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A NOTE ON THE GROUP STRUCTURE OF UNIT REGULAR RING ELEMENTS

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Local properties of unit regular ring elements are investigated. It is shown that an element of a ring R with unity is regular if and only if there exists a unit $u \in R$ and a group G such that $a \in uG$.

1. Introduction. It is well-known that [15, 7] a ring R is strongly regular if and only if every $a \in R$ is a group member. In this note we shall use the basic theorem for group members in a ring to show locally that a ring element $a \in R$ (with unity) is unit regular exactly when there is a unit $u \in R$ and a group G in R such that $a \in uG$. Hence unit regular rings are, as it were locally a "rotated" version of strongly regular rings.

We remind the reader that a ring R is called regular if for every $a \in R$, $a \in aRa$; strongly regular if for every $a \in R$, $a \in a^2R$, and unit regular if for every $a \in R$, there is a unit $u \in R$ such that $aua = a$ [3]. Similar definitions hold locally. A ring with unity is called finite if $ab = 1$ implies $ba = 1$. Any solution a^- to $axa = a$ is called an inner or 1-inverse of [1], while any solution a^+ to $axa = a$ and $xax = x$ is called a reflexive or 1-2 inverse of a.

For idempotents e and f in R, $e \sim f$ denotes the equivalence in the sense of Kaplansky [13] as contrasted with $a \overset{u}{\sim} b$ which denotes that $a = pbq$ with p and q invertible.

As usual, similarity will be denoted by $\approx$, the right and left annihilators of $a \in R$ will be denoted by $a^0 = \{x \in R: ax = 0\}$, ${}^0a = \{x \in R: xa = 0\}$ respectively, while interior direct sums and isomorphisms are denoted by $\dotplus$ and $\cong$ respectively. A ring R is called faithful if $aR = (0)$ implies $a = 0$.

We shall make use of the following fundamental theorem for group members.

THEOREM 1. *Let S be a semigroup and $a \in S$. The following are equivalent.*

1. *a is a group member.*
2. *a has a group inverse $a^\#$ in S which satisfies $axa = a$, $xax = x$ and $ax = xa$.*
3. *a has a commutative inner inverse a^- which satisfies $axa = a$, and $ax = xa$.*
4. *$aS = eS$, $Sa = Se$ and $a \in eSe$ for some idempotent $e \in S$.*
5. *$a \in a^2S \cap Sa^2$.*

6. *$a \in a^- aSaa^=$ for some inner inverses a^-, $a^=$ in S.*
7. *$aS = a^+S$ for some reflexive inverse a^+ in S.*
7a. *$Sa = Sa^+$ for some reflexive inverse a^+ in S.*
8. *$aS = a^-aS$ for some inner inverse a^- in S.*
8a. *$Sa = Saa^-$ for some inner inverse a^- in S.*

If in addition $S = R$ is a faithful ring, these are equivalent to

9. *$R = aR \dotplus a^0$.*
9a. *$R = Ra \dotplus {}^0a$.*

In any of the above cases $a^\#$ and $e = aa^\#$ are unique and the maximal subgroup containing a is given by

$$(1.1)\qquad \begin{aligned} H_a &= \{x \in S: x^\# \text{ exists}, \ xx^\# = aa^\# = e\} \\ &= \{x \in S: xS = aS, \ Sx = Sa, \ x \in aSa\} \ . \end{aligned}$$

Proof. For a proof of the equivalence of (1)—(5); we refer to [14, 7, 8].

(1) $\Rightarrow$ (6): Clearly, $a = a^\# a^3 a^\#$.

(6) $\Rightarrow$ (7): Let $a = a^- azaa^=$ for some $z \in S$ and set $a^+ = a^- aa^=$. Then $a = a^- aa^= azaa^= = a^+ azaa^= \in a^+S$.

On the other hand, since $a^3 = a(a^- azaa^=)a = aza$, we have $a = a^- a^3 a^=$, and $a^3a^= = a^2 = a^-a^3$. Hence $a^+ = a^-aa^= = a^-(a^-a^3a^=)a^= = a^-a^2a^=a^= = a^-a^3(a^=)^3 = a(a^=)^2 \in aS$, and so $a^+S = aS$.

(7) $\Rightarrow$ (8): Obvious, since $a^+S = a^+aS$.

(8) $\Rightarrow$ (1): If $aS = a^-aS$, then $a^2 = a^-ax$ for some x. Hence $a^-aa^2 = a^2$ or $a^-a^3 = a^2$.

Similarly, $a^-a = ay$ for some y, and so $a = a^2y$. By a result of Drazin [2] the index of a equals one and $a^\#$ exists.

