} = \{0\}^{\sigma_e}$.

EXAMPLE 3.4. Let C be a central simple algebra of dimension

n over a field k and choose a separability idempotent e for C over k. If $x_1, \cdots, x_{n-1}$ are indeterminates over k,

$$C^{\sigma_e} \simeq k[x_1, \cdots, x_{n-1}]/\langle\{x_i x_j\}_{i,j=1}^{n-1}\rangle .$$

4. **Rigidity.** Using the method of multiplication alteration by two-cocycles, we introduce a partial order on the class of k-algebras and study a related rigidity property.

DEFINITION 4.1 (Sweedler). Let C and D be algebras over the commutative ring k. We say that C dominates D, written $C > D$, if there is a C-two-cocycle σ with $D \simeq C^\sigma$. C is called rigid if $C > D$ implies that $D \simeq C$.

Since for any k-algebra C the element $1 \otimes 1 \otimes 1$ is a C-two-cocycle, dominance is reflexive. To prove that dominance is transitive we first develop another approach to C-two-cocycles. Let $A(C)$ denote the category of k-algebras over C. The objects of $A(C)$ are k-algebra maps $C \xrightarrow{f} D$ with D a k-algebra. The morphisms are obvious. Let $\mathscr{A}(C)$ denote the category with objects $C \xrightarrow{f} D$, where C, D are k-modules with multiplications (i.e., k-linear maps $C \otimes C \rightarrow C$ and $D \otimes D \rightarrow D$) and f is a multiplicative k-module map. Again take the obvious morphisms. Note that $A(C)$ is a subcategory of $\mathscr{A}(C)$.

Given any element $\sigma = \sum_i a_i \otimes b_i \otimes c_i$ in $C \otimes C \otimes C$ and an object $C \xrightarrow{f} D$ of $A(C)$, we have an object $C^\sigma \xrightarrow{f^\sigma} D^{f(\sigma)}$ of $\mathscr{A}(C^\sigma)$ with the multiplication in $D^{f(\sigma)}$ given by

$$x^{f(\sigma)} * y^{f(\sigma)} = \sum_i f(a_i)xf(b_i)yf(c_i))^{f(\sigma)}$$

and $f^\sigma(x^\sigma) = f(x)^{f(\sigma)}$ for x, y in C. In this manner we obtain a functor $A(C) \xrightarrow{F^\sigma} \mathscr{A}(C^\sigma)$.

NOTATION. For any k-algebra C, we denote the free algebra obtained by adjoining three noncommuting indeterminants X, Y, Z by $C\{X, Y, Z\}$.

The following lemma gives a characterization of a C-two-cocycle σ in terms of the functor F^σ.

LEMMA 4.2 (Sweedler). *Let C be an algebra over the commutative ring k and σ be in $C \otimes C \otimes C$. The following are equivalent:*

(i) *σ is a C-two-cocycle.*

(ii) *The image of F^σ lies in $A(C^\sigma)$,* i.e., *F^σ is a functor from $A(C)$ to $A(C^\sigma)$.*

(iii) *$C\{X, Y, Z\}^\sigma$ is an associative unitary k-algebra.*

Proof. (i) $\Rightarrow$ (ii). If σ is a C-two-cocycle and $C \xrightarrow{f} D$ is in $A(C)$, $f(\sigma)$ is a D-two-cocycle and hence $C^\sigma \xrightarrow{f^\sigma} D^{f(\sigma)}$ is in $A(C^\sigma)$.

(ii) $\Rightarrow$ (iii). $C \xrightarrow{f} C\{X, Y, Z\}$ where $f(c) = c$ for all c in C is an object of $A(C)$ and thus by hypothesis $C^\sigma \xrightarrow{f^\sigma} C\{X, Y, Z\}^{f(\sigma)}$ is an object of $A(C^\sigma)$. Hence $C\{X, Y, Z\}^{f(\sigma)} = C\{X, Y, Z\}^\sigma$ is an associative unitary k-algebra.

(iii) $\Rightarrow$ (i). The unit e_σ^σ of $C\{X, Y, Z\}^\sigma$ must lie in C and we have $X^\sigma * e_\sigma^\sigma = X^\sigma = e_\sigma^\sigma * X^\sigma$ which implies (1.1b). By associativity, $X^\sigma * (Y^\sigma * Z^\sigma) = (X^\sigma * Y^\sigma) * Z^\sigma$ which implies (1.1a). Thus σ is a C-two-cocycle.

PROPOSITION 4.3. *Dominance is transitive.*

Proof. Suppose we have a C-two-cocycle $\sigma = \sum_i a_i \otimes b_i \otimes c_i$ and a C^σ-two-cocycle $\tau = \sum_i d_i^\sigma \otimes e_i^\sigma \otimes f_i^\sigma$. Let x, y be in C. Then writing out $(x^\sigma)^\tau * (y^\sigma)^\tau$ shows that we will be done if we can prove that

$$\gamma = \sum_{i_1, i_2, i_3, i_4, j} a_{i_1} a_{i_2} a_{i_3} a_{i_4} d_j b_{i_4} \otimes c_{i_4} b_{i_3} e_j c_{i_3} b_{i_2} \otimes c_{i_2} b_{i_1} f_j c_{i_1}$$

is a C-two-cocycle with $e_\gamma = e_\tau$ since then $C^\gamma \simeq (C^\sigma)^\tau$ via $x^\gamma \mapsto (x^\sigma)^\tau$. By Lemma 4.2 we have functors $F^\sigma\colon A(C) \to A(C^\sigma)$ and $F^\tau\colon A(C^\sigma) \to A((C^\sigma)^\tau)$. The composite $F^\tau \circ F^\sigma$ is just F^γ. Hence γ is a C-two-cocycle by Lemma 4.2. It is easily checked that $e_\gamma = e_\tau$.

