

Pacific Journal of Mathematics

REAL REPRESENTATIONS OF GROUPS WITH A SINGLE INVOLUTION

I. MARTIN (IRVING) ISAACS

REAL REPRESENTATIONS OF GROUPS WITH A SINGLE INVOLUTION

I. M. ISAACS

If G is a finite group containing just one involution and G has a faithful, absolutely irreducible real representation, then G has order 2.

This was proved by Jerry Malzan [2] using the classification of simple groups with dihedral Sylow 2-subgroups. The purpose of this note is to give a proof of Malzan's theorem which assumes nothing but some elementary character theory.

Let G have the unique involution z and assume $G > \langle z \rangle$. Let $\chi \in \text{Irr}(G)$ be faithful and real valued (where $\text{Irr}(G)$ is the set of complex irreducible characters of G). By the Frobenius-Schur theory (see Lemma 4.4 and Corollary 4.15 of [1]) it follows that in order to prove that χ is not afforded by a real representation, it suffices to show that

$$\sum_{g \in G} \chi(g^2) \neq |G|.$$

THEOREM. *In the above situation we have*

$$\sum_{g \in G} \chi(g^2) < |G|.$$

Proof. Each $g \in G$ may be uniquely factored as $g = \sigma c$ where σ has 2-power order and $c \in C(\sigma)$ has odd order. We write $\sigma = g_2$. For each cyclic 2-subgroup $U \subseteq G$ we set $Y(U) = \{g \in G \mid \langle g_2 \rangle = U\}$. Thus the sets $Y(U)$ partition G . We shall prove

- (1) $\sum_{g \in Y(1)} \chi(g^2) = \sum_{g \in Y(\langle z \rangle)} \chi(g^2) < |G|/2$
- (2) $\sum_{g \in Y(U)} \chi(g^2) \leq 0$ if $|U| = 4$
- (3) $\sum_{g \in Y(U)} \chi(g^2) = 0$ if $|U| \geq 8$.

The theorem will then follow.

Proof of (1). $Y(1)$ is the set of elements of G of odd order and since $z \in Z(G)$, we have $Y(\langle z \rangle) = zY(1)$ and so $\sum_{g \in Y(1)} \chi(g^2) = \sum_{g \in Y(\langle z \rangle)} \chi(g^2)$. Since the map $g \mapsto g^2$ is a permutation of $Y(1)$, the common value of these sums is

$$s = \sum_{g \in Y(1)} \chi(g).$$

If α is any automorphism of the field $\mathbb{Q}(\chi)$, then there exists an integer m with $(m, |G|) = 1$ such that $\chi(g)^\alpha = \chi(g^m)$ for all $g \in G$. Since the map $g \mapsto g^m$ is a permutation of $Y(1)$, it follows that $s^\alpha = s$ and thus s is rational.

Now let $\chi = \chi_1, \chi_2, \dots, \chi_n$ be the distinct Galois conjugates of χ and let $\theta = \sum \chi_i$. Then θ is rational valued and hence $\theta(g) \in \mathbb{Z}$ and $\theta(g) \leq \theta(g)^2$ for all $g \in G$. Furthermore, $s = \sum_{Y(1)} \chi_i(g)$ for all i since s is rational, and thus

$$ns = \sum_{g \in Y(1)} \theta(g) \leq \sum_{g \in Y(1)} \theta(g)^2.$$

Since $\chi(zg) = -\chi(g)$ for all $g \in G$, we have $\sum_{Y(1)} \theta(g)^2 = \sum_{Y(\langle z \rangle)} \theta(g)^2$ and so

$$\begin{aligned} 2ns &\leq \sum_{g \in Y(1) \cup Y(\langle z \rangle)} \theta(g)^2 \\ &\leq \sum_{g \in G} \theta(g)^2 = |G|[\theta, \theta] = n|G|. \end{aligned}$$

Therefore, $s \leq |G|/2$. In fact, this inequality is strict since otherwise $\theta(1) = \theta(1)^2$ and hence $\chi(1) = 1$. Since χ is real-valued and faithful and $|G| > 2$, this is impossible and (1) follows.

Proof of (2). Let $|U| = 4$ with $\langle \sigma \rangle = U$. Since $C(\sigma)$ has a unique involution and a central element of order 4, it follows that $C(\sigma)$ has a cyclic Sylow 2-subgroup and therefore has a normal 2-complement N . Thus $Y(U) = \sigma N \cup \sigma^{-1}N$. Since $\sigma^2 = (\sigma^{-1})^2 = z$ and $\chi(zg) = -\chi(g)$ for all $g \in G$, we have

$$\begin{aligned} \sum_{g \in Y(U)} \chi(g^2) &= -2 \sum_{g \in N} \chi(g^2) \\ &= -2 \sum_{g \in N} \chi(g) = -2|N|[\chi_N, 1_N] \leq 0 \end{aligned}$$

since $g \mapsto g^2$ is a permutation of N .

Proof of (3). Let $|U| \geq 8$ and let V be the subgroup of order 4 in U . If $g \in Y(U)$ and $\tau \in V$, then $\tau g \in Y(U)$ and hence $Y(U)$ is a union of cosets of V of the form Vx with $x \in C(V)$. Now

$$\sum_{g \in Vx} \chi(g^2) = 2\chi(x^2) + 2\chi(zx^2) = 0.$$

REFERENCES

1. I. M. Isaacs, *Character Theory of Finite Groups*, Academic Press, New York, 1976.
2. J. Malzan, *On groups with a single involution*, Pacific J. Math., **57** (1975), 481-489.
3. ———, *Corrections to On groups with a single involution*, Pacific J. Math., **67** (1976), 555.