The results 7a and 8a follow by symmetry.

We remark that an element $a \in R$ for which $aR = a^+R$ or $Ra = Ra^+$ for some a^+, generalizes so called *EP* elements [16, 7, 1] for which $aR = a^-R = a^\dagger R$, R $*$-regular, where $a^\dagger$ is the Moore-Penrose inverse of a. Thus in a $*$-regular ring, an *EP* element belongs to some group G.

For a proof of (9) $\Rightarrow$ (1) for the case where R has a unity 1 or is regular, we refer to [7]. When R is faithful we have to proceed as follows. $R = aR \dotplus a^0 \Rightarrow a = ar + n$, $an = 0 \Rightarrow a = a(as + m) + n$, for some $s \in R$, $m \in a^0$. Hence $a^2 = a^4b$, for some $b \in aR$. Also $a(ax) = 0 \Rightarrow ax \in aR \cap a^0 = (0)$, so that $(a^2)^0 = a^0$. Hence $R = a^2R \dotplus (a^2)^0$. It then follows that $b = (a^2)^\#$, since

$$a^2(a^2 - a^2ba^2) = a^2(a^2b - ba^2) = a^2(ba^2b - b) = 0 \ .$$

Because a^2 commutes with a, it follows by a result of Drazin [2] that $(a^2)^\# a = a(a^2)^\#$. Now $(a - a^2(a^2)^\# a)R = (a - a^2(a^2)^\# a)aR = (0)$ and hence if R is faithful, $a = a^2(a^2)^\# a = a^2(a(a^2)^\#)$. One may now repeat the above argument to show that $a^\# = a(a^2)^\# = a^\# a a^\#$.

That $(1) \Rightarrow (9)$ is clear.

Before giving our main result several remarks should be made here.

REMARK 1. The condition "faithful" may be replaced by the weaker condition

$$\text{(1.2)} \qquad \text{for every} \quad r \in R \quad r^0 \cap {}^0R = (0)\,.$$

This may not be dropped entirely as seen from the example

$$R = \left\{\begin{bmatrix} 0 & \alpha \\ 0 & 0 \end{bmatrix} : \alpha \text{ is a real number}\right\},\ a = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix},\ aR = (0),\ a^0 = R$$

and ${}^0R = R$. Here $R = aR + a^0$, yet $a^\#$ clearly does not exist since $a^2 = 0$.

REMARK 2. For a regular ring R with unity, (1.1) may be written as [10]

$$\text{(1.3)} \qquad H_a = \{x \in R : x = pa = aq \quad \text{for some units } p \text{ and } q\}\,.$$

REMARK 3. If a has a unique reflexive inverse a^+ then $a^\#$ exists, and if a has a unique idempotent of the form aa^+ then $a \in a^2R$. Hence if either of them hold globally, then R is stronly regular. These results are easy consequences of the fact that the class $\{a^+\}$ of *all* reflexive inverses of a is given by [9],

$$[a^+ + (1 - a^+a)R]a[a^+ + R(1 - aa^+)]\,.$$

2. Main results.

We begin with several preliminary results which will be used in our main theorem.

LEMMA 1. *If R is a ring with unity* 1, *and if* $\phi: aR \to bR$ *is a module isomorphism, where a and $p = \phi(a)$ are regular elements, then $Ra = Rp$ and $pR = bR$.*

Proof. $\phi(a) = \phi(aa^-a) = \phi(aa^-)a$ and $\phi(a) = pp^-p \Rightarrow a = \phi^{-1}(pp^-)p = \phi^{-1}(pp^-)\phi(a)$. The following is given in [10].

LEMMA 2. *If a and b are regular elements in a ring R with unity* 1, *then*

$$aR = bR \quad \text{and} \quad Ra = Rb \Longleftrightarrow b = ua = av$$

for some units u, v *in* R.

LEMMA 3. *Let* R *be a ring with unity* 1 *and* a *and* b *be regular elements in* R. *Then the following are equivalent*:

(i) $b \overset{u}{\sim} a$;

(ii) $aa^- \approx bb^-$ *and* $a^-a \approx b^-b$, *for some, and hence all* a^-, b^-;

(iii) $aa^- \sim bb^-, 1 - aa^- \sim 1 - bb^-$, *and* $1 - a^-a \sim 1 - b^-b$, *for some and hence all* a^-, b^-;

(iv) $aR \cong bR$ *and* $R/aR \cong R/bR$, $R/Ra \cong R/Rb$.