Therefore dominance is a partial order on the class of k-algebras. In §6 we study the functors $A(C) \xrightarrow{F^\sigma} A(C^\sigma)$ in detail.

REMARKS 4.4. (a) Example 1.2 shows that commutative k-algebras are rigid.

(b) If $C = M(n, k)$ and $\sigma = \sigma_e$ is a Waterhouse two-cocycle for C, C^σ is commutative (cf. Example 3.4). Hence dominance is not symmetric.

The following two theorems provide a dominance characterization of central simple k-algebras.

THEOREM 4.5. *Let C be an algebra over a field k. If C dominates a separable k-algebra, C is separable over k.*

Proof. There exists a C-two-cocycle σ with C^σ k-separable. Hence by [6, Theorem 3.1] k-dim $C = k$-dim C^σ is finite. It then follows from Lemma 2.1 that $J(C)^\sigma \subseteq J(C^\sigma) = \{0\}^\sigma$, proving C to be semi-simple. Since $Z(C) \hookrightarrow Z(C^\sigma)$ by Lemma 2.1 and $Z(C^\sigma)$ is a commutative separable k-algebra [1, Theorem III.12], $Z(C)$ is k-separable. Therefore C is k-separable, again using [1, Theorem III.21].

THEOREM 4.6. *Let k be a field and C a k-algebra with k-dim $C=n$. The following are equivalent*:

(a) *C dominates a central simple k-algebra.*

(b) *C is a central simple k-algebra.*

(c) *$C > D$ for all k-algebras D with k-dim $D = n$.*

(d) *$C > k \oplus \overset{n}{\cdots} \oplus k$ and $C > k[x]/(x^n)$.*

(e) *C dominates a separable k-algebra and C dominates a purely inseparable k-algebra.*

Proof. Recall that an algebra A over the field k is a purely inseparable k-algebra [8, Definition 1] if the contraction map $A \otimes A^0 \rightarrow A$ given by $a \otimes b^0 \mapsto ab$ provides an $A \otimes A^0$ projective cover of A. If k-dim $A < \infty$, A is purely inseparable over k iff $A/J(A)$ is a purely inseparable (in the usual sense) field extension of k [8, Corollary 13(b)].

(a) $\Leftrightarrow$ (b). This is clear from the reflexive property of dominance and (v) and (vi) of Lemma 2.1.

(b) $\Rightarrow$ (c). Let D be any k-algebra of k-dimension n. We may then identify C and D as k-spaces. Since C is central simple, by [7, 1.3a and 1.6] we have a linear isomorphism $C \otimes C \otimes C \simeq \mathrm{Hom}\,(C \otimes C, C)$ given by

$$x_1 \otimes x_2 \otimes x_3 \longmapsto (y_1 \otimes y_2 \longmapsto x_1 y_1 x_2 y_2 x_3) \ .$$

Since a multiplication on D is a linear map $C \otimes C \rightarrow C$ we have an element σ in $C^{\otimes 3}$ with $C^\sigma \simeq D$ as k-algebras. By [7, Proposition 1.6] σ is a C-two-cocycle, and thus $C > D$.

(c) $\Rightarrow$ (d). Clear.

(d) $\Rightarrow$ (e). Clear since $k \oplus \overset{n}{\cdots} \oplus k$ is k-separable and $k[x]/(x^n)$ is k-purely inseparable.

(e) $\Rightarrow$ (b). By Theorem 4.5, C is k-separable. In particular, C is a finite k-dimensional semi-simple k-algebra and $Z(C)$ is k-separable. We have a C-two-cocycle τ with C^τ purely inseparable over k. Since $Z(C) \hookrightarrow C^\tau$ (cf. 2.1 (iv)), $Z(C)$ is purely inseparable k [8, Corollary 7(c)]. Thus $Z(C)$ is both separable and purely inseparable over k, which implies $Z(C) = k$ [8, Corollary 7(a)]. Since C is semisimple and $Z(C) = k$, it follows that C is simple.

We now study the structure of rigid algebras. The crucial theorem is

THEOREM 4.7. *Let k be a commutative ring and C be a k-algebra. Suppose there is a k-algebra map $p: C \rightarrow C$ such that $p^2 = p$ and $(\mathrm{Ker}\,(p)) \otimes C^0 + C \otimes (\mathrm{Ker}\,(p))^0 \subseteq J(C \otimes C^0)$. Then every C-two-cocycle is equivalent to a $p(C)$-two-cocycle.*

