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**PERMUTATIONS OF THE POSITIVE INTEGERS WITH
RESTRICTIONS ON THE SEQUENCE OF DIFFERENCES**

PETER JOHN SLATER AND WILLIAM YSLAS VÉLEZ

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Let $\{a_k\}$ be a sequence of positive integers and $d_k = |a_{k+1} - a_k|$. We say that $\{a_k\}$ is a permutation if every positive integer appears once and only once in the sequence, $\{a_k\}$. We prove the following: Let $\{m_i\}$ be any sequence of positive integers, then there exists a permutation $\{a_k\}$ such that $|\{k|d_k = i\}| = m_i$.

By a permutation $\{a_k|k \in N\}$, where N denotes the set of positive integers, we shall mean a sequence of positive integers such that every element of N appears once and only once in the sequence $\{a_k|k \in N\}$. Set $d_k = |a_{k+1} - a_k|$. The purpose of this paper is to answer, in the affirmative, two questions which were raised by Roger Entringer at the University of New Mexico.

Question 1. Can one construct a permutation $\{a_k|k \in N\}$ such that given any interger n , $|\{k|d_k = n\}| \leq C$, where C is some fixed constant which is independent of n ?

Question 2. Can one construct a permutation $\{a_k|k \in N\}$ such that $\{d_k|k \in N\}$ is also a permutation?

These questions are similar in nature to a problem described in [2] as having been solved by M. Hall. A solution by J. Browkin appears in [1], and the problem is to find a subset A of N such that every natural number is the difference of precisely one pair of numbers of the set A . Note that in this problem one considers all differences and not just differences formed by adjacent members in a sequence.

Let us consider the following procedure for constructing a sequence. Let $a_1 = 1, a_2 = 2$. We define a_3 as follows: Let a_3 be the smallest integer, which has not already appeared in the sequence, such that the difference $|a_3 - a_2|$ has also not appeared. Clearly, $a_3 = 4$. Assume that a_1, a_2, \dots, a_t have been defined in this way. Define a_{t+1} by the following conditions: (i) $|a_{t+1} - a_t| \neq d_i, i < t$, (ii) $a_{t+1} \neq a_i, i < t + 1$, and (iii) a_{t+1} is the smallest positive integer with properties (i) and (ii).

Clearly, every integer appears at most once in the sequences $\{a_k|k \in N\}$ and $\{d_k|k \in N\}$. But are these sequences permutations? The next theorem settles this question for the sequence $\{a_k|k \in N\}$.

THEOREM 1. *The sequence, $\{a_k|k \in N\}$, constructed above is a permutation.*

Proof. Assume that this sequence is not a permutation. Let i be the smallest integer which does not appear in the sequence. Choose k so that $\{1, 2, \dots, i-1\} \subset \{a_1, \dots, a_k\}$. Choose subscripts k_1, k_2, \dots, k_{i+1} such that $k+1 \leq k_1 < k_2 < \dots < k_{i+1}$ and $a_{k_j} > a_i$, for $l < k_j$, that is, a_{k_j} is the largest integer to appear in $\{a_1, \dots, a_{k_j}\}$. Let $M = \max\{d_j \mid j = 1, \dots, k_{i+1} - 1\}$, $M_1 = \max\{d_j \mid j = 1, \dots, k_1 - 1\}$, $M_2 = \max\{d_j \mid j = k - 1, \dots, k_{i+1} - 1\}$. Then $M = \max\{M_1, M_2\}$. But $M_1 \leq a_{k_1} - 1$ and $M_2 \leq a_{k_{i+1}} - (i+1)$, since the smallest integer appearing in the sequence $\{a_{k+1}, a_{k+2}, \dots, a_{k_{i+1}}\}$ is larger than or equal to $(i+1)$. Hence $M \leq \max\{a_{k_1} - 1, a_{k_{i+1}} - (i+1)\}$. But $a_{k_1} - 1 \leq a_{k_2} - 2 \leq \dots \leq a_{k_{i+1}} - (i+1)$. So $M \leq a_{k_{i+1}} - (i+1) < a_{k_{i+1}} - i$. Hence $a_{k_{i+1}} - i > d_j$, $j = 1, \dots, k_{i+1} - 1$, and i is the smallest integer which has not been used, so we must have that $a_{k_{i+1}+1} = i$, which is a contradiction.

We have not been able to determine whether or not the sequence $\{d_k \mid k \in N\}$ is a permutation.

We next consider another way of constructing permutations so that the differences are also a permutation.

We say that $\{a_1, \dots, a_t\}$ has property 1 if the a_i are distinct and the $d_i = |a_{i+1} - a_i|$, $i = 1, \dots, t-1$, are also distinct.

Let i_t be the smallest integer not appearing in the set $\{a_1, \dots, a_t\}$, e_t is the smallest integer not appearing in the set $\{d_1, \dots, d_{t-1}\}$, $I_t = \max\{a_1, \dots, a_t\}$, $E_t = \max\{d_1, \dots, d_{t-1}\}$. Clearly $E_t < I_t$.