Proof. (i) $\Rightarrow$ (ii): If $b = paq$, for some units p and q, then for any particular a^-, $q^{-1}a^-p^{-1} \in \{b^-\}$, and hence $paa^-p^{-1} \in \{bb^-\}$, $q^{-1}a^-aq \in \{b^-b\}$. Now for any $a^= \in \{a^-\}$, $b^= \in \{b^-\}$, $aa^= \sim aa^-$, $bb^= \sim bb^-$ and thus $aa^= \approx aa^- \approx paa^{-1}p^{-1} \approx bb^=$.

(ii) $\Rightarrow$ (i): Let $aa^- = ubb^-u^{-1}$, $a^-a = v^{-1}b^-bv$. Then $aR = ubvR$, $Ra = Rubv$. Lemma 2 now ensures that $a = ubvp = qubv$ for some units p, q and thus $a \overset{u}{\sim} b$.

The equivalence of (ii) and (iii) is well-known since $aa^- \approx bb^- \Leftrightarrow aa^- \sim bb^-$ and $1 - aa^- \sim 1 - bb^-$, while $aa^- \sim bb^- \Leftrightarrow a^-a \sim b^-b$, [11].

(i) $\Rightarrow$ (iv): If $b = paq$ where p and q are units, then $aR \cong bR$ and $1 - bb^- = p(1 - aa^-)p^{-1} \Rightarrow (1 - bb^-)R = p(1 - aa^-)R \cong (1 - aa^-)R$. Lastly, since $bR \dotplus (1 - bb^-)R = R = aR \dotplus (1 - aa^-)R \Rightarrow R/aR \cong (1 - aa^-)R$ and $R/bR \cong (1 - bb^-)R$, the results follows.

(iv) $\Rightarrow$ (ii): If $aR \cong bR$ and $R/aR \cong R/bR$, then $(1 - aa^-)R \cong R/aR \cong R/bR \cong (1 - bb^-)R$ and so $aa^- \sim bb^-, 1 - aa^- \sim 1 - bb^-$. It follows that $aa^- \approx bb^-$. Similarly, $a^-a \approx b^-b$.

We note in passing that the statement $R/aR \cong R/bR$ is clearly equivalent to the statement "aR and bR have all direct summands isomorphic."

LEMMA 4. *If* $a \in R$ *is a regular element of* R *and* $1 \in R$, *then for all units* $u, v \in R$, $\{(uav)^-\} = v^{-1}\{a^-\}u^{-1}$.

Proof. This is an easy consequence of the fact that the class of *all* inner inverses of b is given by $\{b^-\} = b^- + (1 - b^-b)R + R(1 - bb^-)$.

We now come to the main theorem of this paper, which gives numerous conditions for a ring element to be unit regular.

THEOREM 2A. *Let* R *be a ring with unity* 1 *and let* $a \in R$. *Then the following are equivalent*:

1. $aua = a$ *for some unit* u *in* R.

2. *$(au)^{\#}$ exists for some unit u in R.*

2a. *$(ua)^{\#}$ exists for some unit u in R.*

3. *au has a commutative inner inverse for some unit u in R.*

3a. *ua has a commutative inner inverse for some unit u in R.*

4. *$auR = eR$ and $Rau = Re$ for some unit u and idempotent e in R.*

4a. *$uaR = eR$ and $Rua = Re$ for some unit u and idempotent e in R.*

5. *$a \in auaR \cap Raua$ for some unit u in R.*

6. *$R = aR \dotplus u(a^0)$ for some unit u in R.*

6a. *$R = Ra \dotplus ({}^0a)u$ for some unit u in R.*

Proof. (1) $\Rightarrow$ (2): Clearly, $aua = a \Rightarrow (au)^2 = au \Rightarrow (au)^{\#}$ exists.

(2) $\Rightarrow$ (1): Observe that $au[(au)^{\#} + (1-(au)^{\#}au)]au = au \Rightarrow auva = a$, where $v = (au)^{\#} + (1-(au)^{\#}au)$ and $v^{-1} = au + 1 - (au)^{\#}au$.

(2) $\Leftrightarrow$ (2a): $ua = u(au)u^{-1}$ and so $(ua)^{\#}$ exists exactly when $(au)^{\#}$ exists.

Since idempotents clearly are group members, it is obvious that a is unit regular precisely when $a \in uG$ for some group G and unit u in R. The equivalence of (2) through (6a) follows immediately from Theorem 1, applied to the group members au, and ua. For example, $au \in (au)^2R \cap R(au)^2 \Leftrightarrow a \in auaR \cap Raua$ and $(ua)^{\#}$ exists $\Leftrightarrow R = uaR \dotplus (ua)^0 \Leftrightarrow R = aR \dotplus u^{-1}(a^0)$. If we are given in addition that $a \in R$ is a *regular* element, then several important additional conditions may be given for a to be *unit* regular.