Proof. Let $\sigma = \sum_i a_i \otimes b_i \otimes c_i$ be a C-two-cocycle. Since

$$\begin{aligned}[p \otimes 1^0](\sum_i p(a_i e_o) b_i \otimes c_i^0) &= \sum_i p(a_i e_o) p(b_i) \otimes c_i^0 \\ &= [p \otimes 1^0](\sum_i a_i e_o b_i \otimes c_i^0) \\ &= [p \otimes 1^0](1 \otimes 1^0) = 1 \otimes 1^0\end{aligned}$$

and $\ker(p \otimes 1^0) \subseteq J(C \otimes C^0)$ by hypothesis, $\sum_i p(a_i e_o) b_i \otimes c_i^0$ is invertible in $C \otimes C^0$. Thus $\delta_1 = \sum_i p(a_i e_o) b_i \otimes c_i$ is vertible. Denote its verse by $\sum_i u_i \otimes v_i \cdot \tau_1 \sim^1 \sigma$ defines a C-two-cocycle τ_1 with $e_{\tau_1} = p(e_o)$ and it follows from the associativity relation for σ that

$$\tau_1 = \sum_{i,j} p(a_i) \otimes b_i u_j \otimes v_j c_i \ .$$

By an obvious analog of the argument used for δ_1 above, one may see that $\delta_2 = \sum_{i,j} p(a_i) \otimes b_i u_j p(e_o v_j c_i)$ is vertible. Call its verse $\sum_i x_i \otimes y_i$. Note that $\sum_i x_i \otimes y_i = \sum_i p(x_i) \otimes y_i$ by uniqueness of verse (uniqueness of inverse in $C \otimes C^0$).

$\tau_2 \sim^{\delta_2} \tau_1$ defines a C-two-cocycle τ_2 with $e_{\tau_2} = e_{\tau_1}$ and it follows from the associativity relation for τ_1 that

$$\tau_2 = \sum_{i,j,l} p(a_i) x_j \otimes y_j b_i u_l \otimes p(v_l c_i) \ .$$

Thus τ_2 is in $p(C) \otimes C \otimes p(C)$.

We claim that τ_2 in fact lies in $p(C)^{\otimes 3}$. To see this, apply the map $(1 \otimes m_e \otimes 1) \circ (1 \otimes p \otimes 1 \otimes 1)$ to the associativity relation for τ_2, where $m_e \colon C \otimes C \to C$ is given by $a \otimes b \mapsto a e_{\tau_2} b$. Since $p^2 = p$, this yields that τ_2 is in $p(C)^{\otimes 3}$.

THEOREM 4.8. *k perfect field. Let C be a k-algebra with $J(C)$ nilpotent and $C/J(C)$ locally finite* (i.e., every finite subset of $C/J(C)$ generates a finite dimensional k-algebra). *If every k-separable subalgebra B of C is commutative and* $\operatorname{Ker} \Gamma_e$ *is an ideal of square zero for some separability idempotent e of B, then C is rigid.*

Proof. Let $\sigma = \sum_{i=1}^n a_i \otimes b_i \otimes c_i$ be a C-two-cocycle and let D be the subalgebra of C generated by $\{a_i, b_i, c_i, e_o\}_{i=1}^n \cup J(C)$. Since $J(C)$ is a nilpotent ideal of D, $J(C) \subseteq J(D)$. The locally finiteness of $C/J(C)$ implies that $D/J(C)$ is finite dimensional. Hence the radical $J(D/J(C) = J(D)/J(C)$ of $D/J(C)$ is nilpotent. Since $J(C)$ is nilpotent, it follows that $J(D)$ is nilpotent.

$D/J(D)$ is k-separable. Hence by the Wedderburn Principal Theorem $D = B \oplus J(D)$ for some k-separable subalgebra B of C. (cf. [4, Theorem 72.19]. To remove the finite dimension restriction on D, induct on the index of nilpotency of $J(D)$.) By Theorem 4.7 σ is equivalent to a B-two-cocycle. Hence we need only show $C^\sigma \simeq C$ if σ is in $B^{\otimes 3}$. Since B is commutative by hypothesis, we may assume $e_\sigma = 1$.

Recall that $C = Z_C(B) \oplus \operatorname{Ker} \Gamma_e$. By hypothesis we may assume $\operatorname{Ker} \Gamma_e$ is an ideal of square zero. For a, b in $Z(C)$, x, y in $\operatorname{Ker} \Gamma_e$, we have

$$\begin{aligned} a^\sigma * b^\sigma &= (ab)^\sigma \\ a^\sigma * y^\sigma &= (ay)^\sigma \\ x^\sigma * b^\sigma &= (xb)^\sigma \\ x^\sigma * y^\sigma &= 0^\sigma = (xy)^\sigma . \end{aligned}$$

Thus $C^\sigma \simeq C$ via $c^\sigma \Leftrightarrow c$.

We now study dominance and Waterhouse two-cocycles in order to prove a partial converse of Theorem 4.8.

LEMMA 4.9. *Let C be a k-algebra with $J(C) = \{0\}$. Suppose that B is a k-separable subalgebra of C with separability idempotent e. If $C^{\sigma_e} \simeq C$, $B \subseteq Z(C)$.*

Proof. $J(C^{\sigma_e}) = \{0\}^{\sigma_e}$ so the nilpotent ideal $(\operatorname{Ker} \Gamma_e)^{\sigma_e}$ must be the zero ideal. Hence $C = Z_C(B)$ and B is central.

LEMMA 4.10. *Let C be an algebra over the field k, B a k-separable subalgebra with separability idempotent e. If $C^{\sigma_e} \simeq C$, B is commutative.*

Proof. Consider the canonical projection $C \overset{\pi}{\twoheadrightarrow} C/J(C)$. $\bar{B} = \pi(B)$ is a k-separable subalgebra of $\bar{C} = C/J(C)$ with separability idempotent $\bar{e} = \pi(e)$. Since $C^{\sigma_e} \simeq C$, we have $\bar{C}^{\sigma_{\bar{e}}} \simeq \bar{C}$. Hence by Lemma 4.9 $\bar{B}$ is central in $\bar{C}$. Thus for all x, y in B $xy - yx$ is in $B \cap J(C)$. Since B is finite dimensional over k, $B \cap J(C)$ is a nil ideal of B. Because B is separable, $J(B) = \{0\}$ and hence $B \cap J(C) = \{0\}$. Therefore B is commutative.