REMARK. Note that either $e_t < E_t$ or $e_t = E_t + 1$. In either case we have that $e_t \leq I_t$.

Rule 1. Set $a_{t+1} = 2I_t + 1$. If $e_t \leq i_t$, then set $a_{t+2} = a_{t+1} - e_t$. If $e_t > i_t$, then set $a_{t+2} = i_t$.

LEMMA 1. *If $\{a_1, \dots, a_t\}$ has property 1 and a_{t+1}, a_{t+2} are constructed according to Rule 1, then $\{a_1, \dots, a_t, a_{t+1}, a_{t+2}\}$ also has property 1.*

Proof. Clearly $a_{t+1} \cap \{a_1, \dots, a_t\} = \emptyset$ and $d_t = a_{t+1} - a_t = 2I_t + 1 - a_t = I_t + 1 + (I_t - a_t) \geq I_t + 1 > E_t$, so $d_t \cap \{d_1, \dots, d_{t-1}\} = \emptyset$.

Assume that $e_t \leq i_t$. Then $a_{t+2} = 2I_t + 1 - e_t = I_t + 1 + (I_t - e_t) \geq I_t + 1$. Hence $\{a_{t+2}\} \cap \{a_1, \dots, a_t\} = \emptyset$, so $\{a_1, \dots, a_t, a_{t+1}, a_{t+2}\}$ are $t+2$ distinct integers. Further $d_{t+1} = |a_{t+2} - a_{t+1}| = e_t$, so $\{d_1, \dots, d_{t+1}\}$ are $t+1$ distinct differences, hence $\{a_1, \dots, a_{t+2}\}$ has property 1.

Assume that $i_t < e_t$. Then $a_{t+2} = i_t$, so $\{a_1, \dots, a_{t+2}\}$ are $t+2$ distinct integers. Further $d_{t+1} = 2I_t + 1 - i_t = I_t + 1 + (I_t - i_t) > (I_t + 1) + (I_t - e_t) \geq I_t + 1 > E_t$. So $\{d_{t+1}\} \cap \{d_1, \dots, d_t\} = \emptyset$, hence $\{a_1, \dots, a_t, a_{t+1}, a_{t+2}\}$ has property 1.

Since $\{a_1, \dots, a_{t+2}\}$ now has property 1, we can apply Rule 1 to this sequence and obtain the sequence $\{a_1, \dots, a_{t+4}\}$, which again has property 1.

THEOREM 2. *Let $\{a_1, \dots, a_i\}$ have property 1 and assume that the infinite sequence $\{a_1, \dots, a_i, a_{t+1}, \dots\}$ is obtained from $\{a_1, \dots, a_t\}$ by applying Rule 1 successively, then the sequences $\{a_k | k \in N\}$ and $\{d_k | k \in N\}$ are both permutations.*

Proof. If $e_i \leq i_t$, then $d_{t+1} = e_i$. Hence the smallest difference which has not appeared in $\{d_1, \dots, d_{t+1}\}$ is larger than e_i , while i_t is still the smallest integer which has not appeared in $\{a_1, \dots, a_{t+2}\}$. If $i_t < e_i$, then just the opposite happens. We have that $a_{t+2} = i_t$ while the smallest difference which has not appeared in $\{d_1, \dots, d_{t+1}\}$ is still e_i . From these remarks the theorem follows by induction.

Let $\{m_1, m_2, \dots\}$ be any sequence of positive integers. Then by a slight variation we can obtain a permutation $\{a_k | k \in N\}$ such that $|\{i | d_i = j\}| = m_j$.

We say that $\{a_1, \dots, a_t\}$ has property 2 if the a_i are distinct and $|\{i | d_i = j, i < t\}| \leq m_j$, for all j .

Let i_t, I_t, E_t be defined as before. Let e_t be the smallest integer such that $|\{i | d_i = j, i < t\}| = m_j$, for $j < e_t$, and $|\{i | d_i = e_t, i < t\}| < m_{e_t}$. As before, we have that $E_t < I_t$ and $e_t \leq I_t$.

LEMMA 2. *Assume that $\{a_1, \dots, a_i\}$ has property 2 and that a_{t+1}, a_{t+2} are defined according to Rule 1, then $\{a_1, \dots, a_t, a_{t+1}, a_{t+2}\}$ also has property 2.*

Proof. The proof is exactly the same as Lemma 1.

THEOREM 3. *Let $\{m_1, m_2, \dots\}$ be any infinite sequence of positive integers and let $\{a_1, a_2, \dots, a_i\}$ be a sequence which satisfies property 2. If the sequence $\{a_1, \dots, a_i, a_{t+1}, \dots\}$ is obtained by successively applying Rule 1, then this sequence is a permutation and it also has the property that $|\{i | d_i = j\}| = m_j$.*

Proof. The proof follows by induction.

REMARK. There are sequences which satisfy property 2, for example, $\{a_1, a_2\}$, where $a_1 \neq a_2$.

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