THEOREM 2B. *If R is a ring with unity 1 and $a \in R$ is a regular element, then the following are equivalent to a being unit regular.*

(7) *$a \in u^{-1}a^{-}aRaa^{=}u^{-1}$ for some unit u and some inner inverses a^{-}, $a^{=}$ in R.*

(8) *$a^{-}xa = y$, $aya^{=} = x$, where a^{-}, $a^{=}$ are inner inverses of $a \Rightarrow x \approx y$.*

(9) *$ca = ac$, $c \in R \Rightarrow caa^{=} \approx a^{-}ac$ for some and hence all inner inverses a^{-}, $a^{=}$ in R.*

(10) *$aa^{-} \approx a^{-}a$ for some and hence all inner inverses a^{-} in R.*

(11) *$aR = ua^{-}aR$ for some unit u and some inner inverse a^{-} in R.*

(12) *$aR = ua^{+}R$ for some unit u and some reflexive inverse a^{+} in R.*

(13) *$aR = eR$, with $e^2 = e \Rightarrow au = e$ for some unit u in R.*

(14) *$aR = bR$ with b unit regular $\Rightarrow ag = b$ for some unit g in R.*

(15) *$aR \overset{\phi}{\cong} bR$, with $\phi(a)$, b unit regular $\Rightarrow a \overset{u}{\sim} b$.*

(16) *$aR \overset{\phi}{\cong} bR$, with $\phi(a)$, bunit regular $\Rightarrow R/aR \cong R/bR$,*

together with their left analogues.

Proof. (2) $\Leftrightarrow$ (7): By Theorem 1(6), $(au)^{\#}$

$$\text{exists} \Longleftrightarrow au \in (au)^{-}auRau(au)^{=} \Longleftrightarrow au \in u^{-1}a^{-}aRaa^{=} \Longleftrightarrow a \in u^{-1}a^{-}aRaa^{=}u^{-1},$$

for some inner inverses a^{-}, $a^{=}$ of a. It should be noted that Lemma 4 was also used.

(1) $\Rightarrow$ (8): Let $aua = a$, where u is a unit. Then $y = a^{-}xa = a^{-}aya^{=}a \Rightarrow y = ya^{=}a = a^{-}ay = ya^{-}a$, and $x = aya^{=} = aa^{-}xaa^{=} \Rightarrow aa^{=}x = x = aa^{-}x = xaa^{=}$. Also, clearly, $ay = xa$ and $yua = ya^{=}aua = ya^{=}a = y$, $aux = x$. Now note that $y = a^{-}ay \approx uay$ since $uay(1 - a^{-}a + ua) = uay = (1 - a^{-}a + ua)a^{-}ay$ and so, $y = a^{-}ay \approx uay = uxa = u(xau)u^{-1}$. Next, again $xua \approx xaa^{=} = x$, for

$$(1 - aa^{=} + au)xau = xau = xaa^{=}(1 - aa^{=} + au)\,.$$

And so, $y = q^{-1}xq$, where $q = (1 - aa^{=} + au)u^{-1}(1 - a^{-}a + ua)$.

(8) $\Rightarrow$ (9): Since $a^{-}(caa^{=})a = a^{-}ac$ and $a(a^{-}ac)a^{=} = caa^{=}$, the result follows at once from (9).

(9) $\Rightarrow$ (10): Because $aa^{-} \approx aa^{=}$ for any $a^{-}, a^{=}$, we simply set $c = 1$ in (9).

(10) $\Rightarrow$ (11): $aa^{-} \approx a^{-}a \Rightarrow aa^{-} = ua^{-}au^{-1}$ for some unit $u \Rightarrow aR = ua^{-}aR$ as desired.

(11) $\Rightarrow$ (12): $aR = ua^{-}aa^{-}aR = ua^{+}aR = ua^{+}R$, where $a^{+} = a^{-}aa^{-}$.

(12) $\Rightarrow$ (2a): Let $aR = ua^{+}R$. Then $u^{-1}aR = a^{+}R = a^{+}uR = (u^{-1}a)^{+}R$, and hence by Theorem 1(7), $(u^{-1}a)^{\#}$ exists.

(1) $\Rightarrow$ (13): If $aR = eR$ and $aua = a$, u unit, $e^2 = e$, then $auR = eR \Rightarrow aue = e$. Hence $auv = e$, where $v = 1 - au + e$, $v^{-1} = 1 + au - e$. Thus a and e are right associates.