THEOREM 4.11. *Let k be a perfect field and C be a k-algebra with $J(C)$ nilpotent and k-dimension of $C/J(C)$ finite. If C is rigid every k-separable subalgebra B of C is commutative and* $\operatorname{Ker} \Gamma_e$ *is an ideal of square zero for some separability idempotent e for B.*

Proof. Every k-separable subalgebra of C is commutative by Lemma 4.10. Using the Wedderburn Principal Theorem we have $C = B_0 \oplus J(C)$ for some k-separable subalgebra B_0 of C. For any separability idempotent e_0 for B_0, $C^{\sigma_{e_0}} \simeq C$ implies that there is a k-

separable subalgebra B_1 of C with separability idempotent e_1 such that $B_1 \simeq B_0$ and $\operatorname{Ker} \Gamma_{e_1}$ is an ideal of square zero. Since any two Wedderburn factors of C are isomorphic by an inner automorphism of C (cf [4, p. 491]) and any k-separable subalgebra of C is contained in some Wedderburn factor, we are done.

Combining Theorems 4.8 and 4.11 we have

THEOREM 4.12. *Let k be a perfect field and C be a k-algebra with $J(C)$ nilpotent and k-dimension of $C/J(C)$ finite. Then C is rigid iff every k-separable subalgebra B of C is commutative and* $\operatorname{Ker} \Gamma_e$ *is an ideal of square zero for some separability idempotent e for B.*

5. **Modular rigidity.** Given a k-algebra C, we denote by $M(C)$ the category of C-bimodules. Let $\sigma = \sum_i a_i \otimes b_i \otimes c_i$ be a C-two-cocycle. If M is C-bimodule, we form a C^σ-bimodule from M in the following manner. Starting with an abelian group M^σ isomorphic with M via $m^\sigma \Leftrightarrow m$, we define left and right actions of C^σ on M^σ by

$$x^\sigma * m^\sigma = (\sum_i a_i x b_i m c_i)^\sigma$$

$$m^\sigma * x^\sigma = (\sum_i a_i m b_i x c_i)^\sigma \quad x \text{ in } C,\ m \text{ in } M .$$

Using the defining relations (1.1) of a C-two-cocycle, it is readily checked that this provides M^σ with a C^σ-bimodule structure. Given a C-bimodule map $M \xrightarrow{f} N$ we let $M^\sigma \xrightarrow{f^\sigma} N^\sigma$ by $f^\sigma(m^\sigma) = f(m)^\sigma$. These constructions define a faithful functor from $M(C)$ to $M(C^\sigma)$ which we denote by $(\ )^\sigma$.

We define a linear map

$$C^\sigma \otimes C^{\sigma 0} \xrightarrow{\varphi_\sigma} C \otimes C^0$$

$$x^\sigma \otimes y^{\sigma 0} \longrightarrow \sum_{i,j} a_i a_j x b_j \otimes (c_j b_i y c_i)^0 .$$

LEMMA 5.1. *φ_σ is a map of k-algebras.*

Proof. Since φ_σ is linear and $\varphi_\sigma(e_\sigma^\sigma \otimes e_\sigma^{\sigma 0}) = 1 \otimes 1^0$, we need only check that φ_σ is multiplicative. This follows from the two-cocycle associativity relation for σ:

$$\varphi_\sigma\{(x^\sigma \otimes y^{\sigma 0}) * (x_1^\sigma \otimes y_1^{\sigma 0})\} = \varphi_\sigma\{\sum_{i,j} (a_i x b_i x_1 c_i)^\sigma \otimes (a_j y_1 b_j y c_j)^{\sigma 0}\}$$

$$= \sum_{i,j,m,n} a_m a_n a_i x b_i x_1 c_i b_n \otimes (c_n b_m a_j y_1 b_j y c_j c_m)^0$$

$$= \sum_{i,j,m,n} a_m a_i x b_i x_1 c_i b_m a_n \otimes (b_n a_j y_1 b_j y c_j c_n c_m)^0$$

$$= \sum_{i,j,m,n} a_m x b_m a_i x_1 b_i a_n \otimes (b_n a_j y_1 b_j y c_j c_n c_i c_m)^0$$
$$= \sum_{i,j,m,n} a_m x b_m a_i a_n x_1 b_n \otimes (c_n b_i a_j y_1 b_j y c_j c_i c_m)^0$$
$$= \sum_{i,j,m,n} a_m x b_m a_i a_j a_n x_1 b_n \otimes (c_n b_j y_1 c_j b_i y c_i c_m)^0$$
$$= \{\sum_{m,i} a_m x b_m a_i \otimes (b_i y c_i c_m)^0\}\{\sum_{j,n} a_j a_n x_1 b_n \otimes (c_n b_j y_1 c_j)^0\}$$
$$= \varphi_\sigma(x^\sigma \otimes y^{\sigma^0}) \cdot \varphi_\sigma(x_1^\sigma \otimes y_1^{\sigma^0}) .$$

The change of rings functor induced by φ_σ is the functor that we called $(\ )^\sigma$.

Recall that a functor F from a category $\mathscr{A}$ to a category $\mathscr{B}$ is dense if given any object B of $\mathscr{B}$ there is an object A of $\mathscr{A}$ with $F(A)$ isomorphic to B in $\mathscr{B}$.