Received November 22, 1976. Research supported by Grant MCS 74-06398A02.

UNIVERSITY OF WISCONSIN-MADISON
MADISON, WI 53706

PACIFIC JOURNAL OF MATHEMATICS

EDITORS

RICHARD ARENS (Managing Editor)

University of California
Los Angeles, CA 90024

CHARLES W. CURTIS

University of Oregon
Eugene, OR 97403

C. C. MOORE

University of California
Berkeley, CA 94720

J. DUGUNDJI

Department of Mathematics
University of Southern California
Los Angeles, CA 90007

R. FINN and J. MILGRAM

Stanford University
Stanford, CA 94305

ASSOCIATE EDITORS

E. F. BECKENBACH

B. H. NEUMANN

F. WOLF

K. YOSHIDA

SUPPORTING INSTITUTIONS

UNIVERSITY OF BRITISH COLUMBIA
CALIFORNIA INSTITUTE OF TECHNOLOGY
UNIVERSITY OF CALIFORNIA
MONTANA STATE UNIVERSITY
UNIVERSITY OF NEVADA, RENO
NEW MEXICO STATE UNIVERSITY
OREGON STATE UNIVERSITY
UNIVERSITY OF OREGON
OSAKA UNIVERSITY

UNIVERSITY OF SOUTHERN CALIFORNIA
STANFORD UNIVERSITY
UNIVERSITY OF HAWAII
UNIVERSITY OF TOKYO
UNIVERSITY OF UTAH
WASHINGTON STATE UNIVERSITY
UNIVERSITY OF WASHINGTON
* * *
AMERICAN MATHEMATICAL SOCIETY

The Supporting Institutions listed above contribute to the cost of publication of this Journal, but they are not owners or publishers and have no responsibility for its content or policies.

Mathematical papers intended for publication in the *Pacific Journal of Mathematics* should be in typed form or offset-reproduced, (not dittoed), double spaced with large margins. Please do not use built up fractions in the text of your manuscript. You may however, use them in the displayed equations. Underline Greek letters in red, German in green, and script in blue. The first paragraph or two must be capable of being used separately as a synopsis of the entire paper. Items of the bibliography should not be cited there unless absolutely necessary, in which case they must be identified by author and Journal, rather than by item number. Manuscripts, in triplicate, may be sent to any one of the editors. Please classify according to the scheme of Math. Reviews, Index to Vol. 39. All other communications should be addressed to the managing editor, or Elaine Barth, University of California, Los Angeles, California, 90024.

The Pacific Journal of Mathematics expects the author's institution to pay page charges, and reserves the right to delay publication for nonpayment of charges in case of financial emergency.

100 reprints are provided free for each article, only if page charges have been substantially paid. Additional copies may be obtained at cost in multiples of 50.

The *Pacific Journal of Mathematics* is issued monthly as of January 1966. Regular subscription rate: \$72.00 a year (6 Vols., 12 issues). Special rate: \$36.00 a year to individual members of supporting institutions.

Subscriptions, orders for back numbers, and changes of address should be sent to Pacific Journal of Mathematics, 103 Highland Boulevard, Berkeley, California, 94708.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION

Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.).

8-8, 3-chome, Takadanobaba, Shinjuku-ku, Tokyo 160, Japan.

Copyright © 1975 by Pacific Journal of Mathematics
Manufactured and first issued in Japan

Krishnaswami Alladi and Paul Erdős, <i>On an additive arithmetic function</i>	275
James Bailey and Dale Rolfsen, <i>An unexpected surgery construction of a lens space</i>	295
Lawrence James Brenton, <i>On the Riemann-Roch equation for singular complex surfaces</i>	299
James Glenn Brookshear, <i>Projective ideals in rings of continuous functions</i>	313
Lawrence Gerald Brown, <i>Stable isomorphism of hereditary subalgebras of C^*-algebras</i>	335
Lawrence Gerald Brown, Philip Palmer Green and Marc Aristide Rieffel, <i>Stable isomorphism and strong Morita equivalence of C^*-algebras</i>	349
N. Burgoyne, Robert L. Griess, Jr. and Richard Lyons, <i>Maximal subgroups and automorphisms of Chevalley groups</i>	365
Yuen-Kwok Chan, <i>Constructive foundations of potential theory</i>	405
Peter Fletcher and William Lindgren, <i>On $w\Delta$-spaces, $w\sigma$-spaces and Σ^\sharp-spaces</i>	419
Louis M. Friedler and Dix Hayes Pettey, <i>Inverse limits and mappings of minimal topological spaces</i>	429
Robert E. Hartwig and Jiang Luh, <i>A note on the group structure of unit regular ring elements</i>	449
I. Martin (Irving) Isaacs, <i>Real representations of groups with a single involution</i>	463
Nicolas P. Jewell, <i>The existence of discontinuous module derivations</i>	465
Antonio M. Lopez, <i>The maximal right quotient semigroup of a strong semilattice of semigroups</i>	477
Dennis McGavran, <i>T^n-actions on simply connected $(n + 2)$-manifolds</i>	487
Charles Anthony Micchelli and Allan Pinkus, <i>Total positivity and the exact n-width of certain sets in L^1</i>	499
Barada K. Ray and Billy E. Rhoades, <i>Fixed point-theorems for mappings with a contractive iterate</i>	517
Fred Richman and Elbert A. Walker, <i>Ext in pre-Abelian categories</i>	521
Raymond Craig Roan, <i>Weak* generators of H^∞ and l^1</i>	537
Saburou Saitoh, <i>The exact Bergman kernel and the kernels of Szegő type</i> ...	545
Kung-Wei Yang, <i>Operators invertible modulo the weakly compact operators</i>	559