(13) $\Rightarrow$ (14): If $aR = bR$, $bvb = b$ and v is a unit, then $aR = eR$, where $e = bv$. By (13), $au = e = bv$ for some unit e. Hence $auv^{-1} = b$ as desired.

(14) $\Rightarrow$ (1): Since $aR = aa^{-}R$, and aa^{-} is unit regular, (14) implies that $ag = aa^{-}$ for some unit g. Hence $aga = a$ as requested. It is now clear by symmetry, that the *left* analogues of the above results also are equivalent to element a being unit regular.

(14) $\Rightarrow$ (15): Suppose that (14) and hence its left analogue (14a) both hold.

Now let $aR \cong \phi(a)R = bR$ and $p = \phi(a)$. Then by Lemma 1, $Ra = Rp$ and $pR = bR$, so that by (14) and (14a), $pv = b$ and $ua = p$ for some units u and v. These are in fact given by $u = (p^{=})^{-1}(1 + p^{=}p - a^{=}a)a^{=}$, $v = p^{=}(1 - pp^{=} + bb^{=})(b^{=})^{-1}$, in which $a^{=}$, $b^{=}$, and $p^{=}$ are unit inner inverses. Hence $b = uav$, as desired.

(15) $\Rightarrow$ (16): This follows immediately from Lemma 3.

(16) $\Rightarrow$ (1): Since $aR \overset{\phi}{\cong} a^{-}aR$, where $\phi(a) = a^{-}a = b$, it follows that $aa^{-} \sim b, 1 - aa^{-} \sim 1 - b$, so that $aa^{-} \approx b$.

Hence, by Lemma 3, $uav = b = a^{-}a$ for some units u, v, which implies that $uavuav = uav$ or $a(vu)a = a$, as desired. Alternatively, (10) could be used.

The remaining results follow again by symmetry.

REMARK 1. In (8), we proved the conjecture made in [12] that pseudosimilarity implies similarity in a unit regular ring. Pseudosimilarity, $\underset{\sim}{\approx}$, is defined by

DEFINITION 1. $x \underset{\sim}{\approx} y$ if $a^{-}xa = y, aya^{=} = x$ for some a and its inner inverses $a^{-}, a^{=}$.

REMARK 2. The equivalence of (1) and (6) was also proved by Ehrlich [4] who used endomorphism rings. As shown above it is actually a simple consequence of the fundamental Theorem 1.

REMARK 3. Part (10) should be compared with the *global* result of Vidav [17] and Fuchs [5], which state that a regular ring R is unit regular exactly when $e^2 = e \sim f = f^2 \Rightarrow e \approx f$ [17] or when $aR \cong bR \Rightarrow R/aR \cong R/bR$ [5].

REMARK 4. The global analogue of (16) is that a regular ring R is unit regular exactly when $aR \cong bR$ implies that aR and bR have a *common* direct summand [6].

One final remark is here needed, namely, if R is a unit regular ring and if $\phi: aR \rightarrow bR$ is *any* isomorphism, then, by Lemma 1, $Ra = R\phi(a)$ and hence by (14a) $\phi(a) = ua$ for some unit u.

We have thus shown:

COROLLARY 1. *In a unit regular ring R, all right module isomorphisms $\phi: aR \rightarrow bR$, are of the form $\phi(ar) = uar$, where u is a unit. Similarly, all left module isomorphisms $\phi: Ra \rightarrow Rb$ are of the form $\phi(ra) = rav$, for some unit $v \in R$.*

The converse of these statements always hold.

3. The unit inner inverses. We shall now examine more closely the class $\mathscr{U}_a$ of unit inner inverses of a given element a of a unit regular ring.

We begin by noting that if $aua = a$, with u invertible then $\mathscr{U}_a$ can be represented as

$$\mathscr{U}_a = u\mathscr{U}_{au} = \mathscr{U}_{ua}u \ . \tag{3.1}$$

Indeed, if $w \in \mathscr{U}_{au}$, then $auwau = au$ which implies that $auwa = a$ and hence $uw \in \mathscr{U}_a$, while conversely, if $awa = a$, w a unit, then $au(u^{-1}w)au = au$ which implies that $u^{-1}w \in \mathscr{U}_{au}$ and hence $w \in u\mathscr{U}_{au}$. The second identity follows similarly.

Since $u\mathscr{U}_{au}$ is independent of the choice of the unit inner inverse u of a, we have, for any unit inner inverses u and v of a,

$$\mathscr{U}_a = u\mathscr{U}_{au} = v\mathscr{U}_{av} \ , \tag{3.2}$$

so that in particular, $u^{-1}v \in \mathscr{U}_{au}$.