REMARKS 5.2. (a) $C^\sigma \otimes C^{\sigma^0}$ is a faithful $C^\sigma \otimes C^{\sigma^0}$-module. Thus if $(\ )^\sigma$ is dense, φ_σ is injective since $\varphi_\sigma(\sum_i x_i^\sigma \otimes y_i^{\sigma^0}) = 0$ implies $(\sum_i x_i^\sigma \otimes y_i^{\sigma^0}) * M^\sigma = \{0\}^\sigma$ for all M in $M(C)$.

(b) If φ_σ is an isomorphism, $(\ )^\sigma$ is an equivalence of categories.

(c) If C is a finite dimensional algebra over a field k, parts (a) and (b) imply that $(\ )^\sigma$ is an equivalence iff φ_σ is an isomorphism since k-dim $C \otimes C^0 = k$-dim $C^\sigma \otimes C^{\sigma^0}$.

DEFINITION 5.3 (Sweedler). Let C be a k-algebra. We say that C is modularly rigid (modularly semi-rigid) if $(\ )^\sigma$ is an equivalence (dense) for all C-two-cocycles σ.

Note that modular rigidity implies modular semi-rigidity. We will later show that for certain types of algebras over a field k, e.g., finite dimensional ones, modular rigidity is equivalent to modular semi-rigidity.

EXAMPLES 5.4. (a) Let C be a commutative ring and σ be a C-two-cocycle. By Example 1.2, e_σ is invertible. For x, y in C, $\varphi_\sigma(x^\sigma \otimes y^{\sigma^0}) = (x \otimes y^0)(e_\sigma^{-1} \otimes e_\sigma^{-1^0})$. Hence C is modularly rigid by (5.2b).

(b) Let $C = U(2, k)$, the algebra of upper triangular two by two matrices over k, and take σ_e to be the Waterhouse two-cocycle associated with $B = ke_{11} \oplus ke_{22}$ and separability idempotent $e = e_{11} \otimes e_{11} + e_{22} \otimes e_{22}$. Since $\varphi_{\sigma_e}(e_{12}^{\sigma_e} \otimes e_{12}^{\sigma_e 0}) = 0$ by direct calculation, $(\ )^{\sigma_e}$ is not dense by (5.2a). Thus $U(2, k)$ is rigid (by Theorem 4.7) but not modularly rigid.

The remainder of this section is devoted to studying the structure of modularly rigid algebras over a field k.

LEMMA 5.5. *Let C be an algebra over a commutative ring k. Suppose that $\sigma = \sum_i a_i \otimes b_i \otimes c_i$ is a C-two-cocycle and let*

$$z_\sigma = \sum_{i,j} a_i a_j \otimes_{Z(C)} b_j^0 \otimes c_j b_i \otimes_{Z(C)} c_i^0$$

in $C \otimes_{Z(C)} C^0 \otimes C \otimes_{Z(C)} C^0$. Then if z_σ is invertible, φ_σ is an isomorphism.

Proof. Let $z_\sigma^{-1} = \sum_l p_l \otimes_{Z(C)} r_l^0 \otimes s_l \otimes_{Z(C)} t_l^0$. Define a map Λ_σ: $C \otimes C^0 \to C^\sigma \otimes C^{\sigma 0}$ by

$$\Lambda_\sigma(\sum_i x_i \otimes y_i^0) = \sum_{i,l} (p_l x_i r_l)^\sigma \otimes (s_l y_i t_l)^{\sigma 0} .$$

Then $z_\sigma^{-1} z_\sigma = 1$ implies $\Lambda_\sigma \varphi_\sigma = I_{C^\sigma \otimes C^{\sigma 0}}$ and $z_\sigma z_\sigma^{-1} = 1$ implies $\varphi_\sigma \Lambda_\sigma = I_{C \otimes C^0}$. Hence $\Lambda_\sigma = \varphi_\sigma^{-1}$.

In preparation for the next theorem, we need the following

LEMMA 5.6. *k field. Let C be an algebraic k-algebra with all k-separable subalgebras of $C/J(C)$ central. Then every semi-simple subalgebra of $C/J(C)$ is commutative.*

This lemma may be proved using Wedderburn-Artin structure theory and the Jacobson-Noether theorem [5, Theorem 3.2.1].

THEOREM 5.7. *k field. Let C be a k-algebra with $J(C)$ nilpotent and $C/J(C)$ locally finite. If all k-separable subalgebras of C are central, φ_σ is an isomorphism for all C-two-cocycles σ.*

Proof. As in the proof of Theorem 4.8, let $\sigma = \sum_{i=1}^n a_i \otimes b_i \otimes c_i$ be a C-two-cocycle and let D be the subalgebra of C generated by $\{a_i, b_i, c_i, e_\sigma\}_{i=1}^n \cup J(C)$. If $\bar{B}$ is any k-separable subalgebra of $D/J(D)$, we may lift $\bar{B}$ isomorphically to a k-separable subalgebra $B \subseteq D \subseteq C$ by the Wedderburn Principal Theorem. By hypothesis, $B \subseteq Z(C)$, and so also $B \subseteq Z(D)$ and $\bar{B} \subseteq Z(D/J(D))$. Therefore $D/J(D)$ is commutative by Lemma 5.6.

Let $\bar{D} = D/J(D)$. Since $\bar{D}$ is commutative and finite dimensional, there exists a unique maximal k-separable subalgebra $\bar{S}$ of $\bar{D}$ and $\bar{D}$ is a purely inseparable $\bar{S}$-algebra (to see this, use structure theory to write $\bar{D}$ as a finite product of field extensions of k). Lift $\bar{S}$ via the Wedderburn Principal Theorem to a k-separable subalgebra S in the center of C.

Let $z = \sum_{i,j} a_i a_j \otimes_S b_j^0 \otimes c_j b_i \otimes_S c_i^0$. We claim that z is invertible. Once this is established, we would be able to complete the proof by noting that then the image z_σ of z in $C \otimes_{Z(C)} C^0 \otimes C \otimes_{Z(C)} C^0$ is invertible and hence Lemma 5.5 applies.

Thus we need only show that z is invertible. First, we note that z is in $D \otimes {}_S D^0 \otimes D \otimes {}_S D^0$. Since $J(D)$ is nilpotent, z is invertible in $(D \otimes {}_S D^0)^{\otimes 2}$ iff its image $\bar{z}$ under the natural map

$$D \otimes {}_S D^0 \otimes D \otimes {}_S D^0 \longrightarrow \bar{D} \otimes {}_{\bar{S}} \bar{D} \otimes \bar{D} \otimes {}_{\bar{S}} \bar{D}$$

is invertible.

Because $\bar{D}$ is purely inseparable over $\bar{S}$, the kernel of the contraction map $\bar{D} \otimes {}_{\bar{S}} \bar{D}^0 \xrightarrow{m} \bar{D}$ is contained in $J(\bar{D} \otimes {}_{\bar{S}} \bar{D}^0)$. $J(\bar{D} \otimes {}_{\bar{S}} \bar{D}^0)$ is nilpotent since $\bar{D} \otimes {}_{\bar{S}} \bar{D}^0$ is a finite dimensional k-algebra and hence $\bar{z}$ is invertible iff its image under the map $m \otimes m$ is invertible (note that m is an algebra map since $\bar{D}$ is commutative). Since $\{m \otimes m\}(x)$ clearly has inverse $\bar{e}_\sigma \otimes \bar{e}_\sigma$, we done.

We use Waterhouse two-cocycles to obtain the converse of the above theorem.

LEMMA 5.8. *Let C be an algebra over the field k and B a k-separable subalgebra of C with separability idempotent e. Then if φ_{σ_e} is injective B is central in C.*

Proof. If B were not central, we would have a nonzero x in $\operatorname{Ker} \Gamma_e$. Then $\varphi_{\sigma_e}(x^{\sigma_e} \otimes x^{\sigma_e 0}) = 0$ by explicit calculation.

COROLLARY. *k field. If C is modularly semi-rigid, all k-separable subalgebras B of C are central.*

Proof. Since C is modularly semi-rigid, in particular $(\ )^{\sigma_e}$ is dense for all Waterhouse two-cocycles σ_e. By Remark 5.2a we thus have φ_{σ_e} injective for all σ_e. Hence all k-separable subalgebras of C are central by the lemma.

We have thus shown

THEOREM 5.9. *k field. Let C be a k-algebra with $J(C)$ nilpotent and $C/J(C)$ locally finite. The following are equivalent:*

(a) *C is modularly rigid.*

(b) *C is modularly semi-rigid.*

(c) *All k-separable subalgebras of C are central.*

(d) *φ_σ is an isomorphism for all C-two-cocycles σ.*

6. Categorical rigidity. In this section we take a "functorial" approach to multiplication alteration by two-cocycles. As in §4 we let $A(C)$ denote the category of k-algebras over C. Recall that given a C-two-cocycle σ and an object $C \xrightarrow{f} D$ of $A(C)$, $f(\sigma)$ is a D-two-cocycle and $C^\sigma \xrightarrow{f^\sigma} D^{f(\sigma)}$ is an object of $A(C^\sigma)$ with $f^\sigma(x^\sigma) = f(x)^{f(\sigma)}$

for x in C. This map describes a faithful functor from $A(C)$ to $A(C^\sigma)$ which we denoted F^σ.

DEFINITION 6.1 (Sweedler). Let C be a k-algebra. We say that C is categorically rigid (categorically semi-rigid) if F^σ is an equivalence (dense) for all C-two-cocycles σ.

Note that categorical rigidity implies categorical semi-rigidity. We will later show that for certain types of algebras over a field k, e.g., finite dimensional ones, categorical rigidity is equivalent to categorical semi-rigidity.

Suppose σ, τ are C-two-cocycles with $\sigma \sim^\delta \tau$, $\delta = \sum_i u_i \otimes v_i$. Then the map $R^\delta \colon C^\sigma \to C^\tau$ given by $x^\sigma \mapsto (\sum_i u_i x v_i)^\tau$ induces a functor $A(C^\tau) \xrightarrow{\mathscr{R}^\delta} A(C^\sigma)$ by "composition." For $C \xrightarrow{f} D$ in $A(C)$, define

$$T_{C \xrightarrow{f} D} \colon F^\sigma(C \xrightarrow{f} D) \longrightarrow \mathscr{R}^\delta F^\tau(C \xrightarrow{f} D)$$

to be

$$\begin{array}{ccc} D^{f(\sigma)} & \xrightarrow{R^{f(\delta)}} & D^{f(\tau)} \\ {}^{f^\sigma}\nwarrow & & \nearrow{}_{f^\tau R^\delta} \\ & C^\sigma & \end{array}$$

in $A(C^\sigma)$. T describes a natural transformation F^σ to $\mathscr{R}^\delta F^\tau$. If δ is vertible the reader may check that T is a natural equivalence. Since $\mathscr{R}^\delta$ is an equivalence when δ is vertible we have

LEMMA 6.2. *Let σ, τ be C-two-cocycles with $\sigma \sim^\delta \tau$, δ vertible. Then F^σ is dense* (resp. full) *iff F^τ is dense* (resp. full).