Consequently, the set $\mathscr{U}_a$ is determined by the set of unit inner inverses $\mathscr{U}_e$ of the idempotent element $e = au$. When $e^2 = e$, there are several representations for $\mathscr{U}_e$. In fact, $\mathscr{U}_e$ is the set of *all* units of the form:

$$\begin{array}{lll} \text{(i)} & 1 + (1-e)x + y(1-e) & \text{for some } x, y; \\ \text{(ii)} & e + (1-e)v + s(1-e) & \text{for fome } v, s; \\ \text{(iii)} & 1 + h - ehe & \text{for some } h; \\ \text{(iv)} & e + k - eke & \text{for some } k. \end{array} \tag{3.3}$$

In general, the set $\mathscr{U}_a$ or even $\mathscr{U}_e$ will not be a union of semigroups. For example, if $e = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} \in \boldsymbol{R}_{2\times 2}$, where $\boldsymbol{R}_{2\times 2}$ denotes the two by two matrix ring over the real field, then it is easy to see that $\begin{bmatrix} -1 & 2 \\ 0 & 2 \end{bmatrix} \in \mathscr{U}_e$, but $\begin{bmatrix} -1 & 2 \\ 0 & 2 \end{bmatrix}^2 \notin \mathscr{U}_e$.

In fact, it is only for idempotent elements possible to possess union of semigroups of unit inner inverses.

PROPOSITION 3. *Let a be a unit regular element of a ring R with unity* 1.

(i) *If the set $\mathscr{U}_a$ of unit inner inverses of a is a union of semigroups then $a^2 = a$.*

(ii) *If R is a prime ring and if $\mathscr{U}_a$ forms a semigroup, then $a = 0$ or $a = 1$.*

Proof. (i) Let $aua = a$ with u a unit. Then $u^2 \in \mathscr{U}_a$ and $au^2a = a$. Now consider: $au(1 + au(1 - au))a = aua + au(1 - au)a = a + a - a = a$, which implies that $u(1 + a(1 - au)) \in \mathscr{U}_a$. Thus $(u(1 - a(1 - au)))^2 \in \mathscr{U}_a$. That is, $a = a(u(1 - a(1 - au)))^2a = (au - a(1 - au))u(1 - a(1 - au))a = (au^2 - au + a^2u^2)(a - a^2 + a^2) = au^2a - aua + a^2u^2a = a - a + a^2 = a^2$.

(ii) Now suppose that $a = e = e^2$. Then clearly $1 + eR(1 - e)$ and $1 + (1 - e)Re$ are contained in $\mathscr{U}_e$. Hence by the semigroup assumption, $e(1 + eR(1 - e))(1 + (1 - e)Re)e = e$ which implies that

$$eR(1 - e)Re = 0 \,. \tag{3.4}$$

Since R is prime, it follows that either $e = 0$ or $e = 1$ as desired.

REMARK 1. In (ii), the primeness cannot be dropped as seen from the example of semiprime ring $R = Z_2 \oplus Z_2$, where Z_2 denotes the Galois field of order 2. Here $\mathscr{U}_{(1,0)} = \mathscr{U}_{(0,1)} = \{(1, 1)\}$ is a semigroup, yet (1, 0) and (0, 1) are neither zero element nor unity element.

REMARK 2. The *same* conclusions may be drawn if the element is just regular and the set $\{a^-\}$ of inner inverses forms a semigroup. In fact, if $aba = a$ then $ab^2a = a$ and also $a(b - ba + ba^2b)a = a \Rightarrow a(b - ba + ba^2b)ba = a \Rightarrow a = a - aba + a^2b^2a = a^2$.

The rest follows as in part (ii).

REMARK 3. For an invertible element $1 + h - ehe$ in a unit regular ring, ehe need not lie in H_e. For example, if $e = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} \in \boldsymbol{R}_{2\times 2}$ and $h = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$, then $1 + h - ehe = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ is invertible but $ehe = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \notin H_e$.

There are five sets of units that appear naturally in the study of $\mathscr{U}_e$. These are:

1. $P_e = 1 + (1 - e)Re = \{u \in \mathscr{U}_e \colon e(1 - u)(1 - e) = 0\}$,
2. $Q_e = 1 + eR(1 - e) = \{u \in \mathscr{U}_e \colon (1 - e)(1 - u)e = 0\}$,
3. $V_e = \{v \in \mathscr{U}_e \colon ev = e\}$,
4. $W_e = \{w \in \mathscr{U}_e \colon we = e\}$, and
5. $C_e = \{z \in R \colon ez = ze, z \text{ is a unit}\}$.