We now direct our attention to the structure of algebras over a field k which are categorically rigid.

LEMMA 6.3. *Let C be an algebra over the commutative ring k. Suppose $\sigma = \sum_i a_i \otimes b_i \otimes c_i$ is a C-two-cocycle with $e_\sigma = 1$. Let*

$$w_\sigma = \sum_{i_1, i_2, i_3, i_4} (a_{i_1} a_{i_2} a_{i_3} a_{i_4})^0 \otimes b_{i_4} \otimes (c_{i_4} b_{i_3})^0 \otimes c_{i_3} b_{i_2} \otimes (c_{i_2} b_{i_1})^0 \otimes c_{i_1}$$

in $(C^0 \otimes C)^{\otimes 3}$. Then if w_σ is invertible there is a C^σ-two-cocycle τ with $F^\tau \circ F^\sigma = F^{1 \otimes 1 \otimes 1}$, which is the identity functor on $A(C)$.

Proof. There is an element

$$w_\sigma^{-1} = \sum_j u_j^0 \otimes v_j \otimes w_j^0 \otimes x_j \otimes y_j^0 \otimes z_j$$

in $(C^0 \otimes C)^{\otimes 3}$ with $w_\sigma^{-1}w_\sigma = 1$. This implies

$$\sum a_{i_1}a_{i_2}a_{i_3}a_{i_4}u_jv_jb_{i_4} \otimes c_{i_4}b_{i_3}w_jx_jc_{i_3}b_{i_2} \otimes c_{i_2}b_{i_1}y_jz_jc_{i_1} = 1 \otimes 1 \otimes 1 .$$

Thus we will be done if we show that

$$\tau = \sum_j (u_jv_j)^\sigma \otimes (y_jx_j)^\sigma \otimes (y_jz_j)^\sigma \equiv \sum_j d_j^\sigma \otimes e_j^\sigma \otimes f_j^\sigma$$

is a C^σ-two-cocycle with $e_\tau = 1$.

Let $C\{X, Y, Z\}$ be the free algebra on noncommuting indeterminants X, Y, Z as in Lemma 4.2. From the last paragraph we have $(C\{X, Y, Z\}^\sigma)^\tau \simeq C\{X, Y, Z\}$ as k-algebras via $(x^\sigma)^\tau \mapsto x$ for x in $C\{X, Y, Z\}$. In particular, $(C\{X, Y, Z\}^\sigma)^\tau$ is an associative algebra with unit element 1. This two-cocycle unitary property for τ is then a consequence of $(1^\sigma)^\tau*(X^\sigma)^\tau = (X^\sigma)^\tau = (X^\sigma)^\tau*(1^\sigma)^\tau$. Since $(C\{X, Y, Z\}^\sigma)^\tau$ is associative we have

$$\sum [[[[[(d_i^\sigma*d_j^\sigma)*X^\sigma]*e_j^\sigma]*Y^\sigma]*(f_j^\sigma*e_i^\sigma)]*Z^\sigma]*f_i^\sigma$$
$$= \sum [[[[[d_i^\sigma*X^\sigma]*(e_i^\sigma*d_j^\sigma)]*Y^\sigma]*e_j^\sigma]*Z^\sigma]*(f_j^\sigma*f_i^\sigma) .$$

The two-cocycle associativity relation for τ follows from this.

We have left the tedious verifications to the reader since they are straightforward applications of the two-cocycle relations for σ and the invertibility of w_σ.

COROLLARY. *Let C be commutative k-algebra and σ be an Amitsur two-cocycle* (i.e., an invertible C-two-cocycle). *Then F^σ is an equivalence.*

Proof. Since C is commutative, e_σ is invertible and hence we may assume $e_\sigma = 1$ by Lemma 6.2. w_σ is clearly invertible so by Lemma 6.3 there is a C^σ-two-cocycle τ with $F^\tau \circ F^\sigma = F^{1\otimes 1\otimes 1}$. Thus F^τ is dense and F^σ is full. It is easy to see that τ is an invertible C^σ-two-cocycle by its construction and another application of Lemma 6.3 proves that F^τ is full. Hence F^τ is an equivalence, which implies that F^σ is dense.

THEOREM 6.4. *Let C be an algebra over a field k with $J(C)$ nilpotent and $C/J(C)$ locally finite. Then C has no k-separable subalgebras* (except k) $\Leftrightarrow$ *all C-two-cocycles are invertible.*

Proof. ($\Leftarrow$) If C had a nontrivial k-separable subalgebra B, any Waterhouse B-two-cocycle σ_e would be a nontrivial idempotent element of $C \otimes C \otimes C$ and hence would not be invertible.

($\Rightarrow$) As in the proof of Lemma 4.8, let $\sigma = \sum_{i=1}^{n} a_i \otimes b_i \otimes c_i$ be a C-two-cocycle and let D be the subalgebra of C generated by $\{a_i, b_i, c_i, e_\sigma\}_{i=1}^{n} \cup J(C)$. If $\bar{S}$ is any k-separable subalgebra of $D/J(D)$, we may lift $\bar{S}$ isomorphically to a k-separable subalgebra $S \subseteqq D \subseteqq C$ by the Wedderburn Principal Theorem. By hypothesis, $S = k$ so $\bar{S} = k$. It follows from Wedderburn-Artin structure theory that $D/J(D)$ is a purely inseparable field extension of k. Since $J(D)$ is nilpotent σ is invertible iff $\bar{\sigma} = p(\sigma)$ is invertible, where $p: D \rightarrow D/J(D)$ is the natural map. But $\bar{\sigma}$ is a $D/J(D)$-two-cocycle and hence invertible [7, 2.15].