For example, $1 - aa^= + aa^{\equiv} \in W_{aa^-}$ for any inner inverses $a^-, a^=, a^{\equiv}$ of a.

It is easily seen that

(i) all these sets are semigroups (in fact monoids).

(ii) $P_e \subseteqq V_e \subseteqq \mathscr{U}_e$, $Q_e \subseteqq W_e \subseteqq \mathscr{U}_e$, $V_e \cap W_e = \{1 + (1 - e)x(1 - e) \in \mathscr{U}_e \colon x \in R\}$.

(iii) $P_e \cap Q_e \subseteqq V_e \cap W_e = \mathscr{U}_e \cap C_e \subseteqq C_e$.

In addition it is known that [14]

(iv) $eC_e = H_e$ is the maximal subgroup containing e.

Moreover, it is easily shown that

(v) $V_e\mathscr{U}_eW_e = \mathscr{U}_e = P_e\mathscr{U}_eQ_e$, for let $u \in \mathscr{U}_e$, $v \in V_e$, $w \in W_e$, then $evuwe = eue = e$, while conversely $u = 1 \cdot u \cdot 1$ ensures the first equality. The second equality follows similarly.

It should be remarked here that in general $P_e \neq V_e$, $Q_e \neq W_e$, for again let $e = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ and $x = \begin{bmatrix} x_1 & x_3 \\ x_2 & x_4 \end{bmatrix}$ in $\boldsymbol{R}_{2\times 2}$ with $x_1 \neq 0$ and $1 + x_1$ invertible. Then $1 + (1 - e)x = \begin{bmatrix} 1 + x_1 & x_3 \\ 0 & 1 \end{bmatrix} \in V_e$, while $\begin{bmatrix} x_1 & x_3 \\ 0 & 0 \end{bmatrix} \neq ye$ for any $y \in \boldsymbol{R}_{2\times 2}$.

Before examining the subgroup H_e, let us first prove a global conjecture made in [11]. We start with

LEMMA 4. *Let R be a ring with unity 1. Then the following two conditions are equivalent.*

(i) *R is unit regular such that every nonzero element in R has a unique inner inverse;*

(ii) *R contains only idempotent elements and invertible elements.*

Proof. (i) $\Rightarrow$ (ii): Suppose $a^2 \neq a \in R$ and $aua = a$, u a unit $u \neq 1$. Then

$$au(1 - a(1 - au))a = a = a(1 - (1 - ua)a)ua$$

where $(1 - a(1 - au))^{-1} = 1 + a(1 - au)$ and

$$(1 - (1 - ua)a)^{-1} = 1 + (1 - ua)a .$$

Hence by uniqueness, $u(1 - a(1 - au)) = u = (1 - (1 - ua)a)u$ or $a(1 - au) = 0 = (1 - ua)a$. Now $a^2u = a = ua^2$ implies by Theorem 1, that a has a group inverse $a^\# = uau$. Consequently, $au = aa^\# = a^\# a = ua$. Since $a(a^\# + 1 - aa^\#)a = a$ and $(a^\# + 1 - aa^\#)^{-1} = a + 1 - aa^\#$, it follows by uniqueness that $u = uau + 1 - au$ or $u(1 - au) = 1 - au$. Multiplying this by $1 - au$, we obtain

$$(1 - au)u(1 - au) = 1 - au . \tag{3.7}$$

Now either $1 - au = 0$ or $1 - au \neq 0$. Since $1 - au \neq 0$ is idempotent and $(1 - au)1(1 - au) = 1 - au$, uniqueness implies that $u = 1$, which is impossible. Hence $au = 1 = ua$ and a is a unit.

(ii) $\Rightarrow$ (i): It is clear that R is a regular ring. Now let $a \in R$ and $a \neq 0$. First suppose $a = 1$. Then $aua = a$ implies that $u = 1$ and so is unique. Next, suppose $a \neq 1$. If $a^2 = a$ and $aua = a$, where u is a unit $\neq 1$, then $1 - u$ is also a unit. For otherwise $(1 - u)^2 = 1 - u$ would imply that $u^2 = u$ which forces u to equal 1. Now, since a is not a unit, $a(1 - u)$ is not a unit. Hence $[a(1 - u)]^2 = a(1 - u)$. This implies that $a = a(1 - u)a = a^2 - aua = a^2 - a = 0$, a contradiction. Hence $u = 1$ and the unit inner inverse of a is unique. If a is a not idempotent then a is a unit and clearly a^{-1} is the only unit inner inverse of a, completing the proof.

We may now sharpen this to the following.