Note that $D/J(D)$ commutative implies that e_σ is invertible and σ is equivalent to a C-two-cocycle τ with $e_\tau = 1$ (cf. §2).

THEOREM 6.5. *Let C be an algebra over a field k with $J(C)$ nilpotent and $C/J(C)$ locally finite. If all C-two-cocycles are invertible, C is categorically rigid.*

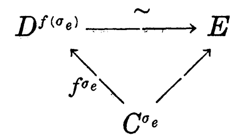
Proof. Let σ be a C-two-cocycle. By the remark at the end of Theorem 6.4 and Lemma 6.2 we may assume that $e_\sigma = 1$. Let D be as in Theorem 6.4 and consider the element w_σ in $(D^0 \otimes D)^{\otimes 3}$ as in Lemma 6.3. Since $J(D)$ is nilpotent, w_σ is invertible iff $\bar{w}_\sigma = p(w_\sigma)$ is invertible, where $p: D \rightarrow D/J(D)$ is the natural map. Because σ is invertible and $D/J(D)$ is commutative, $\bar{w}_\sigma$ is clearly invertible. Therefore by Lemma 6.3 we have a C^σ-two-cocycle τ with $F^\tau \circ F^\sigma = F^{1\otimes 1\otimes 1}$. F^σ is full, F^τ is dense, and we will be done if we show F^τ is also full, i.e. F^τ is an equivalence.

Let E be the subalgebra of C^σ generated by $\{d_j^\sigma, e_j^\sigma, f_j^\sigma\} \cup J(C)^\sigma$. Noting how $\tau = \sum_j d_j^\sigma \otimes e_j^\sigma \otimes f_j^\sigma$ arose, we have $E \subseteqq D^\sigma$. Since $J(C)^\sigma$ is a nilpotent ideal of E, $J(C)^\sigma \subseteqq J(E)$ and $E/J(C)^\sigma \subseteqq D^\sigma/J(C)^\sigma = (D/J(C))^\sigma$. Hence $E/J(C)^\sigma$ is finite dimensional and it follows that $J(E)$ is nilpotent. C^σ has no k-separable subalgebras and thus $E/J(E)$ is commutative by Wedderburn-Artin theory. The invertibility of τ follows easily from the proof of Lemma 6.3 and it follows that w_τ is invertible in $(E^0 \otimes E)^{\otimes 3}$. Thus F^τ is full by Lemma 6.3.

Now we use Waterhouse two-cocycles to prove the converse of the above theorem.

LEMMA 6.6. *Let k be a field, C be a k-algebra, and B a k-separable subalgebra with separability idempotent e and associated Waterhouse two-cocycle σ_e. Then, if F^{σ_e} is dense, $B = k$.*

Proof. Let $E = \mathrm{End}_k(C^{\sigma_e})$ and $C^{\sigma_e} \hookrightarrow E$ be given by $x^{\sigma_e} \mapsto$ (left multiplication by x^{σ_e}). There is an object $C \xrightarrow{f} D$ in $A(C)$ with



commutative. In particular, f is injective.

$\bar{B} = f(B)$ is a k-separable subalgebra of D with separability idempotent $\bar{e} = f(e)$ and associated Waterhouse two-cocycle $\bar{\sigma} = f(\sigma_e)$. Thus we have $D^{\bar{\sigma}} = Z_D(\bar{B})^{\bar{\sigma}} \oplus (\mathrm{Ker}\,\Gamma_{\bar{e}})^{\bar{\sigma}}$ with $(\mathrm{Ker}\,\Gamma_{\bar{e}})^{\bar{\sigma}}$ an ideal of square zero. Since $J(D^{\bar{\sigma}}) = \{0\}^{\bar{\sigma}}$ and $Z(D^{\bar{\sigma}}) = k$, we have $f(B) = \bar{B} \subseteq k$. Because f is an injective k-algebra map we have $B = k$.

Combining the above results we have

THEOREM 6.7. *Let C be an algebra over the field k with $J(C)$ nilpotent.*

Consider the following statements:

(1) $\sigma \sim \delta_\sigma 1 \otimes 1 \otimes 1$ for some vertible δ_σ for all C-two-cocycles σ.
(2) C is categorically rigid.
(3) C is categorically semi-rigid.
(4) C has no k-separable subalgebras (except k).
(5) All C-two-cocycles are invertible.

Then

(a) $(1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (4) \Leftarrow (5)$.
(b) If $C/J(C)$ is locally finite, (2)-(5) are equivalent.

Proof. (a) $(1) \Rightarrow (2)$ follows from Lemma 6.2. $(2) \Rightarrow (3)$ is trivial and $(3) \Rightarrow (4)$ holds by Lemma 6.6. For $(5) \Rightarrow (4)$, see Theorem 6.4, proof of $(\Leftarrow)$.

(b) $(2) \Rightarrow (3) \Rightarrow (4) \Leftarrow (5)$ by part (a). $(4) \Rightarrow (5)$ and $(5) \Rightarrow (2)$ follow from Theorem 6.4 and Theorem 6.5, respectively.

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