THEOREM 4. *Let R be a unit regular ring. If every nonzero element of R has a unique unit inner inverse then either R is a Boolean ring or R is a division ring.*

Proof. Suppose R is neither Boolean nor a division ring. Then there exists $a \in R$ such that $a^2 \neq a$ and there are $x \neq 0$, $y \neq 0$ in R such that $xy = 0$, (since it is well-known that a regular integral domain must be a field). By Lemma 4, a is a unit and x and y are idempotents. Now, consider element ax. If $(ax)^2 = ax$ then

$$a(xa-1)x = 0 \Longrightarrow (xa - 1)x = 0 \Longrightarrow x = xax \Longrightarrow a = 1\ ,$$

by the uniqueness of unit inner inverses of x. This yields a contradiction. On the other hand, if $(ax)^2 \neq ax$ then ax must be a unit which implies that x is a unit and thus that $y = 0$, which again is a contradiction. Thus R must be either a division ring or a Boolean ring.

Let us now consider briefly the maximal subgroup

$$H_e = \{x \in R\colon xR = eR,\ Rx = Re\}$$

which contains the idempotent element $e \in R$. We begin with a global result.

PROPOSITION 5. *If R is a regular ring with unity 1 and e is an idempotent element in R, then*

$$H_e = \{eue\colon eueve = e = eveue,\ u, v \text{ units in } R\}\ . \tag{3.8}$$

This says that the e-units in eRe are all of the form eue for some 1-unit $u \in R$.

Proof. It is well-known that

$$\begin{aligned} H_e &= \{ere\colon erese = e = esere;\ r, s \in R\} \\ &= \{ere\colon ereR = eR,\ Rere = Re\}\ . \end{aligned}$$

By Lemma 3, for $ere \in H_e$ there are units u, v in R such that $ereu = e = vere$, which implies that $(ere)(eue) = e = (eve)(ere)$. The uniqueness of e-inverses ensures that $eue = eve$.

Now again by Lemma 3, since $eueR = eR$ *and* $Reue = Re$, there are units w, z in R, such that $euew = e = zeue$. Consequently, $euewe = e = ezeue$. And so, by uniqueness, $ewe = eze = ere$. Hence we may replace in each element ere the element r by a 1-unit $w \in R$.

Conversely, it is easily seen that this set is contained in H_e.

We remark that when R is a finite regular ring [11] we may shorten this to

$$H_e = \{eue\colon eueve = e;\ u, v \text{ units in } R\}\,. \tag{3.9}$$

Suppose now again that $aua = a = ava$, with u, v units in R. Then if we set $e = au$, $f = av$, we have $a \in H_{au}u^{-1}$, and more generally, $a \in \bigcap \{H_{au}u^{-1}\colon u \in \mathscr{U}_a\}$. Since $eR = fR = aR$, it follows that $ef = f$, $fe = e$ and that $e \approx f$. In fact, if $w = 1-e+f = (1+e-f)^{-1} = 1 - a(u - v)$, then $ew = wf = f$ and thus

$$wH_f w^{-1} = H_e\ , \tag{3.10}$$

that is, the subgroups H_{au} and H_{av} are isomorphic. It follows similarly that

$$H_{ua} = uH_{au}u^{-1}\ , \tag{3.11}$$

because $x \in H_{ua} \Leftrightarrow u^{-1}xu \in H_{au}$. And so, the subgroups H_{au}, H_{ua}, H_{av}, H_{va} are all isomorphic.

4. Conclusions. We have seen that an element $a \in R$ is unit regular exactly when $a \in uG$ for some unit u and group G in R. In the same way that the concept of a Drazin inverse a^d(see [1, 2]) generalizes that of a group inverse $a^\#$ to the case that $(a^k)^\#$ exists for some $k \geqq 1$, we may generalize the concept of a unit regular element.

DEFINITION 2. (i) An element $a \in R$ is k-unit regular if a^k is unit regular for some $k \geqq 1$.

(ii) An element $a \in R$ is unit-Drazin invertible if there is a unit $u \in R$ such that $(ua)^k$ is a group member for some $k \geqq 1$.

By Theorem 2, the former is equivalent to $R = a^kR \dotplus u(a^k)^0$, while the latter reduces to the existence of $(ua)^d$.

In closing we mention of few open problems relating to $\mathscr{U}_a$ in a unit regular ring. Let e be an idempotent element.

1. For what h is $1 + h - ehe$ invertible?
2. For what x is $1 + (1 - e)x$ invertible?
3. How are $\mathscr{U}_e$ and H_e related?
4. What sort of subgroup is $\cap \{H_{au}\colon u \in \mathscr{U}_a\}$?
5. For what type of regular semigroups does Theorem 2, 1–2 remain valid?

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